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STATIONARY SPACIAL PATTERNS FOR A REACTION-DIFFUSION SYSTEM WITH AN EXCITABLE STEADY STATE

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ABSTRACT. In this note, the existence of stationary patterns in $n \geqq 2$ dimensional state space is established for a reaction-diffusion system which exhibits a single-globally attracting, excitable steady state. The system studied is dynamically like the FitzHugh-Nagumo model for nerve conduction but has a large inhibitor diffusion term. Variational methods are applied to an energy functional which give one pattern as a minimum and a second as a saddle point of the functional.

1. Introduction. Consider the system

$$\begin{aligned} u_t &= \Delta u + f(u) - v \\ v_t &= D\,\Delta v + \varepsilon(u - \gamma v) \end{aligned}, \tag{1.1}$$

where $\Delta \equiv \sum_{i=1}^{n} \partial^2/(\partial x_i^2)$, $n \geqq 1$, $t \geqq 0$, $(x_1, \ldots, x_n) \in \Omega \subseteq \mathbf{R}^n$, $f(u) = u(1 - u)(u - a)$, $0 < a < 1/2$, $D > 0$, $\varepsilon > 0$, and $\gamma > 0$. Equations (1.1) are an extension of the simpler FitzHugh-Nagumo [3, 10] equations, namely

$$\begin{aligned} u_t &= u_{xx} + f(u) - v \\ v_t &= \varepsilon(u - \gamma v). \end{aligned} \tag{1.2}$$

The FitzHugh-Nagumo system serves as a prototype for nerve conduction and other chemical and biological systems. The interested reader is referred to [6, 11] for a review of results obtained to this date.

Recently, Ermentrout, Hastings and Troy [2] have proposed system (1.1) as a prototype model for systems which exhibit lateral inhibition and excitability. In this setting u is interpreted as an activator concentration and v is interpreted as an inhibitor concentration. They discuss the physical motivation for the existence of nonconstant stable time independent solutions of (1.1) when $n = 1$ and solutions u and v are defined on all of $\mathbf{R}$ with $u(\pm\infty) = v(\pm\infty) = 0$. Summarizing their discussion, if $0 < \gamma < 4/(1 - a)^2$ then the dynamic equations

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$$(1.3)\qquad \begin{aligned} u_t &= f(u) - v \\ v_t &= \varepsilon(u - \gamma v) \end{aligned},$$

associated with (1.1), have a unique globally attracting steady state (0, 0) as well as a threshold of excitation. If $D = 0$, as is the case in the Fitz-Hugh-Nagumo model, a large enough initial concentration of u will trigger the formation of a travelling wave of u concentration. However, if $D > 0$ is large, the effective rapid diffusion of the inhibitor v should halt the u wave and produce a stable standing wave. Thus, nonconstant time independent solutions of (1.1) are expected either on $\mathbf{R}^n$ or on large bounded subsets $\Omega \subseteq \mathbf{R}^n$ with either Dirichlet or Neumann boundary conditions. In **[2]**, such solutions are proved to exist for $n = 1$ on $\mathbf{R}$, and numerical calculations are made on finite domains with Neumann boundary conditions which exhibit their profiles. For further discussion on these ideas, the reader is referred to Meinhardt **[9]**, Gierer and Meinhardt **[5]**.

The goal of the present article is to show that, in more than one space dimension, i.e., $n \geqq 2$, if $\gamma > 0$ is small and $D > 0$ is large, then system (1.1) has two nontrivial stationary solutions on sufficiently large space domains $\Omega \subseteq \mathbf{R}^n$. These solutions will satisfy either the Dirichlet or Neumann boundary conditions on $\partial\Omega$ and are critical points of an energy functional associated with system (1.1). Variational methods are applied to show that one solution minimizes the functional while the second is a saddle point.

These results compliment earlier work by Klaasen and Mitidieri **[8]** in which solutions of similar problems are shown to exist for the case when $\gamma > 0$ is large. In that case the dynamic equations have three constant solutions—two of which are stable and one unstable.

2. Statement and interpretation of main results. This paper is concerned with the Dirichlet problem for the elliptic system

$$(2.1)\qquad \begin{aligned} -\Delta u &= -\sigma g(u) - \beta v \\ -\Delta v &= \lambda u - \delta v \end{aligned},$$

where σ, β, λ, δ are positive constants and $g(u)$ is a continuous function on $\mathbf{R}$. The space domain Ω_R is a bounded domain in $\mathbf{R}^n$, $n \geqq 2$, containing a closed ball $\overline{B_R(0)}$. We assume the boundary of Ω_R, $\partial\Omega_R$ is of class $C^{2+\alpha}$, for some $0 < \alpha < 1$. Then the boundary value problem consists of equation (2.1) on Ω, with Dirichlet boundary conditions

$$(2.2)\qquad u = v = 0 \qquad \text{on } \partial\Omega_R.$$

System (2.1) reduces to the steady state equations for (1.1) if

(2.3) $\sigma = \beta = 1,\ g(u) = -f(u),\ \lambda = \varepsilon D^{-1}$ and $\delta = \varepsilon\gamma D^{-1}$;

namely,

$$\begin{aligned} -\Delta u &= f(u) - v \\ -D\Delta v &= \varepsilon u - \varepsilon\gamma v. \end{aligned} \tag{2.4}$$

The theorems below are stated for the general system (2.1), (2.2) and are followed by discussion and interpretation for the more specific system (2.4).

The first theorem gives sufficient conditions for the existence of a nontrivial solution of (2.1), (2.2).

THEOREM 1. *Suppose g is Holder continuous of order α, $0 < \alpha < 1$ and satisfies*

(i) *there is a $b > 0$ such that $(g(u))/u > (\beta\lambda)/(\sigma\delta)$ for $|u| \geqq b$;*

(ii) *there is a $u_0 > 0$ such that $\int_0^{u_0} g(s)\,ds < 0$.*

If the parameters δ, β, γ, σ and $R > 1$ are such that

$$\begin{aligned} &\left[\left(\frac{R}{R-1}\right)^n - 1\right]\left[u_0^2 + \frac{\beta\lambda u_0^2 R^4}{1+\delta R^2} + 2\sigma \max_{[0,u_0]}\left(\int_0^u g(s)ds\right)\right] \\ &\qquad + 2\sigma\int_0^{u_0} g(s)ds < 0, \end{aligned} \tag{2.5}$$

then (2.1), (2.2) *has a nontrivial solution pair $u, v \in C^{2+\alpha}(\bar{\Omega}_R)$.*

The second theorem guarantees the existence of a second nontrivial solution.

THEOREM 2. *If, in addition to the hypothesis of Theorem* 1, *$g(0) = 0$ and $g'(0) > 0$, then the BVP* (2.1), (2.2) *has a second nontrivial solution.*

As an example we wish to apply the results of Theorems 1 and 2 to the two diffusion FitzHugh-Nagumo model (2.4). The parameters are chosen in the following sequence of steps.

(2.6) (a) $0 < a < 1/2$ is fixed.

(2.7) (b) $0 < \gamma < 4/(1-a)^2$ is chosen to ensure the existence of a single stable excitable steady state.

(2.8) (c) $\varepsilon > 0$ can be arbitrarily chosen.

(d) Using (2.3), hypothesis (i) of Theorem 1 is satisfied if

$$(u-a)(1-u) < -\frac{1}{\gamma}, \quad \text{for } |u| > b,$$

which is equivalent to requiring that

$$b > \frac{1+a}{2} + \frac{1}{2}\sqrt{(1-a)^2 + 4\gamma^{-1}}.$$

(e) Choose $u_0 = 1$ in hypothesis (ii).

(f) Finally, inequality (2.5) simplifies to the requirement that

$$\left[\left(\frac{R}{R-1}\right)^n - 1\right]\left[1 + \frac{\varepsilon R^4}{D+\varepsilon\gamma R^2} + \frac{a^3}{6}(2-a)\right] < \frac{1-2a}{6}, \tag{2.9}$$

since $\int_0^1 f(s)ds = (1-2a)/12$ and

$$\max_{[0,1]} \int_0^u -f(s)\,ds = \int_0^a -f(s)\,ds = \frac{a^3}{12}(2-a).$$

Thus, for a, γ, ε chosen as in (2.6)—(2.8), we must further restrict D and $R > 1$ to satisfy (2.9). The first observation is that unless $R > 1$ is sufficiently large, even with $\varepsilon R^4/(D + \varepsilon\gamma R^2)$ missing, this inequality is not valid. This is consistent with earlier work of Klaasen and Mitidieri [**8**] showing that no nontrivial solution of (2.4) exists if $R > 0$ is small.

Secondly, for $R > 1$ large with ε and γ fixed, $\varepsilon R^4/(D + \varepsilon\gamma R^2)$ will be small only if D is large. This is consistent with the interpretation of Ermentrout, Hastings and Troy [**2**] which suggests large diffusion is necessary for the formation of a standing wave. In particular, for fixed ε and γ, if R is chosen so that

$$\left[\left(\frac{R}{R-1}\right)^n - 1\right]\left[1 + \frac{a^3}{6}(2-a)\right] < \frac{1-2a}{6},$$

then inequality (2.9) is valid provided D is sufficiently large. We summarize this as

COROLLARY 1. *Let $0 < \gamma < 4/(1-a)^2$, $0 < a < 1/2$ and $\varepsilon > 0$. If $R > 0$ and $D > 0$ are chosen such that*

$$\left[\left(\frac{R}{R-1}\right)^n - 1\right]\left[1 + \frac{\varepsilon R^4}{D+\varepsilon\gamma R^2} + \frac{a^3}{6}(2-a)\right] < \frac{1-2a}{6},$$

then, for any bounded region $\Omega \subset \mathbf{R}^n$ with $C^{2+\alpha}$ boundary which contains a ball of radius R, the Dirichlet problem for (2.4) *has two nontirvial solutions.*

The second solution of Corollary 1 follows from Theorem 2, since $f(0) = 0$ and $f'(0) = -a < 0$.

In a previous paper on this equation, Klaasen and Mitidieri [**8**] showed that, for $D > 0$ sufficiently small, no nontrivial solution exists for any $R > 0$.

3. The proofs of Theorem 1 and Theorem 2. Verification of Theorems 1 and 2 require three lemmas which are first proved. Based on these lemmas, the proofs of Theorems 1 and 2 are the same as the proofs of their corresponding theorems in [**8**] and, hence, we only outline the remainder of the proofs here.

Assume that Ω is a bounded domain with $\partial\Omega$ of class $C^{2+\alpha}$, where α is a constant and $0 < \alpha < 1$. A classical solution of (2.1) is a pair (u, v) such that $u, v \in C^{2+\alpha}(\bar{\Omega})$, and (u, v) satisfy (2.1) in Ω and the Dirichlet boundary conditions (2.2). The first lemma establishes bounds on classical solutions of (2.1), for the Dirichlet *BVP*.

LEMMA 1. *Suppose there exists a* $b > 0$ *such that* $(g(u))/u > (\beta\,\lambda)/(\sigma\,\delta)$, *for all* $|u| \geqq b$. *Then any classical solution* (u, v) *of* (2.1) *satisfies*

$$|u| \leqq b \text{ and } |v| \leqq \frac{\lambda}{\delta} b \qquad \text{on } \bar{\Omega}. \tag{3.1}$$

PROOF. Let $u_M \equiv \max u$, $u_m \equiv \min u$, $v_M \equiv \max v$ and $v_m \equiv \min v$ on $\bar{\Omega}$. An application of a standard maximum principle argument to $-\Delta v = \lambda u - \delta v$ yields

$$v_M \leqq \frac{\lambda}{\delta} u_M \text{ and } \frac{\lambda}{\delta} u_m \leqq v_m. \tag{3.2}$$

Hence, it is easily seen that $|u| \leqq b$ implies $|v| \leqq \lambda/\delta\, b$ on $\bar{\Omega}$ and it suffices to prove $|u| \leqq b$ on $\bar{\Omega}$. Let $u(x_1) = u_M \geqq 0$ and $u(x_2) = u_m \leqq 0$. Then $-\Delta u(x_1) \geqq 0$ and $-\Delta u(x_2) \leqq 0$. Again, an application of a maximum principle argument to $-\Delta u = -\sigma g(u) - \beta v$ implies that $\sigma g(u_M) + \beta v(x_1) \leqq 0$ and $\sigma g(u_m) + \beta v(x_2) \geqq 0$. Consequently, (3.2) implies

$$g(u_M) \leqq -\frac{\beta\lambda}{\sigma\delta} u_m \text{ and } g(u_m) \geqq -\frac{\beta\lambda}{\sigma\delta} u_M. \tag{3.3}$$

Suppose, for contradiction, that $u_M > b$. Then the hypothesis of the Lemma and (3.3) imply $(\beta\lambda)/(\sigma\delta)u_M \leqq g(u_M) \leqq -(\beta\lambda)/(\sigma\delta)u_m$ and, hence,

$$u_m \leqq -u_M \leqq -b. \tag{3.4}$$

But $u_m \leqq -b$ implies similarly that $(\beta\lambda)/(\sigma\delta)u_m > g(u_m) \geqq -(\beta\lambda)/(\sigma\delta)u_M$ and hence $u_m \geqq -u_M$ which contradicts (3.4). Thus $u_M \leqq b$. A similar argument shows that $u_m \geqq -b$.

If g satisfies the hypothesis of Lemma 1, let $\tilde{g}$ be any function which satisfies

(i) $\tilde{g}$ is Holder continuous on $\mathbf{R}$ of order α;
(ii) $\tilde{g}(u) = g(u)$, for $|u| \leqq b$;
(iii) $\tilde{g}(u)/u > (\beta\lambda)/(\sigma\delta)$, for $|u| \geqq b$;
(iv) $|\tilde{g}(u)| \leqq c_1 + c_2 |u|^p$ on $\mathbf{R}$, where c_1, c_2, and p are positive constants and $1 \leqq p \leqq (n+2)/(n-2)$ if $n > 2$, but $1 \leqq p$ if $n = 2$. Such a modification of g is easy to construct. Consider the corresponding modified boundary value problem

$$\begin{aligned} -\Delta u &= -\sigma\tilde{g}(u) - \beta v \\ -\Delta v &= \lambda u - \delta v \end{aligned} \qquad \text{on } \Omega \tag{3.5}$$

with Dirichlet boundary conditions. Lemma 1 implies that the Dirichlet problem for (2.1) and (3.5) have precisely the same classical solutions.

Next, v is eliminated from BVP (3.5) by observing that equation $-\Delta v + \delta v = \lambda u$, together with Dirichlet boundary conditions, is equivalent to an operator equation $v = Bu$ in an appropriate function space, where $B = [(1/\lambda)(-\Delta + \delta)]^{-1}$.

In the classical setting, B is a transformation from $C^{\alpha}(\bar{\Omega})$ into $C^{2+\alpha}(\bar{\Omega})$ and, by substituting $v = Bu$ into the first equation of (3.5), we obtain the system

$$
\begin{aligned}
-\Delta u &= -\sigma\tilde{g}(u) - \beta B(u) && \text{on } \Omega, \\
u &= 0 && \text{on } \partial\Omega
\end{aligned}, \tag{3.6}
$$

which is equivalent to (3.5) and, hence, to (2.1), (2.2).

Our approach, however, will be to prove the existence of weak solutions (solutions in $H_0^1(\Omega)$) of (3.6) by variational methods and then rely on standard regularity results to conclude that these solutions are in fact classical solutions. In this setting, B is viewed as an operator from $H_0^1(\Omega)$ into $H_0^1(\Omega)$, where the Sobolev space $H_0^1(\Omega)$ is the completion of $C_0^{\infty}(\Omega)$ in the norm $\|u\|_0 \equiv (\int_\Omega |\nabla u|^2\,\mathrm{dx})^{1/2}$. We will then use the notation that $\|u\| \equiv (\int_\Omega |u|^2\,\mathrm{dx})^{1/2}$ and $(u, v) \equiv \int uv\,dx$ so that $\|u\|_0 = \|\nabla u\|$. Then, as a consequence of the relationship $-\Delta Bu + \delta Bu = \lambda u$, it is easy to see that B is a positive self-adjoint operator on $H_0^1(\Omega)$ in the sense that $(u, Bv) = (Bu, v)$ and $(Bu, u) \geqq 0$, for all $u, v \in H_0^1(\Omega)$. The following inequality will be important later.

LEMMA 2. *There exists a constant c, depending only on Ω, such that*

$$(u, Bu) = \int_\Omega uBu\,dx \leqq \frac{\lambda c^4}{1+\delta c^2}\|\nabla u\|^2, \qquad \textit{for all } u \in H_0^1(\Omega).$$

In particular, if $\Omega = B_R(0)$, then $C = R$, and we have

$$\int_{B_R(0)} u\,Bu\,dx \leqq \frac{\lambda R^4}{1+\delta R^2}\|\nabla u\|^2 \qquad \forall u \in H_0^1(\Omega).$$

PROOF. We multiply $-\Delta v + \delta v = \lambda u$ by v and integrate to obtain $\|\nabla u\|^2 + \delta\|v\|^2 = \lambda(u, v)$. Next, combine this with the Poincare inequality **[4]**, $\|v\| \leqq c\|\nabla v\|$, and the Cauchy Schwarz inequality to obtain $(1/C^2 + \delta)\|v\|^2 \leqq \|\nabla v\|^2 + \delta\|v\|^2 = \lambda(u, v) \leqq \lambda\|u\|\,\|v\|$ or $\|v\| \leqq C^2\lambda/(1 + C^2\delta)\|u\|$. Next, apply the Cauchy Schwarz inequality again and obtain

$$\int u\,Bu = \int uv = (u, v) \leqq \|u\|\,\|v\| \leqq \frac{C^2\lambda}{1+C^2\delta}\|u\|^2 \leqq \frac{C^4\lambda}{1+C^2\delta}\|\nabla u\|^2.$$

For a discussion of the appropriate choice of $C = R$ when $\Omega = B_R(0)$, see Gilbarg and Trudinger **[4]**.

Define ϕ on $H_0^1(\Omega)$ by

$$\phi(u) \equiv \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\beta}{2}\int_\Omega u\, Bu + \sigma\int_\Omega \tilde{G}(U), \tag{3.7}$$

where $\tilde{G}(u) \equiv \int_0^u \tilde{g}(s)\, ds$.

LEMMA 3. *Let $B_R(0) \subset \Omega$ and define $\tilde{u} \in H_0^1(\Omega)$ by*

$$\tilde{u}(x) = \begin{cases} u_0, & \text{for } |x| \leqq R - 1 \\ u_0\,(R - |x|,) & \text{for } R - 1 \leqq |x| \leqq R \\ 0, & |x| \geqq R. \end{cases}$$

Then inequality (2.5) *implies that* $\phi(\tilde{u}) < 0$.

PROOF OF LEMMA 3. Estimating each term of $\phi(\tilde{u})$ in succession, we have

$$\frac{1}{2}\int_\Omega |\nabla\tilde{u}|^2 = \frac{1}{2}\int_{B_R(0)} |\nabla\tilde{u}|^2 = \frac{1}{2}\int_{C_R} u_0^2 = \frac{w_n u_0^2}{2}[R^n - (R-1)^n], \tag{3.8}$$

where $C_R = B_R(0) - B_{R-1}(0)$ and w_n = volume of $B_1(0)$. Secondly, using Lemma 2,

$$\frac{\beta}{2}\int_\Omega \tilde{u}\, B(\tilde{u}) = \frac{\beta}{2}\int_{B_R(0)} \tilde{u}\, B\tilde{u} \leqq \frac{\beta\lambda\, R^4\, w_n\, u_0^2}{2(1 + \delta R^2)}[R^n - (R - 1)^n]. \tag{3.9}$$

Finally,

$$\begin{aligned} \sigma\int_\Omega \tilde{G}(\tilde{u}) &= \sigma\int_{B_R(0)} \tilde{G}(\tilde{u}) = \sigma\int_{B_{R-1}(0)} \tilde{G}(\tilde{u}) + \sigma\int_{C_R} \tilde{G}(\tilde{u}) \\ &\leq \sigma\, w_n(R-1)^n\, \tilde{G}(u_0) + \sigma(\max_{[0\text{'}u_0]} \tilde{G}(u))\, w_n[R^n - (R-1)^n]. \end{aligned} \tag{3.10}$$

Combining (3.7)–(3.10) with the observation that $\tilde{G}(u) = G(u)$ on $[0, u_0]$, we conclude that

$$\begin{aligned} \frac{2}{w_n(R-1)^n}\phi(\tilde{u}) \leqq \left[\left(\frac{R}{R-1}\right)^n - 1\right]&\left\{u_0^2 + \frac{\beta\lambda\, u_0^2 R^4}{1+\delta\, R^2} + 2\sigma \max_{[0, u_0]} G(u)\right\} \\ &+ 2\sigma G(u_0) < 0, \end{aligned}$$

from inequality (2.5).

PROOF OF THEOREM 1. See [8] for details. First one observes that there is a constant $k > 0$ such that

$$\phi(u) \geqq \frac{1}{2}\|\nabla\|^2 - \sigma k\,|\Omega| \qquad \text{on } H_1^0(\Omega)$$

and, hence, ϕ is bounded below and $\phi(u) \to \infty$ as $\|\nabla u\| \to \infty$.

Secondly, ϕ is lower semicontinuous and Frechet differentiable on $H_1^0(\Omega)$ and, hence, ϕ attains its minimum on $H_0^1(\Omega)$ at a function $u_0 \in H_0^1(\Omega)$.

Lemma 3 implies that $\phi(u_0) < 0$ and hence $u_0 \neq 0$. Thus, u_0, $v_0 = B(u_0)$ is a weak solution pair of (2.1), (2.2), and by a standard "boot strap" argument $u_0, v_0 \in C^{2+\alpha}(\bar{\Omega})$ and are classical solutions of (2.1), (2.2).

PROOF OF THEOREM 2. Under the additional hypothesis that $g(0) = 0$ and $g'(0) > 0$, the function ϕ defined in (3.7) has a local minimum at 0, and, moreover, there is $r > 0$, $\rho > 0$ such that $\phi(u) > 0$ for all $0 \leqq \|u\| \leqq r$, and $\phi(u) \geqq \rho$ for all $\|u\| = r$. The mountain pass theorem of Ambrosetti and Rabinowitz [1] applies to conclude the existence of a second solution u_1 which satisfies

$$\phi(u_1) = \inf_{\sigma \in \Sigma} \max_{u \in \sigma([0,1])} \phi(u) > 0,$$

where $\sum = \{\sigma \in C([0, 1], H_0^1(\Omega)) | \sigma(0) = 0, \sigma(1) = u_0\}$. By similar arguments, u_1 is shown to be a classical solution.

REMARK 1. Since the test function $\tilde{u}$ of Lemma 3 is spherically symmetric, the existence of weak solutions could be established in the closed subspace of $H_0^1(\Omega)$ consisting of spherically symmetric functions. Then the resulting solutions are spherically symmetric.

REMARK 2. Similar results can be established for the Neumann boundary value problem for (2.1) consisting of the restrictions $\partial u/\partial n = \partial v/\partial n = 0$ on $\partial\Omega$ where $\partial/\partial n$ is the outward normal differential operator on $\partial\Omega$.

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