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THE CHARACTERIZATION OF DEGENERATE AND NON-DEGENERATE SYSTEMS

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I. Introduction. In this paper we study the system

$$u' = F(u), \tag{1.1}$$

where $F:U \to \mathbf{R}^2$, $U \subset \mathbf{R}^2$ is open, $0 \in U$ and $F(0) = 0$. We assume that F is C^1 and that the origin is a center of (1.1).

Let $v(t)$ be a non-constant T-periodic solution of (1.1) and consider the corresponding linear variational equation

$$y' = F_u(v(t))y. \tag{1.2}$$

DEFINITION 1.1. We say that v is degenerate if and only if every solution of the corresponding linear variational equation (1.2) is T-periodic.

Since $y = v(t)$ is a T-periodic solution of (1.2) we have that v will be degenerate if and only if there exists a T-periodic solution of (1.2) that is linearly independent of $v(t)$.

DEFINITION 1.2. We say that (1.1) is degenerate in a neighborhood of 0, or simply degenerate, if and only if every non-constant periodic solution in this neighborhood is degenerate.

DISCUSSION. We will see that (1.1) is non-degenerate if and only if the periodic solutions in a neighborhood of 0 have distinct minimum periods, for example, as a function of the maximum amplitude of the solution. Thus, this concept is a generalization of the idea of "hard" and "soft" springs for the equation of a nonlinear spring $x'' + g(x) = 0$. Although this idea is interesting in its own right, it also has important applications in the study of

$$x' = F(x) + \varepsilon g(t, x), \tag{1.3}$$

where g is T-periodic in t. For example, if (1.1) is Hamiltonian and non-

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degenerate, one can use the Moser "small twist" Theorem [9] to show the existence of a family of T-periodic integral manifolds of (1.3) for g in a certain class. For a particular application of this idea, see [2].

In addition, (1.3) has T-periodic solutions if (1.1) is non-degenerate and there is a T-periodic solution of that equation. These results will be announced in a forthcoming paper [8].

In classical terminology, (1.1) is degenerate if and only if 0 is an isochronous center of (1.1). However, Definitions 1.1 and 1.2 are particularly useful in the study of (1.3) and, since this is our eventual goal, we will use our terminology rather than the more conventional one.

The study of the behavior of the period of periodic solutions of

$$x'' + g(x) = 0 \tag{1.4}$$

has a long history. In [6], Loud gave an explicit formula for the derivative of the period with respect to the amplitude, using the formula for the period

$$T(A) = \sqrt{2} \int_{-B}^{A} \frac{dx}{\sqrt{E - G(x)}}, \tag{1.5}$$

where E is the energy of a solution and $G' = g$.

Urabe studied the problem from a slightly different point of view in a series of papers [13], [14], [15], and [16]. Using methods different from ours, he gave in [13], necessary and sufficient conditions on g such that (1.4) is degenerate. Also, he proved our Corollary 4.2 and gave an example of a nonlinear g which results in a degenerate center. In [14–16], are further generalizations where methods are given to construct a g such that the period is a specified function. These papers are summarized in [17] and [18]. Levin and Shatz refined some of Urabe's work in [5].

In 1976, the problem was studied further by Obi in [10], using the representation (1.5) for the period. He sharpened some of the results of Urabe and Levin and Shatz and obtained our Corollary 4.1, but by a less direct method.

The problem of degeneracy or non-degeneracy of the center for the more general planar system has been studied less extensively and most of our results in § III appear to be new, including Theorem 3.4, the remarks following it, and Theorem 3.5. Loud, in [7], studies

$$\begin{aligned} x' &= P(x, y) \\ y' &= Q(x, y), \end{aligned} \tag{1.6}$$

where P and Q are analytic. He uses an implicit function method to determine the first term in the power series expansion of the derivative of the period and gives a necessary condition for degeneracy of the center. Finally, he completely characterizes degenerate centers in the case that

$$P(x, y) = -y + Ax^2 + Bxy + Cy^2$$

and

$$Q(x, y) = x + Dx^2 + Exy + Fy^2.$$

Russian mathematicians have studied the problem of characterizing degenerate systems when F is holomorphic. See **[1]** for an account of results obtained and methods used together with an extensive bibliography. See **[11]** for more recent work and **[12]** for work on a related problem.

II. Basic results. The following theorem is stated without proof.

THEOREM 2.1. (1.1) *is degenerate in a neighborhood of* 0 *if and only if every non-constant periodic solution in that neighborhood has the same minimum period.*

The proof of this theorem for the particular case of a second order scalar equation was communicated to one of the authors **[4]**.

The proof of Theorem 2.1 requires a lemma which will be used later. This lemma is also stated without proof.

LEMMA 2.2. *Let* $v(t, a)$, $v(0, a) = \binom{0}{a}$, $a \geqq 0$, *be a solution of* (1.1) *with least period* $T(a)$. *Then* $T(a)$ *is differentiable in an interval of the form* $(0, a_0)$.

With Theorem 2.1, it is now much easier to give examples of degenerate and non-degenerate systems.

EXAMPLE 2.1.

$$\begin{aligned} x' &= y\, h(x^2 + y^2), \\ y' &= -x\, h(x^2 + y^2) \end{aligned} \tag{2.1}$$

where $h \in C^1(\mathbf{R}, \mathbf{R})$ and h has at most one root, 0, in some neighborhood of the origin. Transforming to polar coordinates, (2.1) becomes

$$\begin{aligned} r' &= 0 \\ \theta' &= -\, h(r^2). \end{aligned} \tag{2.2}$$

However, (2.2) can be integrated easily and one obtains

$$\begin{aligned} r &= c \\ \theta &= h(c^2)\,(t_0 - t) \end{aligned}$$

and

$$\begin{aligned} x &= c \cos\,[h(c^2)\,(t_0 - t)] \\ y &= c \sin\,[h(c^2)\,(t_0 - t)]. \end{aligned}$$

Thus, the orbits are $2\pi/h(c^2)$ – periodic, and if h = constant, (2.1) is degenerate. Otherwise it is non-degenerate.

EXAMPLE 2.2. If F in (1.1) is linear, then (1.1) is degenerate since all periodic solutions of a two-dimensional linear differential equation have the same minimum period.

In the next example, F is not linear but degenerate.

EXAMPLE 2.3. Consider the system

$$\begin{aligned} x' &= -y + xy \\ y' &= x + y^2. \end{aligned} \tag{2.3}$$

It is not difficult to calculate that

$$E(x, y) = \frac{1}{2}\frac{y^2 + 1}{(1 - x)^2} - \frac{1}{1 - x} + \frac{1}{2} \tag{2.4}$$

is a first integral for (2.3). Moreover, (2.4) has a local minimum at the origin and thus the origin is a center. Changing to polar coordinates, we obtain

$$\theta' = 1, \tag{2.5}$$

which verifies that the origin is a degenerate center.

In Theorem 2.1, we have given a complete characterization of degeneracy and nondegeneracy in terms of the behavior of the period of the periodic solutions. We now use that characterization to give conditions on F in (1.1) which result in (1.1) being degenerate or nondegenerate.

For the purposes of the next theorem, let

$$F = \begin{pmatrix} f \\ g \end{pmatrix} \text{and } u = \begin{pmatrix} x \\ y \end{pmatrix} \text{in (1.1)}$$

THEOREM 2.3. *Suppose $c = xg - yf$ has one non-zero sign in some deleted neighborhood of* 0. *Let $D = 1 - x(\partial/\partial x) - y(\partial/\partial y)$. Then, if*

$$\Gamma F = x\, Dg - y\, Df \tag{2.6}$$

is of one sign in a deleted neighborhood of 0, (1.1) *is nondegenerate. If $\Gamma f \equiv 0$ in some deleted neighborhood of* 0, (1.1) *is degenerate. Moreover, in the notation of Lemma* 2.2, *if* sgn $c\, \Gamma F > 0 (< 0)$ *in some deleted neighborhood of* 0, *then $T'(a) > 0\ (< 0)$ in the same neighborhood.*

PROOF. We use polar coordinates. In this case the orbit can be expressed by $r = r(\theta, a)$, $r(\pi/2, a) = a$. Let us define

$$\begin{aligned} G(\theta, r) &= \cos\theta g(r\cos\theta, r\sin\theta) \\ &\quad - \sin\theta f(r\cos\theta, r\sin\theta), \end{aligned}$$

Then it can be proved that

$$T'(a) = \operatorname{sgn} c \int_0^{2\pi} \frac{(G - rG_r)r_a}{G^2}\, d\theta,$$

where G_r denotes the partial derivative of G with respect to r and r_a the partial derivative of r with respect to a. The theorem then follows from the fact that $G - rG_r = \Gamma F/r$.

If F can be expressed in terms of a convergent power series then we obtain the following corollary.

COROLLARY 2.4. *In addition to the hypotheses of Theorem* 2.3, *suppose that*

$$f(x, y) = \sum_{n=1}^{\infty} f_n(x, y)$$

and

$$g(x, y) = \sum_{n=1}^{\infty} g_n(x, y)$$

converge where f_n, g_n *are homogeneous of order n. Let* $h_n = x\, g_n(x, y) - y\, f_n(x, y)$ *and let m be the first index greater than* 1 *for which* h_n *is not identically* 0. *Then, if* h_m *has one sign in a deleted neighborhood of* 0, (1.1) *is non-degenerate.*

PROOF. Observe that

$$c = \sum_{n=1}^{\infty} h_n(x, y),$$

$$x\, D\, x^i y^{n-i} = x(1 - n)x^i y^{n-i}$$

and

$$y\, D\, x^i y^{n-i} = y(1 - n)x^i y^{n-i}.$$

Thus

$$\Gamma F = \sum_{n=1}^{\infty} (1 - n)\,(x\, g_n - y\, f_n)$$
$$= \sum_{n=1}^{\infty} (1 - n)h_n$$

and the corollary is proved.

We observe in particular that Γ applied to the linear terms is always identically 0. Thus the question of degeneracy or non-degeneracy can never be decided on the basis of a linear approximation (unless F is linear). It is an inherently nonlinear phenomenon.

III. Hamiltonian systems. In this section we assume that (1.1) is a Hamiltonian system.

Thus, we suppose that (1.1) takes the form

$$\begin{aligned} x' &= H_y(x, y) \\ y' &= -H_x(x, y) \end{aligned}, \tag{3.1}$$

where $H: U \to [0, \infty)$, $U \subset \mathbf{R}^2$ is open, $0 \in U$ and $H(0, 0) = 0$. We suppose further that H is C^2 and that origin is a center of (3.1). As an immediate corollary of Theorem 2.3, we have

COROLLARY 3.1. *Suppose $x H_x + y H_y > 0$ in some deleted neighborhood of 0. Let*

$$\Delta = x^2 \frac{\partial^2}{\partial x^2} + 2xy \frac{\partial^2}{\partial x \partial y} + y^2 \frac{\partial^2}{\partial y^2} - x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}. \tag{3.2}$$

Then, if $\Delta H \neq 0$ and has one sign in a deleted neighborhood of 0, (3.1) is non-degenerate. If $\Delta H \equiv 0$ in a deleted neighborhood of 0, then (3.1) is degenerate. Furthermore, let the orbits be given implicitly by $H(x, y) = h$, h a non-negative constant. Then $\Delta H > 0 (< 0)$ in some deleted neighborhood of 0 implies that $T'(h) < 0 \,(> 0)$ in that neighborhood.

PROOF. In this case $c < 0$ and $\Gamma\binom{H_y}{-H_x} = \Delta H$, which proves the theorem.

In the case that H can be expanded as a convergent power series, Corollary 3.1 has several corollaries which are easy to apply. In what follows, we suppose that

$$H(x, y) = \sum_{n=2}^{\infty} H_n(x, y)$$

converges in some neighborhood of 0 where $H_n(x, y)$ is a homogeneous form of degree n. We observe first that

$$\Delta H_n = (n^2 - 2n) H_n = n(n - 2) H_n.$$

COROLLARY 3.2. *Let H_m be the first non zero term in the expansion of H, with $m > 2$. Then, if H_m is of one sign in a deleted neighborhood of the origin, (3.1) is non-degenerate. If $H_m > 0 \;(< 0)$ in this neighborhood, $T'(h) < 0 (> 0)$ in the neighborhood.*

PROOF. If $H = \sum_{n=2}^{\infty} H_n$, then $\Delta H = \sum_{n=m}^{\infty} n(n - 2) H_n$, and the result follows from Corollary 3.1.

COROLLARY 3.3. *Suppose $H(x, y) = \sum_{n=2}^{\infty} H_n(x, y)$ is convergent where $H_n \geqq 0$ for all n. Then (3.1) is degenerate if and only if $H(x, y) = H_2(x, y)$.*

PROOF. If $H = H_2$, $\Delta H = 0$ and (3.1) is degenerate. If $H \neq H_2$, then

$\Delta H \geqq 0$ and $\Delta H > 0$, at least over some sector. Thus, (3.1) is non-degenerate.

Unfortunately, Corollaries 3.2 and 3.3 are indeterminate in a variety of cases, including the case that m, from Corollary 3.2., is odd. To resolve most of these cases, a more powerful result is developed in Theorem 3.5. We begin by studying some preliminary results. In polar coordinates $\theta' = -H_r/r$ and, thus,

$$T(h) = \int_0^{2\pi} \frac{r}{H_r}\, d\theta, \tag{3.3}$$

where the path of integration is the curve $H(r, \theta) = h$ and $T(h)$ is the least period of this orbit. If $H_r = (x\, H_x + y\, H_y)/r > 0$ in a deleted neighborhood of the origin, then we can solve $H(r, \theta) = h$ to obtain $r = r(\theta, h)$, an equation for the orbit with energy level h. By use of the Implicit Function Theorem, r is differentiable in h. Thus,

$$H_r(r(\theta, h), \theta) r_h(\theta, h) = 1$$

and

$$T(h) = \int_0^{2\pi} r(\theta, h) r_h(\theta, h) d\theta. \tag{3.4}$$

(3.4) leads us to a theorem and corollary remarks after which we will establish our main result, Theorem 3.5.

From (3.4), follows immediately

THEOREM 3.4. *Suppose* $x\, H_x + y\, H_y > 0$ *in a deleted neighborhood of the origin. Then* (3.1) *is degenerate if and only if*

$$\int_0^{2\pi} r(\theta, h) r_h(\theta, h) d\theta = T(h)$$

is constant in h.

REMARK 1. Since H is C^2, we can use the Implicit Function Theorem to show that $r(\theta, h)$ is twice differentiable in h. Thus, under the hypothesis of Theorem 3.4, (3.1) is degenerate if and only if

$$T'(h) = \int_0^{2\pi} [r_h(\theta, h)^2 + r(\theta, h) r_{hh}(\theta, h] d\theta = 0. \tag{3.5}$$

REMARK 2. If, in addition, $r_{hh}(\theta, h) > 0$ in some deleted neighborhood of the origin, (3.1) is non-degenerate.

Thus, if we could evaluate $r(\theta, h)$, we would have complete knowledge of $T(h)$ and would be able to decide the degeneracy or non-degeneracy of (3.1) immediately.

Suppose next that H can be expanded in a convergent power series

$$H(x, y) = \sum_{n=2}^{\infty} H_n(x, y)$$

with

$$H_n(x, y) = \sum_{i=0}^{n} \alpha_{ni} x^i y^{n-i},$$

where the α_{ni} are real constants. Then

$$\begin{aligned} H(r, \theta) &= H(r \cos\theta, r \sin\theta) \\ &= \sum_{n=2}^{\infty} a_n(\theta) r^n, \end{aligned} \tag{3.6}$$

where

$$a_n(\theta) = \sum_{i=0}^{n} \alpha_{ni} \cos^i \theta \sin^{n-i} \theta. \tag{3.7}$$

We consider the integrals

$$B_1 = \int_0^{2\pi} \{[3a_3^2/2 - a_2 a_4]/a_2^4\} \, d\theta$$

and

$$\begin{aligned} B_2 = \int_0^{2\pi} \{3[5\, a_3^4 &- 10\, a_2 a_3^2 a_4 + 4a_2^2 a_3 a_5 \\ &+ 2\, a_2^2 a_4^2 - a_2^3 a_6]/2a_2^7\} d\theta. \end{aligned} \tag{3.8}$$

THEOREM 3.5. *In the above notation, suppose that* $a_2(\theta) > 0$. *If* $B_1 \neq 0$, *then* (3.1) *is non-degenerate. Moreover, if* $B_1 < 0$ (> 0), *then* $T'(h) < 0$ (> 0) *in a deleted neighborhood of the origin. If* $B_1 = 0$ *and* $B_2 \neq 0$, *then* (3.1) *is non-degenerate. Furthermore, in this case, if* $B_2 < 0$ (> 0), *then* $T'(h) < 0$ (> 0) *in a deleted neighborhood of the origin.*

PROOF. From (3.4), we know that

$$T(h) = \int_0^{2\pi} r(\theta, h) r_h(\theta, h) d\theta,$$

where $r(\theta, h)$ is the solution of $H(r, \theta) = h$ and we know the solution exists, since $a_2(\theta) > 0$ implies $H_r > 0$ in a deleted neighborhood of 0. Thus, we must solve the equation

$$\sum_{n=2}^{\infty} a_n(\theta) r^n = h \tag{3.9}$$

for r. By standard techniques of power series inversion (see [3, Theorems 104 and 107]), we obtain

$$r\,r_h = \frac{1}{2}b_1^2 + \frac{3}{2}b_1 b_2 h^{1/2} + (2b_1 b_3 + b_2^2)h$$
$$+ \frac{5}{2}(b_1b_4 + b_2b_3)h^{3/2} \tag{3.10}$$
$$+ 3(b_1b_5 + b_2b_4 + \frac{1}{2}b_3^2)h^2 + \cdots,$$

where,

$$\frac{1}{2}b_1^2 = \frac{1}{2a_2},$$
$$\frac{3}{2}b_1 b_2 = -3a_3/4a_2^{5/2}, \qquad 2b_1 b_3 + b_2^2 = \left(\frac{3}{2}a_2^3 - a_2a_4\right)/a_4^2,$$
$$\frac{5}{2}(b_1 b_4 + b_2 b_3) = -(105\,a_3^3 - 140\,a_2 a_3 a_4 \tag{3.11}$$
$$+ 40\,a_2^2 a_5)/32\,a_2^{11/2},\ 3\left(b_1b_5 + b_2 b_4 + \frac{1}{2}b_3^2\right) = 3\,(5a_3^4$$
$$-10\,a_2 a_3^2 a_4 + 4a_2^2 a_3 a_5 + 2a_2^2 a_4^2 - a_2^3 a_6)/2\,a_2^7.$$

It is easy to verify that a_3 and a_5 satisfy

$$a_3(\theta + \pi) = -a_3(\theta)$$

and

$$a_5(\theta + \pi) = -a_5(\theta)$$

and, therefore,

$$\int_0^{2\pi} \frac{3}{2}b_1 b_2\, d\theta = 0$$

and

$$\int_0^{2\pi} \frac{5}{2}(b_1 b_4 + b_2 b_3)d\theta = 0.$$

Using (3.4), one then has, for $h > 0$,

$$T(h) = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a_2(\theta)} + B_1 h + B_2 h^2 + \mathcal{O}(h^{5/2}). \tag{3.12}$$

The theorem follows easily from (3.12).

With Theorem 3.5, the great majority of the indeterminate cases mentioned after Corollary 3.3 can be resolved. If $B_1 = B_2 = 0$, it is then necessary to calculate more terms of the power series (3.12). However, the algebra is tedious and this task is left as an exercise for the reader.

IV. Second order equations. A particular case of the Hamiltonian system is the equation

$$x'' + g(x) = 0, \tag{4.1}$$

where $g\colon (-\lambda, \lambda) \to \mathbf{R}$, for some $\lambda > 0$, $g(0) = 0$, $xg(x) > 0$, for $x \neq 0$, and g is C'. The equation (4.1) can be rewritten as a system

$$\begin{aligned} x' &= y \\ y' &= -\,g(x). \end{aligned} \tag{4.2}$$

If

$$H(x, y) = y^2/2 + G(x), \tag{4.3}$$

where

$$G(x) = \int_0^x g(s)ds,$$

then (4.2) can be rewritten as a Hamiltonian system

$$\begin{aligned} x' &= H_y \\ y' &= -H_x \end{aligned} \tag{4.4}$$

and the theorems of § III can be applied to (4.4). We recall that orbits of (4.4) are level curves of $H(x, y) = h$, $h \geqq 0$.

The results in § IV are all corollaries of results in § III and the proofs are left to the reader.

COROLLARY 4.1. *Let*

$$\delta = x^2\frac{d}{dx} - x. \tag{4.5}$$

If $\delta g = 0$ *in some neighborhood of* 0, *then* (4.2) *is degenerate. If* $\delta g \neq 0$ *and is of one sign in some deleted neighborhood of* 0, *then* (4.2) *is non-degenerate. If the orbits are given implicitly by* $H(x, y) = h$, $h > 0$, *then* $\delta g > 0$ (< 0) *in some deleted neighborhood of* 0 *implies* $T'(h) < 0\,(> 0)$ *in that neighborhood. Furthermore, if g is odd, the above conditions need only hold in an interval of the form* $(0, x_0)$, $x_0 > 0$.

In the case that g can be expanded as a convergent power series, we have

REMARK. If $g(x) = \sum_{n=1}^{\infty} c_n x^n$ is convergent and if the first non-zero coefficient (except c_1) has an odd subscript, then (4.2) is non-degenerate.

EXAMPLE 4.1. ($g(x) = x^3$). $\delta g = 2x^4 > 0$ if $x \neq 0$ and thus (4.2) is non-degenerate.

EXAMPLE 4.2. ($g(x) = \sin x = x - x^3/3! + x^5/5! - + \cdots$). The first non-zero coefficient except c_1 is $c_3 = -1/3!$. By the Remark, (4.2) is non-degenerate.

EXAMPLE 4.3. ($g(x) = x + x^2$). $\delta g = x^3$ is not of one sign and the test fails.

COROLLARY 4.2. *If g is odd and admits a convergent power series expansion, then* (4.2) *is degenerate if and only if g is linear.*

COROLLARY 4.3. *Suppose* $g(x) = x + \sum_{n=2}^{\infty} c_n x^n$ *is convergent and let*

$$\begin{aligned} C_1 &= 5c_2^2/9 - c_3/2 \\ C_2 &= \frac{7\cdot 11}{2\cdot 9\cdot 12} c_2^4 - \frac{7}{8} c_2^2 c_3 + \frac{7}{15} c_2 c_4 \\ &\quad + \frac{7}{32} c_3^2 - \frac{1}{6} c_5. \end{aligned} \tag{4.6}$$

If $C_1 \neq 0$, *then* (4.2) *is non-degenerate. Moreover, if* $C_1 < 0$ (> 0), *then* $T'(h) < 0\,(> 0)$ *in a deleted neighborhood of the origin. If* $C_1 = 0$ *and* $C_2 \neq 0$, *then* (4.2) *is non-degenerate. Furthermore, in this case, if* $C_2 < 0$ (> 0), *then* $T'(h) < 0\,(> 0)$ *in a deleted neighborhood of the origin.*

EXAMPLE 4.4. If $g(x) = x + cx^2$, $C_1 = 5c^2/9$, then (4.2) is nondegenerate if $c \neq 0$ and degenerate if $c = 0$.

EXAMPLE 4.5. ($g(x) = x + cx^2 + 10/9\, c^2x^3$). $C_1 = 0$ but $C_2 = -c\,(7\cdot 23/16\cdot 27)$ and (4.2) is non-degenerate if $c \neq 0$, degenerate if $c = 0$.

EXAMPLE 4.6. ($g(x) = x + cx^2 + dx^3$).

$$C_1 = \frac{5c^2}{9} - \frac{d}{2}$$

and

$$C_2 = \frac{7\cdot 11}{18\cdot 12} c^4 - \frac{7}{8} c^2 d + \frac{7}{32} d^2.$$

If $5c^2/9 \neq d/2$, (4.2) is non-degenerate. If $5c^2/9 = d/2$, then we must use C_2, where $C_2 \neq 0$ implies that (4.2) is non-degenerate.

Note that, in general, a countably infinite number of calculations, of which Corollary 4.3 describes only the first two, must be done in order to verify degeneracy.

NOTE. The assumption that $g(x) = x + \sum_{n=2}^{\infty} c_n x^n$ is made only for convenience and implies no restriction on the coefficient of x, since this coefficient can always be made to be 1 through a rescaling of time.

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