ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 16, Number 1, Winter 1986

THE SPACE $C_0(p)$ OVER VALUED FIELDS

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Introduction. Let K be a non-trivially (rank 1) valued complete field (if necessary we shall specify the nature of the valuation depending on the context). For a sequence $p = \{p_k\}$ of positive real numbers, we define the space

$$C_0(p) = \{x = \{x_k\}: x_k \in K, |x_k|^{p_k} \to 0, k \to \infty\},$$

where $|\cdot|$ denotes the valuation on K. Clearly, $C_0(p)$ is a linear space if and only if p is a bounded sequence and we shall assume henceforth that p is a bounded sequence without explicit mention. Define

$$g(x) = \sup_{k\geq 1} \{|x_k|^{p_k/H}\}, \qquad H = \max(1, \sup_{k\geq 1} p_k).$$

Then g defines a paranorm on $C_0(p)$ and so $d(x, y) = g(x - y)$ defines a metric on $C_0(p)$ with respect to which $C_0(p)$ is a complete metric linear space. On the other hand, we can also define seminorms

$$\mathscr{P}_n(x) = \sup_{k\geq 1} \{|x_k|n^{1/p_k}\}, \qquad n = 1, 2, \ldots, x \in C_0(p),$$

so that the metric d is compatible with the locally convex (locally K-convex) topology defined by these seminorms. In other words, $C_0(p)$ is a Frechet space. Furthermore, the dual $C_0(p)^*$ of $C_0(p)$ consists of functionals f given by:

i) $f(x) = \sum_{k=1}^{\infty} a_k x_k$, $a_k \in K$ such that $\sum_{k=1}^{\infty} |a_k|\, N^{-1/p_k} < \infty$ for some $N > 1$, when the valuation is archimedean (see [6]);

ii) $f(x) = \sum_{k=1}^{\infty} a_k x_k$, $a_k \in K$ such that $\sup_{k\geq 1} |a_k|^{p_k} < \infty$ when the valuation is non-archimedean.

Also, $C_0(p)^*$ is endowed with the topology of uniform convergence over bounded subsets of $C_0(p)$. We write

$$1_\infty(p) = \{\{x_k\}, x_k \in K, \sup_{k\geq 1} |x_k|^{p_k} < \infty\}.$$

(*) The results obtained by the second author independently of the first, and now presented for publication jointly, were included in his thesis approved for Ph. D. by the University of Madras in March 1981.

Received by the editors on September 13, 1982 and in revised form on March 26, 1984.

In this paper, we study first the equivalence (Theorem 1.2) of weak and strong convergence in $C_0(p)$ and its normability (Theorem 1.3). We then discuss briefly the nuclearity of $C_0(p)$ when $K = \mathbf{R}$ or $\mathbf{C}$ and obtain necessary and sufficient conditions for $C_0(p)$ to be a Schwartz space when K is a spherically complete non-archimedean non-trivially valued field. Finally, we indicate the ideal structure in the metric linear algebra $C_0(p)$.

1. Weak and strong convergence in $C_0(p)$ and normability of $C_0(p)$. We start with the following lemma to establish the fact that weak and strong convergence coincide in $C_0(p)$.

LEMMA 1.1. *If $A = (a_{nk})$, $a_{nk} \in K$, $n, k = 1, 2, \ldots$ is an infinite matrix over K, a non-archimedean valued field, then $Ax = \{\sum_{k=1}^{\infty} a_{nk}x_k\} = \{y_n\} \in c_0$, the space of all null sequences over K, for every $x = \{x_k\} \in 1_\infty(p)$ if and only if*

i) $|a_{nk}|^{p_k} \to 0, k \to \infty, n = 1, 2, \ldots$;

ii) $\sup_{k\geqq 1} |a_{nk}|\, N^{1/p_k} \to 0$, $n \to \infty$, *for every integer* $N > 1$.

PROOF. We shall prove that necessity as sufficiency of the conditions is easily checked. Suppose $Ax \in c_0$ for every $x \in 1_\infty(p)$. Then, clearly, i) holds. Suppose ii) does not hold when, for some integer $N > 1$, $\sup_{k\geqq 1} |a_{nk}|\, N^{1/p_k} \nrightarrow 0$, $n \to \infty$. Let $\pi \in K$ be such that $0 < |\pi| = \rho < 1$. Consider the sequence $x = \{x_k = \pi^{\alpha_k}\}$, where α_k are integers satisfying $\rho^{\alpha_k+1} \leqq N^{1/p_k} < \rho^{\alpha_k}$. Now, $\sup_{k\geqq 1} |a_{nk}|\, |\pi|^{\alpha_k} > \sup_{k\geqq 1} |a_{nk}|\, N^{1/p_k} \nrightarrow 0$, so that the matrix $B = (a_{nk}\pi^{\alpha_k})$, does not transform bounded sequences to null sequences (see [11]). Thus, there exists a bounded sequence $x = \{x_k\}$ such that $Bx \notin c_0$. Now, $y = \{y_k = \pi^{\alpha_k} x_k\} \in 1_\infty(p)$ and $Ay \notin c_0$; this is a contradiction.

NOTE. The case $K = \mathbf{R}$ or $\mathbf{C}$ is discussed in [5].

We are now in a position to prove the assertion on weak and strong convergence.

THEOREM 1.2. *If the valuation on K is non-archimedean, weak and strong convergence are equivalent in $C_0(p)$.*

PROOF. It suffices to show that weak convergence implies strong convergence. Let $y_n \to y$ weakly in $C_0(p)$, i. e., if $f \in C_0(p)^*$, then $f(y_n - y) \to 0$, $n \to \infty$. But $f(x) = \sum_{k=1}^{\infty} a_k x_k$, $\{a_k\} \in 1_\infty(p)$, $x \in C_0(p)$. Now,

$$f(y_n - y) = \sum_{k=1}^{\infty} a_k(y_{nk} - y_k).$$

Thus, if $b_{nk} = (y_{nk} - y_k)$ and $B = (b_{nk})$, then $Bx \in c_0$, for every $x \in 1_\infty(p)$. By Lemma 1.1, $\sup_{k\geqq 1} |b_{nk}|^{p_k} \to 0$, $n \to \infty$, which means that $g(y_n - y) \to 0$, $n \to \infty$. The proof is complete.

In contrast to the unconditional equivalence of weak and strong convergence, we have the necessary and sufficient condition for the metric linear space $C_0(p)$ to be normable to be $\inf_k p_k > 0$, whatever K is. The Case $K = \mathbf{R}$ or $\mathbf{C}$ has been dealt with by Maddox and Roles [7]. The proof presented below is applicable to any K with a non-trivial valuation.

THEOREM 1.3. *$C_0(p)$ is normable if and only if* $\inf_k p_k > 0$. *Moreover, in such a case,* $C_0(p) = c_0$.

PROOF. If $\inf_k p_k = 0$ and $U(\varepsilon) = \{x \in C_0(p): g(x) < \varepsilon\}$, g is the paranorm, let $\alpha \in K$, $\alpha \neq 0$ and $k(i)$ be positive integers such that $\lim_{i\to\infty} p_{k(i)}/H = 0$, $H = \max(1, \sup_{k\geq 1} p_k)$. Thus, for some $k(m)$, $|\alpha| p_{k(m)}/H < 2$. By non-triviality of the valuation, there exists $\pi \in K$ and a positive integer n (depending on $k(m)$) such that $0 < |\pi| < 1$ and $|\pi|^{n+1} \leqq (\varepsilon/2)^{H/p_{k(m)}} < |\pi|^n$. If $x = \{0, \ldots, \pi^{n+1}, 0, \ldots\}$, where the non-zero entry is in the $k(m)^{\text{th}}$ place, then $x \in C_0(p)$ and $g(x) < \varepsilon$. But $g(x/\alpha) \geqq (\varepsilon/4)\,|\pi|$, and so, taking $\beta = (\varepsilon/4)\,|\pi|$, it follows that $x \notin \alpha U(\beta)$, i.e., $U(\varepsilon) \not\subset \alpha U(\beta)$. Thus $\inf_k p_k > 0$ when $C_0(p)$ is normable.

Conversely, consider $U(1)$. Take any $U(\beta)$ and choose $\alpha \in K$ such that $|\alpha| > \max(1, 1/\beta^{H/p})$, $p = \inf_k p_k$. Then it is easily seen that $U(1) \subset \alpha\, U(\beta)$, i.e., $U(1)$ is a convex (K-convex) bounded neighbourhood of 0. By the Kolmogorov criteria (see [**4, 8**]) for normability, $C_0(p)$ is normable.

Under this condition it is routine to check that $C_0(p) = c_0$. The proof of the theorem is complete.

NOTE. When $C_0(p)$ is normable, we have $C_0(p) = c_0$ not only as vector spaces but topologically, too. In other words, the above theorem implies that there can not be a proper subspace of c_0 of the form $C_0(p)$ which is topologically isomorphic to c_0.

2. Nuclearity of $C_0(p)$. In this section, K is $\mathbf{R}$ or $\mathbf{C}$. We shall consider Frechet spaces (complete metrizable locally convex spaces) and we shall assume that the topology is given by an increasing sequence $\{\mathscr{P}_n\}$ of semi-norms.

DEFINITION 2.1 (see [**12**]). A continuous linear map T from a normed linear space E into a Banach space F is called nuclear if there are elements $f_n \in E^*$, $y_n \in F$, $n = 1, 2, \ldots$ such that $T(x) = \sum_{n=1}^{\infty} f_n(x) y_n$, $x \in E$ and $\sum_{n=1}^{\infty} \|f_n\|\,\|y_n\| < \infty$.

DEFINITION 2.2 (see [**12**]). A Frenchet space F is said to be nuclear if, given m, there exists $n > m$ such that the canonical map $(F/\mathscr{P}_n^{-1}(0))^\sim \to (F/\mathscr{P}_m^{-1}(0))^\sim$ obtained by extending the map $x + \mathscr{P}_n^{-1}(0) \to x + \mathscr{P}_m^{-1}(0)$ is nuclear, where $\sim$ stands for the completion of the normed linear space in question.

Familiar sequence spaces such as l_∞ and c_0 are not nuclear spaces as they are Banach spaces of infinite dimension. In view of Theorem 1.3, $\inf_k p_k = 0$ is necessary for nuclearity of the Frechet space $C_0(p)$. Schaefer [**13**, p. 107, Example 4] has remarked that the space of entire functions, $C_0(p)(p_k = 1/k)$ is a nuclear space. Thus, it is worthwhile to find necessary and sufficient conditions on the sequence $\{p_k\}$ which make $C_0(p)$ nuclear. Necessary and sufficient conditions for nuclearity of certain sequence spaces (e.g., the α-dual or Köthe-Toeplitz dual [**4**, p. 405], power series spaces of finite and infinite type [**12**, p. 99] are already known [**12**, p. 98, Theorem 6.1.2; Theorem 6.1.4]. Using these results and that in [**5**, Theorem 13(2)], we obtain

THEOREM 2.3. *$C_0(p)$ is nuclear if one of the following equivalent conditions hold*:

i) *The power series space of infinite type defined by the sequence $\alpha_k = 1/p_k$ is the same as $C_0(p)$ (as topological vector spaces)*;

ii) *There exists q, $0 < q < 1$, such that $\sum_{k=1}^{\infty} q^{1/p_k} < \infty$; and*

iii) $C_0(p) \subset l_1$.

REMARKS 2.4. 1) There are sequences $\{p_k\}$ for which $C_0(p) \not\subset l_1$, e.g., $p_k = 1/(\log k)^r$, $0 < r < 1$. On the other hand, if $p_k = O(1/k^r)$, $0 < r < 1$, then $C_0(p) \subset l_1$. 2) If ii) of Theorem 2.3 is satisfied, then $\lim_{k\to\infty} p_k = 0$.

3. $C_0(p)$ as a Schwartz space. In this section, K is a non-archimedean, non-trivially valued complete field. We assume that the linear spaces are complete non-archimedean metrizable locally K-convex spaces so that the topology is induced by an increasing sequence $\{\mathcal{P}_n\}$ of seminorms. $C_0(p)$ is one such space.

A subset A of a non-archimedean normed linear space X (see [**18**, p. 134]) is said to be a compactoid if, for every $\varepsilon > 0$, there exist a finite number of elements $x_1, x_2, \ldots, x_n$ of X such that $A \subset B_\varepsilon(0) + Co(x_1, x_2, \ldots, x_n)$, where $B_\varepsilon(0) = \{x \in X\colon \|x\| \leqq \varepsilon\}$ and $Co(Y)$ denotes the closed absolute convex hull of Y. A linear map from a non-archimedean normed linear space X_1 to a non-archimedean Banach space X_2 is said to be compact if it maps the unit ball of X_1 into a compactoid in X_2 (see [**18**, p. 142]). Prof. De Grande De Kimpe suggested during a discussion that the concept analogous to nuclear spaces relevant to non-archimedean analysis is that of Schwartz spaces introduced in

DEFINITION 3.1. Let X be a locally K-convex space. X is said to be a Schwartz space, if given m, there exists $n > m$ such that the canonical map $(X/\mathcal{P}_n^{-1}(0))^\sim \to (X/\mathcal{P}_m^{-1}(0))^\sim$, obtained by extending the map $x + \mathcal{P}_n^{-1}(0) \to x + \mathcal{P}_m^{-1}(0)$, $x \in X$, is a compact operator, where $\sim$ denotes the completion of the basic normed linear space.

We are interested in finding conditions on $\{p_k\}$ which make $C_0(p)$ a

Schwartz space. To this end, we need the analogue of the Köthe-Toeplitz dual in the non-archimedean set up and also an auxiliary result of De Grande De Kimpe communicated in a discussion.

Let P be a sequence $\{\alpha^{(i)}\}$ of real sequences, $\alpha^{(i)} = \{\alpha_k^{(i)}\}_{k=1}^{\infty}$, $i = 1, 2, \ldots$, such that:

1) $\alpha_k^{(i)} > 0$, $i, k = 1, 2, \ldots$;
2) $\alpha_k^{(i+1)} \geqq \alpha_k^{(i)}$, $i = 1, 2, \ldots$.

The dual $\hat{P}$ is defined to be the space of sequences $\{x_k\}$, $x_k \in K$, $k = 1,2, \ldots$ such that $\lim_{k\to\infty} |x_k|\, \alpha_k^{(i)} = 0$, $i = 1,2, \ldots$. It is easily seen that $\hat{P}$ is a linear space with respect to co-ordinate-wise operations and is also a complete locally K-convex space with the non-archimedean seminorms (in fact norms), given by

$$\mathscr{P}_i(x) = \sup_{k\geqq 1} |x_k|\, \alpha_k^{(i)}, \qquad x \in \hat{P},\ i = 1, 2, \ldots.$$

We shall now prove the auxillary result.

THEOREM 3.2. *If K is spherically complete, $\hat{P}$ is a Schwartz space if and only if, for every $i = 1, 2, \ldots$, there exists $i' > i$ such that $\alpha_k^{(i)}/\alpha_k^{(i')} \to 0$, $k \to \infty$.*

PROOF. Let $\hat{P}$ be a Schwartz space. For each i, there exists $i' > i$ such that the canonical mapping $\eta{:}(\hat{P}, \mathscr{P}_{i'})^{\sim} \to (\hat{P}, \mathscr{P}_i)^{\sim}$, which is the inclusion map, is a compact operator. If U is the unit sphere in $(\hat{P}, \mathscr{P}_{i'})^{\sim}$, then $\bar{U}$, the closure of U in $(\hat{P}, \mathscr{P}_i)^{\sim}$, is a compactoid. So, for $\varepsilon > 0$, there exist $\varphi^{(1)}, \ldots, \varphi^{(n)}$ in $(\hat{P}, \mathscr{P}_i)^{\sim}$ such that $\bar{U} \subset B_\varepsilon(0) + Co\,\{\varphi^{(1)}, \ldots, \varphi^{(n)}\}$. But we can choose $\varphi_1^{(j)}$, $j = 1, 2, \ldots, n$ in $(\hat{P}, \mathscr{P}_i)$ such that $\varphi_1^{(j)} = \{\varphi_{11}^{(j)}, \ldots, \varphi_{1k_0}^{(j)}, 0, \ldots\}$, $j = 1, 2, \ldots, n$ and $B_\varepsilon(0) + Co\,\{\varphi^{(1)}, \ldots, \varphi^{(n)}\} = B_\varepsilon(0) + Co\,\{\varphi_1^{(1)}, \ldots, \varphi_1^{(n)}\}$. Now, $\xi^{(k)} = \{0, \ldots, \pi^{-n(k)}, 0, \ldots\}$, the non-zero entry in the k^{th} place, where π, $n(k)$ are such that $0 < |\pi| < 1$ and $|\pi|^{n(k)+1} \leqq \alpha_k^{(i')} < |\pi|^{n(k)}$, $k = 1, 2, \ldots$, are in U and so are in $\bar{U}$ in $(\hat{P}, \mathscr{P}_i)^{\sim}$. Hence,

$$\xi^{(k)} = \beta^{(k)} + \sum_{j=1}^{n} \alpha_{kj}\, \varphi_1^{(j)},\ \beta^{(k)} \in B_\varepsilon(0), \qquad k = 1, \ldots, n.$$

Thus, for $k > k_0$, $\alpha_k^{(i)}/|\pi|^{n(k)} < \varepsilon$ and so $\alpha_k^{(i)}/\alpha_k^{(i')} < \varepsilon/|\pi|$, i.e., $\alpha_k^{(i)}/\alpha_k^{(i')} \to 0$, $k \to \infty$.

To prove the converse, let i be any given index. Then there exists $i' > i$ such that, given $\varepsilon > 0$, there exists k_0 such that, for all $k > k_0$, $\alpha_k^{(i)}/\alpha_k^{(i')} < \varepsilon$. We shall prove that the canonical map $\eta{:}\,(\hat{P}, \mathscr{P}_{i'})^{\sim} \to (\hat{P}, \mathscr{P}_i)^{\sim}$ is compact. On K_0^k, define $\|\cdot\|^*$ by

$$\|(\beta_1, \ldots, \beta_{k_0})\|^* = \max_{1\leqq k\leqq k_0} |\beta_k|\, \alpha_k^{(i)}.$$

Clearly, $\|\cdot\|^*$ is a norm on K_0^k and the unit sphere in K_0^k, in the topology induced by $\|\cdot\|^*$ is c-compact. Consequently, by Theorem 4.56, [18, p. 161], the unit sphere is a compactoid. That is, there exist points $\varphi^{(i)} = (\varphi_1^{(i)}, \ldots, \varphi_{k_0}^{(i)})$, $i = 1, 2, \ldots, n$ such that the unit sphere is contained in $B'_\varepsilon(0) + Co\ \{\varphi^{(1)}, \ldots, \varphi^{(n)}\}$, where $B'_\varepsilon = \{\varphi \in K_0^k\colon \|\varphi\|^* \leqq \varepsilon\}$. If $\{\xi_j\}$ is any element of $\hat{P}$ with $\mathscr{P}_{i'}(\{\xi_j\}) \leqq 1$, then $\{0, \ldots \xi_{k_0+1}, \xi_{k_0+2}, \ldots\} \in B_\varepsilon(0)$ in $(\hat{P}, \mathscr{P}_i)^\sim$ and $\{\xi_1, \ldots, \xi_{k_0}, 0, \ldots\} \in B_\varepsilon(0) + Co\ \{\gamma^{(1)}, \ldots, \gamma^{(n)}\}$, where $\gamma^{(i)} = \{\varphi_1^{(i)}, \ldots, \varphi_{k_0}^{(i)}, 0, \ldots\}$, $i = 1, 2, \ldots, n$, and $\{\xi_j\} \in B_\varepsilon(0) + Co\ \{\gamma^{(1)}, \ldots, \gamma^{(n)}\}$. This shows that $\eta\colon (\hat{P}, \mathscr{P}_{i'}) \to (\hat{P}, \mathscr{P}_i)^\sim$ is compact. In general if X, Y are non-archimedean normed linear spaces of which Y is complete and if $f\colon X \to Y$ is compact, then f can be uniquely extended to a compact map $\tilde{f}\colon \tilde{X} \to Y$. The proof of the theorem is now complete.

THEOREM 3.3. *Let K be a non-archimedean non-trivially valued spherically complete field. $C_0(p)$ is a Schwartz space if and only if* $\lim_{k\to\infty} p_k = 0$.

PROOF. We first note that $C_0(p) = \hat{P}$, where $P = \{\{i^{1/p_k}\}\colon i = 1, 2, \ldots\}$ as locally convex spaces. If $\lim_{k\to\infty} p_k = 0$, then $(i^{1/p_k})/(i^{2/p_k}) = 1/i^{1/p_k} \to 0$, $k \to \infty$, $i = 1, 2, \ldots$, so, by Theorem 3.2, $C_0(p)$ is a Schwartz space. Conversely, if $i' > i$, $i = 1, 2, \ldots$ and $(i/i')^{1/p_k} \to 0$, $k \to \infty$, it is obvious that $\lim_{k\to\infty} p_k = 0$.

REMARKS 3.4. i) If $P = \{\{(i/i+1)^{1/p_k}\}: i = 1, 2, \ldots\}$ and K is spherically complete, then $\hat{P}$ is a Schwartz space if and only if $\lim_{k\to\infty} p_k = 0$.

ii) Analogously, as in the case $K = \mathbf{R}$ or $\mathbf{C}$, we can define $f\colon E \to F$, where E, F are non-archimedean Banach spaces over a complete non-archimedean non-trivially valued field, to be nuclear if there exist $y_n \in F$, $f_n \in E^*$, $n = 1, 2, \ldots$ such that $\lim \|f_n\|\,\|y_n\| = 0$ and $f(x) = \sum_{n=1}^\infty f_n(x)y_n$, $x \in E$. Such a map is precisely a compact map as defined in the second paragraph of this section (cf. [18, p. 143, Theorem 4.40]). Consequently, nuclear and Schwartz spaces are the same among locally K-convex spaces. It has been shown by van der Put and van Tiel [17] that when K is spherically complete, all locally K-convex spaces are nuclear in the sense that the ε and π topologies on the algebraic tensor product $X \otimes Y$ are the same, where X is the given locally K-convex space and Y is an arbitrary locally K-convex space (we note that this definition of nucelarity does not involve the concept of a nuclear map). Accordingly, any infinite dimensional Banach space over Q_p, the p-adic field, is nuclear while it is not a Schwartz space. Thus it is worthwhile to note that the concept of Schwartz space (Definition 3.1) is different from that of nuclear space in the sense of [17].

iii) In this context, one should note that Serre [14], is not explicit about his definition of nuclearity of a map $f \in \mathscr{L}(X, Y)$, where X, Y are p-adic Banach spaces. If Serre had in mind the definition of a nuclear map as defined in the beginning of Remark ii) above, he would have considered only completely continuous operators (or equivalently compact operators (vide [18])). Hence, the identification of completely continuous operators and nuclear operators mentioned by Serre is meaningless. This being the case, the consideration of the space $C(X, Y)$ of compact operators and $N(X, Y)$ of nuclear operators, X, Y non-archimedean Banach spaces by De Grande De Kimpe [1], without spelling out what her nuclear operators are, makes the situation more confusing. Still more confusing is the remark by a reviewer (MR 46 # 2488) that the counter example given by De Grande De Kimpe to the effect that a compact operator need not be a nuclear operator is wrong, when even the definition of nuclearity of a map is not known. If one were to depend on the result claimed by Serre without caring for an explicit definition of nuclearity of an operator, there does not seem to be any need to introduce nuclear spaces over non-archimedean valued fields in preference to Schwartz spaces. The authors feel that earlier literature in this respect may mislead future workers and are therefore constrained to make the above forthright comments. Obviously, the issue on hand is not semantic.

4. $C_0(p)$ as a metric linear algebra. We have already seen that $C_0(p)$ is a metric linear space, where the metric is induced by the paranorm g. In fact, g also satisfies $g(x \cdot y) \leqq g(x)g(y)$, where $x \cdot y = \{x_k y_k\}$ (Hadamard product), $x = \{x_k\}$, $y = \{y_k\} \in C_0(p)$. Srinivasan [15, 16] has studied the ideal structure of $C_0(p)$ for the case $p_k = 1/k$, $k = 1, 2, \ldots$, when $K = \mathbf{R}$ or $\mathbf{C}$ or a non-archimedean non-trivially valued complete field. Let $e_n = \{0, \ldots, 1, 0, \ldots\}$, 1 at the nth place and 0 elsewhere and $\mathscr{E}$ be the collection of all e_n, $n = 1, 2, \ldots$. A non-zero ideal I of $C_0(p)$ determines a subset $\mathscr{I} = \{e_n \in \mathscr{E}: e_n \in I\}$ of $\mathscr{E}$. We note that a nonzero ideal I of $C_0(p)$ is closed if and only if I is the closed linear span of $\mathscr{I}$. Also, a closed ideal is maximal if and only if it is the closed linear span of $\mathscr{I} = \mathscr{E} \backslash \{e_n\}$, for some n. Consequently, $C_0(p)$ is semi-simple. Further, we observe that every maximal closed ideal of $C_0(p)$ is a step space, usually called an echelon space, (see [2, p. 53]) and every echelon space is a closed ideal. If I is a closed ideal of $C_0(p)$, $\mathscr{M}$ the class of all maximal closed ideals of $C_0(p)$ containing it, and $\mathscr{S}$ the class of all echelon spaces containing it, then

$$\bigcap_{S \in \mathscr{S}} S \subset \bigcap_{M \in \mathscr{M}} M = I \subset \bigcap_{S \in \mathscr{S}} S,$$

i.e., every closed ideal of $C_0(p)$ is the intersection of all echelon spaces containing it.

The authors are thankful to Dr. M. S. Rangachari for his help in the preparation of this paper. They are also thankful to the referees for valuable suggestions.

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