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ON FACTORIZATION OF OPERATOR POLYNOMIALS AND ANALYTIC OPERATOR FUNCTIONS

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ABSTRACT It is proved that the set of biquasitriangular monic operator polynomials which admit factorization into monic linear factors, is dense in the set of all biquasitriangular monic operator polynomials. An extension of this result to the factorization of analytic operator functions with compact spectrum is obtained as well. These results generalize a known factorization property of monic matrix polynomials.

1. Introduction. Let $L(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ be a ploynomial whose coefficients A_j are (linear bounded) operators $H \to H$, where H is a fixed separable (complex) Hilbert space. We shall assume always that the operator polynomial $L(\lambda)$ is monic, i.e., with leading coefficient $A_\ell = I$. The problem of factorization of $L(\lambda)$ into a product of several operater polynomials is an important one and has attracted much attention recently. This problem was studied in [**9**] in connection with oscillations of continua, and in [**12**, **1**, **4**, **11**] (the list is far from being complete). In case H is finite dimensional, a comprehensive treatment of this problem can be found in [**3**].

It turns out that, in case $H = \mathbf{C}^n$, not every monic operator polynomial $L(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$, admits a factorization into a product of linear factors

$$L(\lambda) = (\lambda I + X_1)(\lambda I + X_2) \cdots (\lambda I + X_\ell), \tag{1}$$

where X_ℓ: $\mathbf{C}^n \to \mathbf{C}^n$ are operators (unless, of course, $n = 1$). However, if the companion operator

$$C_L = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & & \cdots & I \\ -A_0 & -A_1 & & \cdots & -A_{\ell-1} \end{bmatrix} : \mathbf{C}^{n\ell} \to \mathbf{C}^{n\ell}$$

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of $L(\lambda)$ is diagonable, then a factorization (1) exists. The proofs of these facts can be found in [3, §3.7]. Observe that the set of diagonable linear operators in a finite dimensional (complex) linear space is dense. So, for a dense set of monic operator polynomials of degree l acting in a finite dimensional space, a factorization (1) exists.

The purpose of this paper is to extend this observation to the case of infinite dimensional H, as well as to factorizations of analytic operator functions.

2. Factorization of operator polynomials. Denoting, by $L(H)$, the algebra of all (linear bounded) operators $H \to H$ with the norm topology, we introduce the natural topology in $L(H) \times \cdots \times L(H)$ (ℓ times). We identify a monic operator polynomial $L(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ with an element $(A_0, A_1, \ldots, A_{\ell-1}) \in L(H) \times \cdots \times L(H)$.

An operator $A \in L(H)$ is called biquasitriangular if $\operatorname{ind}(\lambda I - A) = 0$, for all $\lambda \in \mathbf{C}$ such that $\lambda I - A$ is semifredholm (i.e., $\operatorname{Im}(\lambda I - A)$ is closed) and the semifredholm index makes sense (i.e., at least one of the numbers $\dim \operatorname{Ker}(\lambda I - A)$ and $\operatorname{codim} \operatorname{Im}(\lambda I - A)$ is finite). See Chapter 6 in [5] for the properties of biquasitriangular operators. Analoguosly, an operator polynomial $L(\lambda)$ will be called biquasitriangular if $\operatorname{ind} L(\lambda) = 0$ whenever the semifredholm index makes sense. Denote, by $(BQT)_\ell$, the set of all biquasitriangular operator polynomials of degree ℓ with the induced topology. We say that an operator polynomial $L(\lambda) \in (BQT)_\ell$ is factorable if there exist $X_1, \ldots, X_\ell \in L(H)$ such that

$$L(\lambda) = (\lambda I + X_1) \cdots (\lambda I + X_\ell).$$

The main result of this section is the following

THEOREM 1. *The set of all factorable biquasitriangular monic operator polynomials of degree l is dense in $(BQT)_\ell$.*

For the proof of Theorem 1 we need the following facts.

PROPOSITION 2. (see Chapter 5 in [5]). *Let S be the set of all $A \in L(H)$ with the property that there is a decomposition $H = H_1 \dotplus \cdots \dotplus H_k$ into the direct sum of a finite number of (closed) A-invariant subspaces H_i, $i = 1, \ldots, k$, such that $A|_{H_i} = \lambda_i I$, for some complex numbers $\lambda_1, \ldots, \lambda_k$ (the number k and the subspaces H_i, as well as the numbers λ_i, may depend on A). Then the closure of S (in the norm topology) coincides with the set of all biquasitriangular operators.*

PROPOSITION 3. *Let $H = H_1 \dotplus \cdots \dotplus H_k$ be a decomposition of H into the direct sum of (closed) subspaces H_i, $i = 1, \ldots, k$. Then, for every finite chain of subspaces $N_1 \supset N_2 \supset \cdots \supset N_\ell$ in H, there exist chains of subspaces $M_{i1} \subset M_{i2} \subset \cdots \subset M_{i\ell} \subset H_i$, $i = 1, \ldots, k$, such that $M_j \overset{\text{def}}{=}$*

$M_{1j} + M_{2j} + \cdots + M_{kj}$ is a direct complement to N_j in H, for $j = 1, \ldots, \ell$.

PROOF. Induction on ℓ. For the case $\ell = 1$, Proposition 3 is due to D. Gurarie, and its proof can be found in **[12]**. Assume Proposition 3 has been proved with ℓ replaced by $\ell - 1$. Let $M_{1\ell} \subset H_1, \ldots, M_{k\ell} \subset H_k$ be subspaces with the property that $M_\ell = M_{1\ell} \dotplus \cdots \dotplus M_{k\ell}$ is a direct complement to N_ℓ in H. By the induction hypothesis, there exist chains of subspaces $M_{i1} \subset M_{i2} \subset \cdots \subset M_{i,\ell-1}$ in $M_{i\ell}$, $i = 1, \ldots, k$, such that M_j is a direct complement to $M_\ell \cap N_j$ in M_ℓ, $j = 1, \ldots, \ell - 1$. As $N_j = N_\ell \dotplus (M_\ell \cap N_j)$, it follows that M_j is also a direct complement to N_j in $H, j = 1, \ldots, \ell - 1$.

PROPOSITION 4. *For every monic operator polynomial $L(\lambda)$ of degree ℓ there exist positive constants ε and K such that any operator $B \in L(H^\ell)$ with $\|B - C_L\| < \varepsilon$, where C_L is the companion operator of $L(\lambda)$, is similar to the companion operator C_M of some monic operator polynomial $M(\lambda)$ of degree ℓ; moreover,*

$$\|C_M - C_L\| \leqq K\|B - C_L\|. \tag{2}$$

PROOF. We shall use the ideas developed in **[2]**. Write B in the block matrix form $B = [B_{ij}]_{i,j=1}^{\ell}$, where $B_{ij} \in L(H)$. Letting $P_1 = [I\, 0 \cdots 0]$: $H^\ell \to H$, observe that

$$\begin{bmatrix} P_1 \\ P_1 C_L \\ \vdots \\ P_1 C_L^\ell \end{bmatrix} = I.$$

Hence, there exists $\varepsilon > 0$ such that, for every $B \in L(H^\ell)$ with $\|B - C_L\| < \varepsilon$, the operator

$$Q(B) \overset{\text{def}}{=} \begin{bmatrix} P_1 \\ P_1 B \\ \vdots \\ P_1 B^{\ell-1} \end{bmatrix} : H^\ell \to H^\ell$$

is invertible. Put $M(\lambda) = \lambda^\ell I - P_1 B(V_\ell + V_2\lambda + \cdots + V_\ell \lambda^{\ell-1})$, where $[V_1\, V_2 \cdots V_\ell] = Q(B)^{-1}$, V_i: $H \to J^\ell$. Then one easily verifies the equality $Q(B)B = C_M Q(B)$. Further,

$$\|Q(B)^{-1} - I\| \leqq K_1\, \|B - C_L\|, \qquad \|B^\ell - C_L^\ell\| \leqq K_2\, \|B - C_L\|,$$

for some positive constants K_1 and K_2 (where $B \in L(H^\ell)$ is such that $\|B - C_L\| < \varepsilon$). Taking into account the equality

$$L(\lambda) = \lambda^\ell I - P_1 C_L^\ell (U_1 + U_2\,\lambda + \cdots + U_\ell \lambda^{\ell-1}),$$

where $[U_1\ U_2 \cdots U_\ell] = I$, U_i: $H \to H^\ell$, we obtain (2).

PROOF OF THEOREM 1. Let $L(\lambda) = \sum_{j=0}^{\ell} \lambda^j A_j$ be a biquasitriangular monic operator polynomial ($A_\ell = I$), and let

$$C_L = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & I \\ -A_0 & -A_1 & -A_2 & \cdots & -A_{\ell-1} \end{bmatrix}$$

be its companion operator. There exist everywhere invertible operator polynomials $E(\lambda)$: $H^\ell \to H^\ell$ and $F(\lambda)$: $H^\ell \to H^\ell$ such that

$$E(\lambda)(\lambda I - C_L)F(\lambda) = \begin{bmatrix} L(\lambda) & & & \bigcirc \\ & I & & \\ & & \ddots & \\ \bigcirc & & & I \end{bmatrix}$$

(see, e.g., Theorem 1.1 in [3]). So C_L is biquasitriangular as well. As follows from the factorization theory for monic operator polynomials (see [2]), the existence of a factorization $L(\lambda) = (\lambda I + X_1) \cdots (\lambda I + X_\ell)$ is equivalent to the existence of a chain of C_L-invariant subspaces ($H^\ell \supset$) $M_{\ell-1} \supset M_{\ell-2} \supset \cdots \supset M_1$ such that the operators

$$P_i|_{M_i}: M_i \to H^i, \qquad i = 1, \ldots, \ell - 1,$$

where

$$P_i = \begin{bmatrix} I & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I & 0 & \cdots & 0 \end{bmatrix}$$

are invertible (i.e., one-to-one and onto).

Using Proposition 2, for given $\varepsilon > 0$, find $B \in L(H^\ell)$ such that $\|B - C_L\| < \varepsilon$ and $B|_{M_i} = \lambda_i I$, $i = 1, \ldots, k$, for some decomposition into the direct sum $H^\ell = H_i \dotplus \cdots \dotplus H_k$. Taking ε small enough, in view of Proposition 4, we can ensure that B is similar to the companion operator $C_{\tilde{L}}$ of a monic operator polynomial $\tilde{L}(\lambda)$ of degree ℓ; moreover,

$$\text{(3)} \qquad \|C_{\tilde{L}} - C_L\| \leqq K\|B - C_L\|,$$

where the positive constant K depends on C_L only. We have

$$C_{\tilde{L}}|_{G_i} = \lambda_i I, \qquad i = 1, \ldots, k,$$

where G_i, $i = 1, \ldots, k$, are subspaces such that $H^\ell = G_1 \dot{+} \cdots \dot{+} G_k$. By Proposition 3, for $j = 1, \ldots, \ell - 1$, there is a direct complement M_j to Ker P_j in H^ℓ of the form $M_j = M_{j1} \dot{+} \cdots \dot{+} M_{kj}$, where $M_{ij} \subset G_i$ $(i = 1, \ldots, k)$; moreover, $M_{\ell-1} \supset \cdots \supset M_1$. Obviously, M_j is $C_{\tilde{L}}$-invariant and $P_j|_{M_j}$ is invertible. This implies existence of a factorization $\tilde{L}(\lambda) = (\lambda I + X_1) \cdots (\lambda I + X_\ell)$, which proves (in views of (3)) Theorem 1.

3. Factorization of analytic operator functions. In this section we shall extend the result of Theorem 1 to the frame-work of analytic operator functions.

Let Ω be a domain in the complex plane, and let $W(\lambda)\colon \Omega \to L(H)$ be an analytic operator valued function. We say that $W(\lambda)$ has compact spectrum if the spectrum $\sigma(W)$ of $W(\lambda)$, i.e., the set of points $\lambda \in \Omega$ such that $W(\lambda)$ does not have a (bounded) inverse, is compact. Denote, by $CS(\Omega)$, the set of all analytic operator functions on Ω with compact spectrum. A spectral theory of such operator functions was developed recently in **[6, 7, 8]**. We shall recall briefly the basic facts of this theory which will be used later.

A quintet $\theta = (A, B, C; G, H)$ is called a spectral node on Ω, for $W(\lambda) \in CS(\Omega)$, if G is a separable Hilbert space, $A\colon G \to G$, $B\colon H \to G$, $C\colon G \to H$ are linear bounded operators, and the following properties hold true:

(a) $\sigma(A) \subset \Omega$;

(b) $W(\lambda)^{-1} - C(\lambda I - A)^{-1}B$ has an analytic extension on Ω;

(c) $W(\lambda)\, C(\lambda I - A)^{-1}$ has an analytic extension on Ω;

(d) $\bigcap_{j=0}^{\infty}$ Ker $CA^j = (0)$.

A spectral node exists and is unique up to similarity (Theorem 1.2 in **[6]**), i.e., any other spectral node for $W(\lambda)$ on Ω has the form $(S^{-1}AS, S^{-1}B, CS; G, H)$, for some invertible operator $S\colon G \to G$. Also, $\sigma(A) = \sigma(W)$.

PROPOSITION 5. *Let $(A, B, C; G, H)$ be a spectral node for $W(\lambda)$ on Ω. Then*:

(i) *for some integer $m > 0$, the operators*

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{m-1} \end{bmatrix} : G \to H^m, \ [B, AB, \ldots, A^{m-1}B]\colon H^m \to G$$

are left invertible and right invertible, respectively;

(ii) *there exist analytic and invertible operator valued fundtions $E(\lambda)$: $\Omega \to H \oplus G$, $G(\lambda)\colon \Omega \to H \oplus G$ such that*

$$E(\lambda)\begin{bmatrix} W(\lambda) & 0 \\ 0 & I_G \end{bmatrix} F(\lambda) = \begin{bmatrix} I_H & 0 \\ 0 & \lambda I - A \end{bmatrix}, \qquad \lambda \in \Omega.$$

For the proof of (i) see §6 in [7]; part (ii) is Corollary 4.2 in [6].

Divisibility is characterized in terms of spectral nodes as follows (see [7]):

PROPOSITION 6. *Let $W(\lambda)$, $W_1(\lambda) \in CS(\Omega)$ and let $(A, B, C; G, H)$ and $(A_1, B_1, C_1; G_1, H)$ be the spectral nodes of $W(\lambda)$ and $W_1(\lambda)$, respectively. Then $W(\lambda) = U(\lambda)W_1(\lambda)$, $\lambda \in \Omega$, for some $U(\lambda) \in CS(\Omega)$, if and only if there exist an A-invariant suspace $\hat{G} \subset G$ and an invertible operator $S\colon \hat{G} \to G_1$ such that $A|_{\hat{G}} = S^{-1}A_1S$, $C|_{\hat{G}} = C_1S$.*

Given $W(\lambda) \in CS(\Omega)$, let Δ be a rectifiable contour in Ω such that $\sigma(W)$ is inside Δ. Let

$$W_{pq} = \frac{1}{2\pi i}\int_\Delta \begin{bmatrix} W(\lambda)^{-1} & \lambda W(\lambda)^{-1} & \cdots & \lambda^{q-1}W(\lambda)^{-1} \\ \lambda W(\lambda)^{-1} & \lambda^2 W(\lambda)^{-1} & \cdots & \lambda^q W(\lambda)^{-1} \\ \vdots & \vdots & & \vdots \\ \lambda^{p-1}W(\lambda)^{-1} & \lambda^p W(\lambda)^{-1} & \cdots & \lambda^{p+q-2}W(\lambda)^{-1} \end{bmatrix} : H^q \to H^p$$

$(p, q = 1, 2, \ldots)$. Clearly, W_{pq} does not depend on the choice of Δ. We have

PROPOSITION 7. *A necessary (but not sufficient) condition for existence of a factorization*

$$W(\lambda) = V(\lambda)N(\lambda), \tag{4}$$

where $V(\lambda) \in CS(\Omega)$ and $N(\lambda)$ is a monic operator polynomial of degree ℓ with $\sigma(N) \subset \Omega$, is that the operator $W_{\ell q}$ is onto, for some q.

Note that (because H is a Hilbert space) W_{pq} is onto if and only if it is right invertible; also, if W_{pq} is onto, then so are $W_{p-1,q}, \ldots, W_{1q}$. In case $W(\lambda)$ is a monic operator polynomial of degree m and $\Omega = \mathbf{C}$, the operators W_{pq} are easily seen to be onto for $1 \leqq p \leqq m$, $q \geqq m$. Indeed, in this case,

$$\left(C_W, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix}, [I\, 0 \cdots 0]; H^m, H\right)$$

is a spectral node for $W(\lambda)$ (here C_W stands for the companion operator of $W(\lambda)$, see [2]). Consequently, W_{pq} has the form

$$W_{pq} = \begin{bmatrix} 0 & \cdots & 0 & 0 & & \cdots & I \\ \vdots & & \vdots & \vdots & & \cdot^{\cdot^{\cdot}} & \\ & & & & I & & \\ 0 & \cdots & 0 & I & & & * \end{bmatrix}, \qquad 1 \leqq p \leqq m,\ q \geqq m.$$

PROOF OF PROPOSITION 7. Let $(A, B, C; G, H)$ be a spectral node for $W(\lambda)$. Then

$$W_{pq} = Q_p(C, A) \cdot [B, AB, \ldots, A^{q-1}B], \text{ where } Q_p(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{p-1} \end{bmatrix}.$$

By Proposition 5(i), for q large enough, W_{pq} is onto if and only if $Q_p(C, A)$ is such. If (4) holds, then, by Proposition 6, there is an A-invariant subspace $\hat{G} \subset G$ and an invertible operator $S\colon \hat{G} \to H^{\ell}$ such that

$$A\Big|_{\hat{G}} = S^{-1}C_N S,\ C\Big|_{\hat{G}} = [I\, 0 \ldots 0]S.$$

Hence

$$Q_{\ell}(C, A)\Big|_{\hat{G}} = S$$

is invertible, and, consequently, $W_{\ell q}$ (for q large enough) is onto.

In view of Proposition 7, we shall introduce the following definition. An operator function $W(\lambda) \in CS(\Omega)$ will be called ℓ-complete if the operator $W_{\ell q}$ is onto, for some q.

A natural topology is introduced in $CS\ (\Omega)$. Namely, a sequence $W_n(\lambda) \in CS(\Omega)$, $n = 1, 2, \ldots$, is said to converge to $W(\lambda) \in CS(\Omega)$ if $\bigcup_{n=1}^{\infty} \sigma(W_n)$ is contained in some compact set in Ω and, for every compact $K \subset \Omega$, we have

$$\lim_{n\to\infty} \sup_{\lambda\in K} \| W_n(\lambda) - W(\lambda) \| = 0.$$

As the right invertibility of a Hilbert space operator is stable under small perturbations, it follows that the set of ℓ-complete analytic operator functions with compact spectrum is open in $CS(\Omega)$. However, this set is not dense in $CS(\Omega)$.

As for operator polynomials, we say that a $W(\lambda) \in CS(\Omega)$ is biquasitriangular if ind $W(\lambda) = 0$, for every $\lambda \in \Omega$, such that the semifredholm index makes sense. In view of Proposition 5(ii), $W(\lambda)$ is biquasitirangular if and only if the operator A from a spectral node $(A, B, C; G, H)$ is such.

We now state the extension of Theorem 1 to the framework of analytic operator functions.

THEOREM 8. *Assume the domain Ω is simply connected and $\bar{\Omega} \neq C$. Then the set of ℓ-complete biquastiriangular operator functions $W(\lambda) \in CS(\Omega)$ which admit a factorization of type*

$$(5) \qquad W(\lambda) = V(\lambda)(\lambda I - X_1) \cdots (\lambda I - X_\ell), \qquad X_i \in L(H)$$

wlth $V(\lambda) \in CS(\Omega)$ and $\sigma(X_i) \subset \Omega, i = 1, \ldots, \ell$, is dense in the set of all ℓ-complete biquasitriangular operator functions in $CS(\Omega)$.

PROOF. Let $W(\lambda) \in CS(\Omega)$ be ℓ-complete and biquasitriangular, and let $(A, B, C; G, H)$ be a spectral node for $W(\lambda)$ on ℓ. As A is biquasitriangular, by Proposition 2, there exists a sequence $A_m \in L(G)$, $m = 1, 2, \ldots$, which converges to A and such that, for each A_m, there is a decomposition $G = G_{1m} \dotplus \cdots \dotplus G_{k_m, m}$ into a direct sum of A_m-invariant subspaces G_{jm} such that $A_m|_{G_{jm}} = \lambda_{jm} I$, for some complex numbers $\lambda_{1m}, \ldots, \lambda_{k_m, m}$. We can assume that $\bigcup_{m=0}^{\infty} \sigma(A_m)$ is contained in a compact set in Ω.

Let $\delta > 0$ and $z_0 \in \mathbf{C}$ be such that

$$\Omega \cap \{\lambda \in \mathbf{C} \mid |\lambda - z_0| < \delta\} = \varnothing.$$

By Theorem 2.1 in [8], the operator function

$$\tilde{W}(\lambda) = I + CF^{-1}[\delta^2 - (\lambda - z_0)(A^* - \bar{z}_0)]^{-1}C^*,$$

where

$$F = \sum_{n=0}^{\infty} \delta^{2n}(A^* - \bar{z}_0)^{-n-1}C^*C(A - z_0)^{-n-1},$$

belongs to $CS(\Omega)$ and has the spectral node $(A, \tilde{B}, C; G, H)$, for some $\tilde{B}: H \to G$. Proposition 6 now gives

$$W(\lambda) = U(\lambda)\tilde{W}(\lambda),$$

where $U(\lambda)$ is analytic and invertible in Ω. Analogously, the operator function $\tilde{W}_m(\lambda) = I + CF_m^{-1}[\delta^2 - (\lambda - z_0)(A_m^* - \bar{z}_0)]^{-1}C^*$, where $F_m = \sum_{n=0}^{\infty} \delta^{2n}(A_m - \bar{z}_0)^{-n-1}C^*C(A_m - z_0)^{-n-1}$ belongs to $CS(\Omega)$ (for m large enough) and has the spectral node $(A_m, \tilde{B}_m, C; G, H)$, for some $\tilde{B}_m: H \to G$. Putting $W_m(\lambda) = U(\lambda)\tilde{W}_m(\lambda)$, it is easy to see that the sequence $W_m(\lambda)$, $m = 1, 2, \ldots$, converges to $W(\lambda)$ (in the indicated topology in $CS(\Omega)$).

To complete the proof of Theorem 8 we shall show that $W_m(\lambda)$ admits factorization of type (4), for sufficiently large m. Note that because $W(\lambda)$ is ℓ-complete, so is $\tilde{W}(\lambda)$ (cf. the proof of Proposition 7), and hence also $\tilde{W}_m(\lambda)$ (at least for m large enough). So the operators

$$Q_p(C, A_m) = \begin{bmatrix} C \\ CA_m \\ \vdots \\ CA_m^{p-1} \end{bmatrix}, \qquad p = 1, \ldots, \ell$$

are onto. By Proposition 3, there exist A_m-invariant subspaces $M_\ell \supset \cdots \supset M_1$ such that M_j is a direct complement to Ker $Q_j(C, A_m)$ in G, $j = 1, \ldots, \ell$. In particular, $Q_\ell(C, A_m)|_{M_\ell}$ is invertible. Let

$$L_m(\lambda) = \lambda^\ell I - C(A_m|_{M_\ell})^\ell(T_1 + T_2\lambda + \cdots + T_\ell\lambda^{\ell-1}),$$

where $[T_1 \cdots T] = (Q\ (C, A_m)|_{M_\ell})^{-1}$(so T_i: $H \to M_\ell$, $i = 1, \ldots, \ell$). It follows from [2] that $(A_m|_{M_\ell}, T_\ell, C|_{M_\ell}; M_\ell, H)$ is a spectral node for $L_m(\lambda)$. So

$$\sigma(L_m) = \sigma(A_m|_{M_\ell}) \subset \Omega$$

(at this point we use the simple connectedness of Ω). Also, $A_m|_{M_\ell}$ is similar to the companion operator C_{L_m} of $L_m(\lambda)$: $Q_\ell(C, A_m)|_{M_\ell} \cdot A_m|_{M_\ell} = C_{L_m} \cdot Q_\ell(C, A_m)|_{M_\ell}$. Now we show, as in the proof of Theorem 1, that $L_m(\lambda)$ admits a factorization

$$L_m(\lambda) = (\lambda I - X_1) \cdots (\lambda I - X_\ell), \qquad X_i \in L(H),$$

and the simple connectedness of Ω ensures again that $\sigma(X_i) \subset \Omega$, $i = 1, \ldots, \ell$. It remains to note that, by Proposition 7, $\tilde{W}_m(\lambda) = V(\lambda)L_m(\lambda)$, for some $V(\lambda) \in CS(\Omega)$.

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