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UNIVALENT FUNCTIONS HAVING UNIVALENT DERIVATIVES

HERB SILVERMAN

ABSTRACT. For functions of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geqq 0$, that are analytic and univalent in the unit disk, we investigate subclasses of functions having some or all of their derivatives univalent. Sufficient conditions are given for the functions to be in the various classes and a sharp upper bound for the second coefficient of functions whose derivatives are all univalent is found. Surprisingly, there is no function in the class whose second coefficient attains this sharp upper lound.

1. Introduction. A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is said to be in the family S if it is analytic and univalent in the unit disk $\Delta = \{|z| < 1\}$. If f may further be expressed as

$$(1) \qquad f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \geqq 0.$$

then f is said to be in the family T. In **[6]** it is shown that functions of the form (1) are in T if and only if $\sum_{n=2}^{\infty} n\, a_n \leqq 1$. This enabled us to show that the extreme points of T were z and $z - z^n/n$ $(n = 2, 3, \ldots)$.

Denote, by T_1, the subfamily of T consisting of functions f for which f' is also univalent in Δ. Since the second coefficient of a function in T_1 cannot vanish, the only extreme point of T that is also a member of T_1 is $z - z^2/2$. Although T_1, unlike T, cannot easily be characterized by its coefficients, we do find separate sufficient and necessary conditions that lead to various coefficient bounds. We also investigate subfamilies of T for which higher order derivatives are univalent and obtain a sharp upper bound for the second coefficient when all derivatives are univalent.

2. The family T .

THEOREM 1. *If $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T_1$, then $a_2 \leqq 1/2$ and*

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$$r - r^2/2 \leqq |f(z)| \leqq r + r^2/2, \qquad (|z| = r < 1),$$
$$1 - r \leqq |f'(z)| \leqq 1 + r, \qquad (|z| = r < 1).$$

Equality holds only for $f(z) = z - z^2/2$.

PROOF. In [6], it is shown that these bounds are valid for T, and, consequently, for the subfamily T_1.

We now give a sufficient condition for functions of the form (1) to have a univalent derivative.

THEOREM 2. *If* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T$, $a_2 > 0$, *then* $f \in T_1$ *if*

$$\sum_{n=3}^{\infty} (n-1)n\, a_n \leqq 2a_2. \tag{2}$$

PROOF. The function $f'(z) = 1 - \sum_{n=2}^{\infty} n\, a_n z^{n-1}$ is univalent in Δ if and only if the normalized function

$$g(z) = \frac{1 - f'(z)}{2a_2} = z + \sum_{n=2}^{\infty} \frac{(n+1)a_{n+1}}{2a_2} z u = z + \sum_{n=2}^{\infty} b_n z^n$$

is in S. The result follows upon noting that $\sum_{n=2}^{\infty} n\, b_n \leqq 1$ is a sufficient condition for g to be in S.

Inequality (2) is sharp in the following sense: Given $\varepsilon > 0$, there exists a sequence $\{a_n\}$, $0 < a_2 < 1/2$, such that $\sum_{n=3}^{\infty}(n-1)n\, a_n = 2a_2 + \varepsilon$ and $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T - T_1$. To see this, set $f(z) = z - a_2 z^2 - ((2a_2 + \varepsilon)/n(n+1))z^{n+1}$, where n is large enough to insure that $2a_2 + (2a_2 + \varepsilon)/n \leqq 1$. Then $f \in T$, but $g(z) = (1 - f'(z))/2a_2$ is not in S because $g'(z) = 1 + (1 + \varepsilon/2a_2)z^{n-1} = 0$, for a point z_0, $|z_0| < 1$.

COROLLARY 1. *If* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ $(a_n \geqq 0, 0 < a_2 \leqq 1/3)$, *then* (2) *is a sufficient condition for f to be in* T_1.

PROOF. In view of Theorem 2, we need only show that $f \in T$. Since $2\sum_{n=3}^{\infty} n\, a_n \leqq \sum_{n=3}^{\infty}(n-1)n\, a_n \leqq 2a_2$, we have $\sum_{n=2}^{\infty} n\, a_n \leqq 2a_2 + a_2 \leqq 1$, for $a_2 \leqq 1/3$.

COROLLARY 2. *If* $\{a_n\}$ *is a complex sequence,* $a_2 \neq 0$, *and* $a_n/a_2 \leqq 0$ *for* $n \geqq 3$, *then* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *has a univalent derivative in* Δ *if and only if* $\sum_{n=3}^{\infty}(n-1)n\,|a_n| \leqq 2\,|a_2|$. *If, further,* $|a_2| \leqq 1/3$, *then both f and f' are univalent in* Δ.

PROOF. We have f' univelant in Δ if and only if

$$g(z) = \frac{1 - f'(z)}{2a_2} = z - \sum_{n=2}^{\infty} \frac{(n+1)\,|a_{n+1}|}{2|a_2|} z^n = z - \sum_{n=2}^{\infty} b_n z^n$$

is in T. Since $\sum_{n=2}^{\infty} n\, b_n \leqq 1$ is necessary and sufficient for g to be in T, the result follows. As in Corollary 1, f will also be in S if $2\,|a_2| + |a_2| \leqq 1$.

3. Necessary Conditions for T_1. We next prove that (2) is necessary, as well as sufficient, for f in T to be in T_1 when f is a cubic polynomial.

THEOREM 3. *If $f(z) = z - a_2z^2 - a_3z^3$ ($a_2 > 0$, $a_3 \geqq 0$), then $f \in T_1$ if and only if*

$$3a_3 \leqq \min \begin{cases} 1 - 2a_2 \\ a_2 \end{cases}.$$

PROOF. We have $3a_3 \leqq 1 - 2a_2$ if and only if $f \in T$. On the other hand, f' is univalent in Δ if and only if $g(z) = z + (3a_3/2a_2)z^2 \in S$. But a necessary and sufficient condition for g to be in S is that $3a_3/2a_2 \leqq 1/2$. This completes the proof.

COROLLARY. *If $f(z) = z - a_2z^2 - a_3z^3 \in T_1$, then $a_3 \leqq 1/9$. Equality holds if and only if $f(z) = z - (1/3)z^2 - (1/9)z^3$.*

PROOF. By Theorem 3, we must have $3a_3 \leqq a_2 \leqq (1 - 3a_3)/2$, which means that $a_3 \leqq 1/9$. The only time this double inequality can hold, when $a_3 = 1/9$, is when $a_2 = 1/3$.

From Theorem 3 and its corollary, one might be led to believe that (2) is both necessary and sufficient for a function to be in T_1 and that 1/9 is an upper bound for the third coefficient. We shall show that neither of these is the case for quartics. But first we need a result due to Brannan and Brickman.

THEOREM A [1] *The function $g(z) = z + b_2z^2 + b_3z^3$, b_2 and b_3 real, is in S if and only if*

$$|b_2| \leqq (1 + 3b_3)/2, \qquad -1/3 \leqq b_3 \leqq 1/5, \text{ and}$$
$$|b_2| \leqq 2\sqrt{b_3(1 - b_3)}, \qquad 1/5 \leqq b_3 \leqq 1/3.$$

Note that g in Theorem A cannot be in S if $|b_3| > 1/3$ because g' would then have a root in Δ. We are now ready to prove

THEOREM 4. *If $f(z) = z - a_2z^2 - a_3z^3 - a_4z^4 \in T_1$, then $a_3 \leqq (\sqrt{2} - 1)/3$. The result is sharp, with equality for*

$$f(z) = z - \frac{3}{8}(2 - \sqrt{2})z^2 - \left(\frac{\sqrt{2} - 1}{3}\right)z^3 - \left(\frac{2 - \sqrt{2}}{16}\right)z^4.$$

PROOF. For $a_2 > 0$, $a_3 \geqq 0$, and $a_4 \geqq 0$, we have f' univalent if and only if $g(z) = z + (3a_3/2a_2)z^2 + (2a_4/a_2)z^3 \in S$ which, according to Theorem A, is equivalent to

$$3a_3 \leqq a_2 + 6a_4, \qquad 0 \leqq a_4 \leqq a_2/10, \text{ and}$$
$$3a_3 \leqq 4\sqrt{2a_4(a_2 - 2a_4)}, \qquad a_2/10 \leqq a_4 \leqq a_2/6.$$

For f to be in T, we must also have $3a_3 \leqq 1 - 2a_2 - 4a_4$. Thus, $f \in T_1$ if and only if

$$(3) \qquad 3a_3 \leqq \min \begin{cases} a_2 + 6a_4 \\ 1 - 2a_2 - 4a_4 \end{cases}, \qquad 0 \leqq a_4 \leqq a_2/10$$

and

$$(4) \qquad 3a_3 \leqq \min \begin{cases} 4\sqrt{2a_4(a_2 - 2a_4)} \\ 1 - 2a_2 - 4a_4 \end{cases}, \qquad a_2/10 \leqq a_4 \leqq a_2/6.$$

The right side of (3) will be maximized at a point where $a_2 + 6a_4 = 1 - 2a_2 - 4a_4$, i.e., where

$$(5) \qquad 3a_2 + 10a_4 = 1.$$

Set $a_4 = a_2 t/10, 0 \leqq t \leqq 1$, and note that (5) yields $a_2 = a_2(t) = 1/(3 + t)$. Now $h(t) = a_2(t) + 6a_4(t) = (1 + 3t/5)/(3 + t) \leqq h(1) = 2/5$, so that

$$(6) \qquad 3a_3 \leqq 2/5, \qquad 0 \leqq a_4 \leqq a_2/10.$$

Similarly, the right side of (4) will be maximized at a point where

$$(7) \qquad 4\sqrt{2a_4(a_2 - 2a_4)} = 1 - 2a_2 - 4a_4.$$

Setting

$$(8) \qquad a_4 = a_2 t/6, \qquad 3/5 \leqq t \leqq 1,$$

we see that (7) is equivalent to

$$(9) \qquad a_2 = \frac{3}{2}\left[\frac{1}{(3 + t) + 2\sqrt{t(3 - t)}}\right], \qquad 3/5 \leqq t \leqq 1.$$

With $a_2 = a_2(t)$, defined by (9), and $a_4 = a_4(t)$, defined by (8), either side of (7) is maximized with $t = 1$. Thus, from (4), we have

$$(10) \quad 3a_3 \leqq 1 - 2a_2(1) - 4a_4(1) = \sqrt{2} - 1, \qquad a_2/10 \leqq a_4 \leqq a_2/6.$$

Combining (6) and (10), it follows that $a_3 \leqq (\sqrt{2} - 1)/3$ when f is in T. Equality holds if and only if $f(z) = z - a_2(1)z^2 - ((\sqrt{2} - 1)/3)z^3 - (a_2(1)/6)z^4$.

Before giving an upper bound for the third coefficient in the entire class T_1, we state a result due to Pick [3].

THEOREM B [3]. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ *with* $|f(z)| \leqq M$, *for* z *in* Δ, *then* $|a_2| \leqq 2(1 - 1/M)$.

THEOREM 5. *If* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T_1$, *then* $a_3 < 1/6$.

PROOF. If $f \in T_1$, then $g(z) = (1 - f'(z))/2a_2 = z + \sum_{n=2}^{\infty}((n+1)a_{n+1}/2a_2)z^n \in S$. Since $|g(z)| \leqq g(1) \leqq 1 + (1 - 2a_2)/2a_2 = 1/2a_2$, for z in Δ, according to Theorem B, we have $3a_3/2a_2 \leqq 2(1 - 2a_2)$, or

$$3a_3/4 \leqq a_2(1 - 2a_2). \tag{11}$$

But the right side of (11) is maximized when $a_2 = 1/4$, from which we conlcude that $a_3 \leqq 1/6$. Suppose $a_3 = 1/6$. From (11), $1 \leqq 8a_2(1 - 2a_2) = 1 - (1 - 4a_2)^2$. So $a_2 = 1/4$. Since $f \in T_1$, $\sum_{n=4}^{\infty} na_n \leqq 1 - 2a_2 - 3a_3 = 0$. Hence, $a_n = 0$ for $n \geqq 4$. But then the corollary to Theorem 3 shows that $a_3 \leqq 1/9$. Therefore, it can never be that $a_3 = 1/6$. This completes the proof.

Combining Theorems 4 and 5, we have the following

COROLLARY. *Set* $\beta = \sup\{a_3: f \in T_1\}$. *Then* $(\sqrt{2} - 1)/3 \leqq \beta \leqq 1/6$.

4. Higher order derivatives. We now look at functions in T for which derivatives of higher order are also univalent.

A function $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geqq 0$, is said to be in T_m if f and its first m derivatives are univalent. If $f \in T_m$ for every integer m, then f is said to be in T_∞.

THEOREM 6. *If* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T$ *with* $\prod_{n=2}^{m+1} a_n \neq 0$, *then* $f \in T_m$ *if*

$$\sum_{n=k+2}^{\infty} (n-k)(n-k+1)\cdots na_n \leqq (k+1)!\, a_{k+1}, \tag{12}$$

for $k = 1, 2, \ldots, m$.

PROOF. The case $m = 1$ was proved in Theorem 2. For $k > 1$, $f^{(k)}(z)$ is univalent in Δ if and only if

$$g_k(z) = -\frac{(f^{(k)}(z) + k!a_k)}{(k+1)!a_{k+1}} = z + \sum_{n=2}^{\infty} \frac{(n+1)(n+2)\cdots(n+k)}{(k+1)!a_{k+1}} a_{n+k} z^n$$
$$= z + \sum_{n=2}^{\infty} b_n z^n \in S.$$

The result follows upon noting that $\sum_{n=2}^{\infty} nb_n \leqq 1$ is a sufficient condition for $g_k(z)$ to be in S.

COROLLARY. *If* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T$, $a_n \neq 0$, *and* (12) *holds for every* k, *then* $f \in T_\infty$.

In [5] Shah and Trimble investigated the family E consisting of functions in S having univalent derivatives of all orders. They showed that f in E must be an entire function and that $\alpha = \sup\{|a_2|: f \in E\}$ must satisfy $\pi/2 \leqq \alpha < 1.7208$. They further conjectured that $\alpha = \pi/2$, with extremal function $f(z) = (e^{\pi z} - 1)/\pi$. This conjecture sas disproved by Lachance

[2], using a perturbation of $e^{\pi z}$. The best known lower bound for α may be found in [4].

We close with the sharp upper bound on the second coefficient of a function in T_∞ for which, surprisingly, there is no extremal function.

THEOREM 7. *If* $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in T_\infty$, *then* $a_2 < 1/2$. *The result is sharp in that the value* $1/2$ *may be replaced by any smaller constant.*

PROOF. The function $z - z^2/2$, which is not in T_∞, is shown in Theorem 1 to be the unique extremal function in $T(\supset T_\infty)$ for the second coefficient. It thus suffices to show, for $\varepsilon(>0)$ arbitrarily small, that there exists a function in T_∞ whose second coefficient is $(1-\varepsilon)/2$. For any ε, $0 < \varepsilon \leqq \varepsilon_0 = (2e^{1/2}-1)^{-1}$, set

$$F_\varepsilon(z) = z - \frac{1-\varepsilon}{2} z^2 - \varepsilon \sum_{n=3}^{\infty} \frac{z^n}{2^{n-3} n!}.$$

Then $F_\varepsilon \in T$ because

$$\frac{2(1-\varepsilon)}{2} + \varepsilon \sum_{n=3}^{\infty} \frac{n}{2^{n-3} n!} < 1 - \varepsilon + \varepsilon \sum_{n=3}^{\infty} \frac{1}{2^{n-2}} = 1.$$

In view of the corollary to Theorem 6, F_ε is in T_∞ if its coefficients satisfy (12), for every k. Now, $f \in T_1$, because

$$\varepsilon \sum_{n=3}^{\infty} \frac{n(n-1)}{2^{n-3} n!} = 2\varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n n!} = 2\varepsilon(e^{1/2}-1) \leqq 2\left(\frac{1-\varepsilon}{2}\right),$$

when $\varepsilon \leqq \varepsilon_0$.

For $k > 1$, we have

$$\sum_{n=k+2}^{\infty} (n-k)(n-k+1)\cdots n a_n = \sum_{n=k+2}^{\infty} \frac{(n-k)(n-k+1)\cdots n\varepsilon}{2^{n-3} n!}$$

$$< \varepsilon \sum_{n=k+2}^{\infty} \frac{1}{2^{n-3}} = \frac{\varepsilon}{2^{k-2}} = (k+1)!\, a_{k+1}.$$

Hence, (12) is satisfied, for every k, and $F_\varepsilon \in T_\infty$, for every ε, $0 < \varepsilon \leqq \varepsilon_0$. Since $(1-\varepsilon)/2 \to 1/2$ as $\varepsilon \to 0$, the proof is complete.

REMARK. That there is no extremal function in Theorem 7 shows that T_∞ is not a compact family. In fact, the sequence $\{f_k(z)\}$ defined by

$$f_k(z) = z - \left(\frac{1-1/k}{2}\right) z^2 - \frac{1}{k} \sum_{n=3}^{\infty} \frac{z^n}{2^{n-3} n!}$$

is in T_∞, for every integer $k \geqq 3$, yet $\{f_k(z)\}$ converges uniformly on compact subsets of Δ to $f(z) = z - z^2/2 \notin T_\infty$. In particular, in the general class E of functions in S with all derivatives univalent, which was investigated in [5], there need not, a priori, be an extremal function for sup $\{|a_2| : f \in E\}$.

It would be of interest to determine a sharp upper bound on the third coefficient for functions in T_∞. We know from the bound on $T_1 \supset T_\infty$ given in Theorem 5 that the third coefficient cannot exceed 1/6 and, since F_{ε_0}, defined in Theorem 7, is in T_∞, that an upper bound on the third coefficient is no less than $\varepsilon_0/6 \approx .076$.

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Department of mathematics
College of Charleston
Charleston, SC 29424