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# CONVOLUTION THEOREMS FOR A CLASS OF BOUNDED CONVEX FUNCTIONS

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ABSTRACT. Let $C$, $S_{1/2}$, $S_0$ and $K$ denote respectively the subclasses of normalized univalent functions which are convex, starlike of order 1/2, starlike and close-to-convex. Further, let $C_\gamma$ be the subclass of $C$ defined by $|zf''(z)/f'(z) - \gamma| < 1 + \gamma$, $\gamma \geqq -1/2$. The following results are established: (i) If $f \in C_\gamma$ and $g \in K$, then we have $h = f * g \in S_0$ for $\gamma < .13$; (ii) If $f \in C_\gamma$ and $g \in S_0$, then $h = f * g$ satisfies $|zh'(z)/h(z) - 1 - \gamma| < 1 + \gamma$; and (iii) If $f \in C_\gamma$ and $g \in S_{1/2}$, then $h = f * g$ satisfies $\operatorname{Re}(zh'(z)/h(z)) > 1/[(1 + \alpha)((1 + \alpha)^{1/\alpha} - 1)]$, where $\alpha = \gamma/(1 + \gamma)$. Here, $*$ denotes the Hadamard product of analytic functions.

**1. Introduction and statement of results.** Let $E = \{z \mid |z| < 1\}$ denote the unit disc and let $A$ denote the space of functions analytic in $E$ with the topology of local uniform convergence. If $f$ and $g$ are in $A$ and have the power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $g(z) = \sum_{k=0}^{\infty} b_k z^k$ about the origin, then the convolution or Hadamard product of $f$ and $g$ is defined by

$$h(z) = (f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

A function $f \in A$ and normalized by $f(0) = f'(0) - 1 = 0$:

(i) belongs to the class $S_\alpha$ of functions starlike of order $\alpha$, $\alpha < 1$, if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \qquad z \in E; \tag{1}$$

(ii) belongs to the class $C$ of convex functions if

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0, \qquad z \in E; \tag{2}$$

(iii) belongs to the class $K$ of close-to-convex functions if, for some $g \in S_0$ and some real $\alpha$,

$$\operatorname{Re} e^{i\alpha} \frac{zf'(z)}{g(z)} > 0, \qquad z \in E; \tag{3}$$

and

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(iv) belongs to the class $M$ if, for every $g \in S_0$,

$$\frac{1}{z}(f * g)(z) \neq 0, \qquad z \in E. \tag{4}$$

It is well known that $C \subset S_{1/2} \subset S_0 \subset K \subset M$ and that each of these classes is a subclass of the class $S$ of normalized regular univalent functions in $E$. In 1973, Ruscheweyh and Sheil-Small [7] established the following

THEOREM A.
(i) *If $f$ and $g$ are in $C$, then so is $f * g$.*
(ii) *If $f \in C$ and $g \in K$, then $f * g \in K$.*

In view of the above theorem it is of interest to investigate whether convolution by a function belonging to some suitable subclass of $C$ maps one of the classes mentioned above to another interesting subclass of the same function class. In the present paper, we consider the following subclass of $C$.

DEFINITION. A function $f \in C$ is said to belong to the class $C_\gamma$ if it satisfies the condition

$$\left| \frac{zf''(z)}{f'(z)} - \gamma \right| < 1 + \gamma, \gamma \geqq -1/2, \qquad z \in E. \tag{5}$$

It is readily seen that $C_\infty = C$ and that functions in $C_\gamma$ are bounded convex functions in $E$. We shall establish the following

THEOREM 1. *If $f \in C_\gamma$ and $-1/2 \leqq \gamma < .13$ and $g \in M$, then $f * g \in S_0$. Further, for $\gamma \geqq -1/2$, $|(f * g)(z)| \leqq \sqrt{2}\, \max|zf'(z)|$.*

In particular, we note that if $|f''(z)/f'(z)| < 1$, i.e., $\gamma = 0$, and if $g$ is close-to convex, then $f * g$ is starlike. We may also remark that the upper bound on $\gamma$ as given in this theorem is not best possible.
In view of inequality (13) below, $f * g$ of Theorem 1 is bounded. We expect that the factor $\sqrt{2}$ in the upper bound can be replaced by 1. If $g \in S_0$, this certainly holds, because then $g(z) = z\varphi'(z)$, for some $\varphi \in C$ and Re $\varphi(z)/z > 1/2$. Moreover, functions whose real part is greater than 1/2 and take the value 1 at the origin are bound preserving.

THEOREM 2. *If $f \in C_\gamma$, $\gamma \geqq -1/2$, $g \in S_0$, and $h(z) = (f * g)(z)$, then*

$$\left| \frac{zh'(z)}{h(z)} - 1 - \gamma \right| < 1 + \gamma, \qquad z \in E. \tag{6}$$

THEOREM 3. *If $f \in C_\gamma$, $\gamma > -1/2$, $g \in S_{1/2}$, and $h(z) = (f * g)(z)$, then $zh'(z)/h(z)$ lies in the convex hull of the range of values of the univalent function*

$$z/((1 - \alpha z)(1 - (1 - \alpha z)^{1/\alpha})), \qquad \alpha = \gamma/(1 + \gamma). \tag{7}$$

In particular,

$$\text{(8)} \qquad \operatorname{Re} \frac{zh'(z)}{h(z)} \geqq 1/((1 + \alpha)((1 + \alpha)^{1/\alpha} - 1)) \geqq 1/2.$$

*Equality here is attained for*

$$\text{(9)} \qquad f(z) = f_0(z) = (1 - \alpha z)^{-1/\alpha} - 1$$

*and* $g = z/(1 - z)$.

The ranges of values of $zh'(z)/h(z)$ given by Theorems 2 and 3 are sharp. Further, the class of functions $h$ satisfying (6) is a subclass of $S_0$ and has been extensively studied in **[8]**. We need the following Theorem for the proof of Theorem 1.

THEOREM B. ([**5**]) *If $L_1$ and $L_2$ are continuous linear functionals on $A$ with $0 \notin L_2(M)$, then to each $f \in M$ there corresponds a function*

$$\text{(10)} \qquad g(z) = \frac{1}{1+i\mu}\left(\frac{z}{(1 - zx)^2} + i\mu \frac{z}{1-xz}\right), \qquad \mu \in \mathbf{R},\ |x| \leqq 1,$$

*such that*

$$\text{(11)} \qquad \frac{L_1(g)}{L_2(g)} = \frac{L_1(f)}{L_2(f)}.$$

In order to obtain the upper bound on $\gamma$ in Theorem 1 we need to study the subordination properties of functions in $C_\gamma$. In conformity with common usage, for $f \in A$ and $g \in S$, we denote by $f \prec g$ the fact that $f$ is subordinate to $g$. In the present paper we also establish the following

THEOREM 4. *If $f \in C_\gamma$, $\gamma \geqq -1/2$, $\alpha = \gamma/(1 + \gamma)$, then*

$$\text{(12)} \qquad f(z)/z \prec f_0(z)/z = ((1 - \alpha z)^{-1/\alpha} - 1)/z, \qquad \alpha \leqq 0,$$

$$\text{(13)} \qquad f'(z) \prec f_0'(z) = (1 - \alpha z)^{-(1+\alpha)/\alpha},$$

$$\text{(14)} \qquad zf'(z)/f(z) \prec zf_0'(z)/f_0(z) = z/((1 - \alpha z)(1 - (1 - \alpha z)^{1/\alpha})),$$

*where $f_0(z)$ is the element of $C_\gamma$ as defined in* (9). *Further,*

$$\text{(15)} \qquad \operatorname{Re}(zf'(z)/f(z)) \geqq r/((1 + \alpha r)((1 + \alpha r)^{1/\alpha} - 1)), \qquad |z| = r < 1.$$

The proof of (14) is on the lines given in **[4]**, but for (12) and (13) we need the following theorem.

THEOREM C. ([*6*]) *Let $k(z)$ be a convex conformal mapping of $E$, $k(0) = 1$, and let*

$$\text{(16)} \qquad m(z) = z \exp\left(\int_0^z (k(x) - 1)\frac{dx}{x}\right).$$

*If* $f \in A$ *and* $f(0) = f'(0) - 1 = 0$, *then*

$$zf'(z)/f(z) \prec k(z) \tag{17}$$

*if and only if, for all* $|s| \leqq 1$, $|t| \leqq 1$

$$\frac{tf(sz)}{sf(tz)} \prec \frac{tm(sz)}{sm(tz)}. \tag{18}$$

THEOREM D: ([7]) *Let* $\varphi$ *and* $\psi$ *be convex in* $E$, *and suppose* $f \prec \psi$. *Then* $\varphi * f \prec \varphi * \psi$.

For the sake of completeness, we give below a result from [7] in the form we need it for proving Theorems 2 and 3.

THEOREM E. *If* $\varphi$ *and* $\psi$ *in* $A$ *satisfy the condition*

$$\varphi(z) * \frac{1 + xz}{1 + yz}\psi(z) \neq 0, \qquad |x| = |y| = 1,\ z \in E, \tag{19}$$

*then, for* $f \in A$ *and* $f(0) = 1$, $(\varphi * (\psi f))/(\varphi * \psi)$ *takes all its values in the convex hull of the range of* $f$.

We shall need to use Theorem $E$ when (i) $z\varphi \in C$ and $z\psi \in S_0$ or (ii) $z\varphi$ and $z\psi$ are in $S_{1/2}$. In both of these situations the proof is available in [7], Lemma 2.7 and 3.5, respectively. It may be observed that in Theorem 4 we have not been able to prove the subordination for $f(z)/z$ and $\alpha < 0$. However, we shall show that $f_0(z)/z$ is a convex univalent function which follows from

THEOREM 5. *Let* $f \in A$ *with* $f(0) = 0$ *and, for real* $\mu > 0$, *let*

$$F(z) = \frac{\mu + 1}{z^{\mu}} \int_0^z t^{\mu - 1} f(t)\, dt. \tag{20}$$

*Then* $F(z)$ *is convex univalent if* $f(z)$ *satisfies*

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \begin{cases} -\mu/2, & \text{if } 0 \leqq \mu \leqq 1, \\ -1/2\mu, & \text{if } \mu > 1. \end{cases} \tag{21}$$

The case $\mu = 1$ is interesting enough to be stated separately.

COROLLARY. *If* $f \in A$ *and* $\operatorname{Re}(zf''(z)/f'(z) + 1) > -1/2$, *then* $F(z)$ *defined by* (20), *with* $\mu = 1$, *is a convex univalent function.*

This is an extension of a result of Libera [3] who established the conclusion of the corollary when $f(z)$ is convex. This enables us to extend Theorem 11 from [1] to the case $0 \leqq \alpha < 1/2$.

THEOREM 6. *If* $f$ *satisfies* $\operatorname{Re}(zf''(z)/f'(z) + 1) > \alpha$, $z \in E$, *and* $f(0) = f'(0) - 1 = 0$, *then, for* $0 \leqq \alpha < 1/2$,

$$(22) \qquad f(z)/z \prec ((1 - z)^{2\alpha-1} - 1)/((1 - 2\alpha)z).$$

In view of the remark in [1, p. 427] it is only necessary to show that the function in the right hand side of (22) is convex. But this follows from the fact that

$$\frac{(1 - z)^{2\alpha-1} - 1}{(1 - 2\alpha)z} = \frac{1}{z}\int_0^z (1 - t)^{2\alpha-2}\,dt$$

and the integrand satisfies the conditions of the corollary.

## 2.

PROOF OF THEOREM 1. For $f \in C_T$ and $g_1 \in M$, let $h = f * g_1$. In view of Theorem $B$, the value region of $zh'/h$, $z \in E$, is attained for $g_1 = g$ given by (10). Since

$$(23) \qquad (g * f)(z) = \frac{xzf'(xz) + i\mu f(xz)}{x(1 + i\mu)}, \qquad \mu \in \mathbf{R},\ |x| \leqq 1,$$

we need to find the conditions which assure that

$$\frac{zf'(z) + i\mu f(z)}{1 + i\mu} \in S_0, \qquad \text{for every real } \mu.$$

LEMMA 1. *For every real $\mu$, the function $zf' + i\mu f$ is starlike if and only if $f \in C$ and*

$$(24) \qquad 4\,\mathrm{Re}\frac{1}{v}\,\mathrm{Re}\frac{1}{w} \geqq \left(\mathrm{Im}\left(\frac{\bar{v}}{|v|}\,\frac{w}{|w|}\right)\right)^2,$$

*where $v = f/zf'$ and $w = 1 + zf''/f'$.*

PROOF OF LEMMA 1. It is easily seen that, for every real $\mu$,

$$\mathrm{Re}\frac{w + i\mu}{1 + i\mu v} > 0$$

if and only if

$$\mu^2\,\mathrm{Re}\, v - \mu\,\mathrm{Im}(v\bar{w}) + \mathrm{Re}\, w > 0,$$

for every real $\mu$. Hence, we must have $\mathrm{Re}\, w > 0$, which shows that $f \in C$ and $\mathrm{Re}\, v > 0$. Therefore, we obtain

$$(25) \qquad 4\,\mathrm{Re}\, v\,\mathrm{Re}\, w \geqq (\mathrm{Im}(\bar{v}w))^2.$$

This is equivalent to (24). We note that this condition can also be put in the form

$$(26) \qquad |vw - 1| \leqq 1 + \mathrm{Re}(\bar{v}w).$$

We notice that if $f \in C_0$, i.e., $|zf''/f'| < 1$, then $\mathrm{Re}\, 1/w \geqq 1/2$ and $\mathrm{Re}\, 1/v$

$\geqq 1/2$. Hence, (24) is certainly fulfilled if $f \in C_0$. However, if $f \in C_\gamma$, then, from the proof of Theorem 4 below, it follows that

$$w \prec (1 + z)/(1 - \alpha z), \qquad \alpha = \gamma/(1 + \gamma), \tag{27}$$

and

$$\operatorname{Re} 1/v \geqq 1((1 + \alpha)\,((1 + \alpha)^{1/\alpha} - 1)). \tag{28}$$

Hence,

$$4 \operatorname{Re} \frac{1}{v} \operatorname{Re} \frac{1}{w} \geqq 2(1 - \alpha)/((1 + \alpha)\,((1 + \alpha)^{1/\alpha} - 1)), \tag{29}$$

and an evaluation on a calculator shows that the right hand side of (29) is greater than 1, for $\gamma < .13$. Further, from (23), we obtain

$$|(g * f)\,(z)| \leqq \frac{1 + |\mu|}{\sqrt{1 + \mu^2}} \max |zf'(z)| \leqq \sqrt{2} \max|zf'(z)|,$$

because $\max |f(z)| \leqq \max |zf'(z)|$.

It is clear that the given bound on $\gamma$ is a very crude one, but it has not been possible for us to use either of the conditions (24) or (26) to obtain a sharp bound on $\gamma$.

## 3.

PROOF OF THEOREM 2. Let $g \in S_0$ and let $g_1 \in C$ such that $zg_1'(z) = g(z)$. Then, for $h = g * f, f \in C_\gamma$,

$$\frac{zh'(z)}{h(z)} = \frac{g(z) * zf'(z)}{g(z) * f(z)} = \frac{g_1(z) * z(zf'(z))'}{g_1(z) * zf'(z)}$$

and

$$\frac{zh'(z)}{h(z)} - 1 - \gamma = \frac{g_1(z) * zf'(z)\left(\dfrac{zf''(z)}{f'(z)} - \gamma\right)}{g_1(z) * zf'(z)}. \tag{30}$$

Since $g_1 \in C$ and $zf'(z) \in S_0$, by Theorem $E$, the right hand side of (30) lies in the convex hull of the range of $zf''/f' - \gamma$. This proves Theorem 2.

## 4.

PROOF OF THEOREM 4. It is easily seen that $f \in C_\gamma$ if and only if

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 + \varphi(z)}{1 - \alpha\varphi(z)}, \qquad \alpha = \gamma/(1 + \gamma), \tag{31}$$

where $\varphi(0) = 0$ and $|\varphi(z)| < 1$, $z \in E$. Hence,

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 + z}{1 - \alpha z}. \tag{32}$$

Applying Theorem $C$ with $k(z) = (1 + z)/(1 - \alpha z)$ and taking $s = 1$, $t = 0$, we obtain

$$f'(z) \prec (1 - \alpha z)^{-(1+\alpha)/\alpha}. \tag{33}$$

An elementary calculation shows that the function

$$\varphi(z) = ((1 - \alpha z)^{-(1+\alpha)/\alpha} - 1)/(1 + \alpha) \tag{34}$$

satisfies

$$\operatorname{Re}\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > 0, \qquad \text{for } \alpha \leqq 0 \tag{35}$$

and

$$\operatorname{Re}\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > -1/2, \qquad \text{for } 0 < \alpha < 1. \tag{36}$$

Hence, the right hand side of (33) maps $E$ onto a convex domain, for $\alpha \leqq 0$. Thus, convoluting both sides of (33) by the convex function $-(1/z) \log(1 - z)$ and using Theorem $D$, we obtain (12). We shall show subsequently that the right hand side of (12) maps $E$ onto a convex domain. Now, we shall prove (14). As the method of proof is similar to the one in **[4]**, we shall give the essential steps only. We first notice that if $G(z) = zf'(z)/f(z)$ and

$$1 + \frac{zf''(z)}{f'(z)} = \frac{1 + z}{1 - \alpha z}, \tag{37}$$

then

$$G(z) = z/((1 - \alpha z)(1 - \alpha z)^{1/\alpha}) = 1/(1 + (1 + \alpha)G_1(z)), \tag{38}$$

where

$$-G_1(z) = 1 - \frac{1-(1-\alpha z)^{(1+\alpha)/\alpha}}{(1 + \alpha)z} = \frac{1}{z}\int_0^z (1-(1-\alpha t)^{1/\alpha})dt. \tag{39}$$

Since the integrand on the right hand side of (39) is a convex function, the function $G_1(z)$ being a constant multiple of the Libera transform **[3]** of a convex function is also a convex univalent function. Consequently, $G(z)$ is univalent in $E$. We shall establish

LEMMA 2. *If* $G(z)$ *is defined by* (38), $H(z) = (1 + z)/(1 - \alpha z)$ *and* $k \geqq 1$, *then* $H_k(z)$, *defined by*

$$H_k(z) = kH(z) + (1 - k)G(z), \tag{40}$$

*is univalent in E and* $H(z) \prec H_k(z)$.

PROOF OF LEMMA 2. Following **[4]**, it is enough to show that

$$\operatorname{Re}\,(G'(z)/H'(z)) < 1. \tag{41}$$

Towards this, we notice that

$$G'(z)/H'(z) = T(z)/((1 + \alpha)S(z)), \tag{42}$$

where

$$T(z) = 1 - (1 + z)\,(1 - \alpha z)^{1/\alpha} \tag{43}$$

and

$$S(z) = (1 - (1 - \alpha z)^{1/\alpha})^2 \tag{44}$$

is a bivalent convex function. Further, because

$$T'(z)/S'(z) = (1 + \alpha)z/(2(1 - (1 - \alpha z)^{1/\alpha}) = (1 + \alpha)z/(2u(z)), \tag{45}$$

where $u(z)$ is a convex univalent function,

$$\operatorname{Re}(T'(z)/S'(z)) = (1 + \alpha)\operatorname{Re}(z/(2u(z)) < 1 + \alpha \tag{46}$$

since for any normalized convex univalent function $u(z)$, we have Re $(u(z)/z) \geqq 1/2$ and, consequently, Re $(z/u(z)) < 2$. Hence, a result of Libera **[3]** implies (41).

The remaining part of the proof of (14) is exactly similar to **[4]** and the details will therefore be omitted.

In order to establish (15), we notice that (38) gives

$$\begin{aligned}\operatorname{Re}\left(G(z) - \frac{1}{2}\right) &= \operatorname{Re}\frac{1 - (1 + \alpha)G_1(z)}{1 + (1 + \alpha)G_1(z)} \\ &\geqq \frac{1 - (1 + \alpha)\max|G_1(z)|}{1 + (1 + \alpha)\max|G_1(z)|}.\end{aligned} \tag{47}$$

Further, in view of (39),

$$G_1(\mathrm{re}^{i\varphi}) = -\frac{e^{i\varphi}}{r}\int_0^r\int_0^t (1 - \alpha x e^{i\varphi})^{(1-\alpha)/\alpha}\,dx\,dt. \tag{48}$$

Hence, for $-1 \leqq \alpha \leqq 1$,

$$|G_1(re^{i\varphi})| \leqq \int_0^r\int_0^t (1 + \alpha x)^{(1-\alpha)/\alpha}\,dx\,dt, \tag{49}$$

equality being attained for $\varphi = \pi$. The $\max|G_1(re^{i\varphi})|$ is attained at $\varphi = \pi$, and substituting this in (47), we obtain (15).

**5.**

PROOF OF THEOREM 5. From (20), we obtain

$$(\mu + 1)F'(z) + zF''(z) = (\mu + 1)f'(z)$$

and, therefore,

$$\mathrm{Re}\left(\varphi(z) + \frac{z\varphi'(z)}{\varphi(z) + \mu}\right) = \mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right),$$

where $\varphi(z) = 1 + zF''(z)/F'(z)$. Then, using Jack's Lemma **[2]**, it is seen that $\mathrm{Re}\,\varphi(z) > 0$ when (21) holds.

Since

$$((1 - \alpha z)^{-1/\alpha} - 1)/z = \frac{1}{z}\int_0^z (1 - \alpha t)^{-(1+\alpha)/\alpha}\, dt$$

and (34) and (36) show that the integrand satisfies the conditions of the corollary, it follows that the function in the right hand side of (12) is a convex function for $\alpha > 0$.

**6.**

PROOF OF THEOREM 3. We need to note that

$$\frac{zh'(z)}{h(z)} = \frac{g * zf'}{g * f} = \frac{g * f\frac{zf'}{f}}{g * f},$$

and because $g \in S_{1/2}$ and $f \in C_\gamma \subset S_{1/2}$, in view of Theorem $E$, $zh'/h$ will lie in the convex hull of $\varphi(z) = zf'(z)/f(z)$. Since, by Theorem 4,

(50) $$\varphi(z) \prec z/[(1 - \alpha z)(1 - (1 - \alpha z)^{1/\alpha})] = \psi(z)$$

the convex hull of $\varphi$ will be contained in the convex hull of $\psi$. The inequality (15) therefore establishes the theorem.

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**Remark.** This manuscript was written and circulated as a preprint in 1978 but has not before been published. Since then, it has several times been quoted in papers of various authors. Thus it appeared to be useful to have also this paper is print, although, some of the results have been generalized in the meantime.