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ON REGULAR GROWTH AND ASYMPOTIC STABILITY

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We are concerned with the global asymptotic stability of the trivial solution to the question

$$y'' + q(x)y = 0, \qquad 0 \leqq x \leqq X, \tag{1}$$

when $q(x)$ is right-continuous, nondecreasing, and $\lim_{x\to X}q(x) = +\infty$. It is well known that (1) possesses a zero-tending solution under these assumptions, and examples exist which show that not all solutions need satisfy

$$\lim_{x\to X} y(x) = 0. \tag{2}$$

What is needed is an extra assumption preventing $q(x)$ from doing most of its growth on arbitrarily small sets, i.e., a regular growth assumption. The paper of Macki [1] surveys the various distinct notions of regular growth. In this paper we present a definition of regular growth which improves and unifies the various existing concepts.

Such regular growth conditions are obtained by a converse path, by assuming that (1) has a solution not satisfying (2), and making deductions about the behaviour of $q(x)$ in relation to certain sequences of x-values. These deductions, when used as hypotheses, then form necessary conditions for (1) to have a solution not satisfying (2) and provide a concept of irregular growth. The more such deductions are utilised in this way, the narrower may be the class of $q(x)$ of irregular growth, and so the wider the class of $q(x)$ of regular growth for which (2) must be the case.

We shall work in terms of various notions of "irregular growth". Roughly speaking, these have the character that there must be a family of sequences satisfying certain conditions (see, e.g., (i), (iii), (iv), (v) below) for which a certain series (see (ii)) converges. The converse notion of regular growth, as generally presented in the literature, takes the form that, for every such set of sequences, the series in question must diverge. We will not present these converse notions explicitly here, since they are obtainable by immediate translation from the associated con-

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cepts of irregular growth. Our aim is to obtain as narrow as possible a formulation of the latter idea.

The following theorem gives a notion of irregular growth close to that of McShane ([**2**] or [**1**, p. 365, Definition 4]), but more strict, so that it yields a more general idea of regular growth, and so a more general sufficient criterion for all solutions to satisfy (2).

Subsequently, we add conditions to yield a still stricter notion of irregular growth, and so a concept of regular growth more general, not only than that of McShane, but also that due to Sansone ([**3**] or [**1**] p. 363, Definition 2]).

Most writers on this topic take the case $X = \infty$ in (1), (2). Our treatment of a McShane-type condition does not, however require this, so long as the equation is oscillatory at X.

THEOREM. *Let* $q(x)$, $x_0 \leqq x \leqq X \leqq +\infty$, *be positive, nondecreasing and right continuous, with* $\lim_{x\to X} q(x) = +\infty$. *If* $X < +\infty$, *we also assume that all solutions of* (1) *are oscillatory at* X. *Suppose that* (1) *possesses a solution* $y(x)$ *violating* (2), *with zeros* $\{x_n\}$ *of* $y(x)$ *and zeros* $\{z_n\}$ *of* $y'(x)$. *Then* $q(x)$ *is of irregular growth in the sense that, for any* ε *in* $(0, \pi/2)$, *there exist sequences* $\{b_n\}$, $\{c_n\}$ *such that*

(i) $\cdots < z_n < b_n < x_n < c_n < z_{n+1} < \cdots$;

(ii) $\sum_{n=1}^{\infty} \log[q(c_n)/q(b_n)] < \infty$ (*equivalently*, $\sum_{n=1}^{\infty}\{[q(c_n)/q(b_n)]-1\} < \infty$, *or again* $\sum_{n=1}^{\infty} \{1 - q(b_n)/q(c_n)\} < \infty$);

(iii) $\int_{b_n}^{c_n} q^{1/2}(x)dx \to \pi - 2\varepsilon$;

(iv) $\int_{c_n}^{z_{n+1}} q^{1/2}(x)dx \to \varepsilon$; *and*

(v) $\int_{z_n}^{b_n}(b_n - x)q(x)dx \to \varepsilon^2/2$.

PROOF. Let $r(x) > 0$ be defined by $r^2 = y^2 + (y')^2/q$, where $y(x)$ solves (1) but violates (2). Then $y(x)$ is oscillatory with zeros $\{x_n\}$ of $y(x)$, and zeros $\{z_n\}$ of $y'(x)$ (Figure 1). Now, $d(r^2) = (y')^2 d/q^{-1}) \leqq 0$ and $r(z_n) = y(z_n)$, so that amplitudes of $y(x)$ are decreasing to a nonzero limit, which implies in turn that $r(x)$ decreases to $R > 0$ as $x \to X$.

For a given ε in $(0, \pi/2)$, we define $\{b_n\}$, $\{c_n\}$ by

(3) $$y(b_n) = y(z_n)\cos\varepsilon, \qquad y(c_n) = y(z_{n+1})\cos\varepsilon,$$

with $z_n < b_n < x_n < c_n < z_{n+1}$(Figure 1).

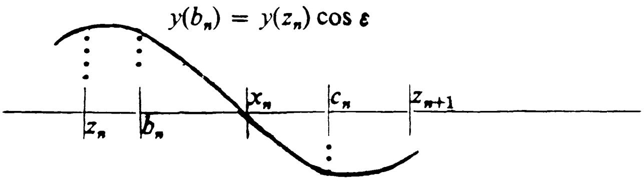

Figure 1.

For x in $[b_n, c_n]$ we have $|y(x)| \leqq |y(b_n)| = r(z_n)\cos\varepsilon$, so

(4) $x \in [b_n, c_n] \Rightarrow (y')^2/q = r^2 - y^2 \geqq r^2(z_{n+1}) - r^2(z_n)\cos^2\varepsilon \to R^2\sin^2\varepsilon.$

Therefore $(y')^2 \geqq (1/2)R^2(\sin^2\varepsilon)q$ for x in $[b_n, c_n]$, $n > N_\varepsilon$, with N_ε suitably large. This lower bounded $(y')^2$ allows the estimate

(5)
$$\begin{aligned} r^2(b_n) - r^2(c_n) &= -\int_{b_n}^{c_n} d(r^2) = \int_{b_n}^{c_n} (y')^2(-dq^{-1}) \\ &\geqq (1/2)R^2\sin^2\varepsilon q(b_n)\{q^{-1}(b_n) - q^{-1}(c_n)\}, \qquad n > N_\varepsilon. \end{aligned}$$

This implies (ii), since

$$\infty > r^2(0) - R^2 \geqq \sum_{N_\varepsilon}^{\infty}\left[-\int_{b_n}^{c_n} d(r^2)\right] \geqq (1/2)R^2\sin^2\varepsilon \sum_{N_\varepsilon}^{\infty}\{1 - q(b_n)/q(c_n)\}.$$

To prove (iii), (iv) and (v) we introduce the phase angle $\theta(x)$ by

(6) $$\tan\theta(x) = y(x)q^{1/2}(x_n)/y'(x), \qquad x \in [b_n, c_n],$$

with $\theta(x_n) = n\pi$. Then

(7) $$\theta'(x) = q^{1/2}(x_n)\cos^2\theta + q(x)q^{-1/2}(x_n)\sin^2\theta, \qquad x \in [b_n, c_n].$$

Note that (ii) and the fact that $q(x)$ is increasing imply that $q(x) = q(x_n)[1 + o(1)]$ uniformly for $x \in [b_n, c_n]$, as $n \to \infty$. Thus $\theta'(x) = q^{1/2}(x)[1 + o(1)]$ uniformly on $[b_n, c_n]$ as $n \to \infty$.

If we define $r_n^2(x) = y^2(x) + (y'(x))^2/q(x_n)$ on $[b_n, c_n]$, then

(8) $$y(b_n) = r_n(b_n)\sin\theta(b_n) = r(z_n)\cos\varepsilon,$$

so $\sin\theta(b_n) = r(z_n)r_n^{-1}(b_n)\cos\varepsilon = [R + o(1)]\,[R + o(1)]^{-1}\cos\varepsilon$. This implies that

(9) $$\theta(b_n) = \left(n - \frac{1}{2}\right)\pi + \varepsilon + o(1)$$

and a similar argument shows that

(10) $$\theta(c_n) = \left(n + \frac{1}{2}\right)\pi - \varepsilon + o(1).$$

Now, (iii) easily follows.

(11)
$$\begin{aligned} \pi - 2\varepsilon + o(1) &= \int_{b_n}^{c_n} \theta'(x)dx = \int_{b_n}^{c_n} q^{1/2}(x)[1 + o(1)]dx \\ &= [1 + o(1)]\int_{b_n}^{c_n} q^{1/2}(x)dx. \end{aligned}$$

To prove (iv) and (v), we choose $0 < \delta < \varepsilon$ and introduce points B_n, C_n (Figure 2), such that $y(B_n) = y(z_n)\cos\delta$, $y(C_n) = y(z_{n+1})\cos\delta$.

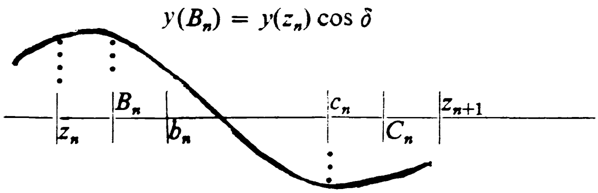

Figure 2.

Our previous arguments show that $q(x) = q(x_n)\,[1 + o(1)]$ and $\theta'(x) = q^{1/2}(x)\,[1 + o(1)]$ on $[B_n, C_n]$ as $n \to \infty$. The analogues of (9), (10) hold (with ε replaced by δ), so we conclude that

(12) $$\int_{c_n}^{C_n} q^{1/2}(x)dx \to \varepsilon - \delta, \quad \text{as } n \to \infty.$$

Thus, (iv) will follow (from letting $\delta \to 0$) if we can show $\lim_{\delta\to 0} \lim\sup_{n\to\infty} \int_{C_n}^{z_n+1} q^{1/2}(x)dx = 0$. To this end, note that

$$\int_x^{z_n+1} (t - x)q(t)y(t)dt = y(z_{n+1}) - y(x),$$

so that

(14) $$\int_{C_n}^{z_n+1} (t - C_n)q(t)dt \leqq |y(C_n)|^{-1}\, y(z|_{n+1}) - y(C_n)| = \sec\delta - 1,$$

and so

(15) $$\lim_{\delta\to 0} \lim\sup_{n\to\infty} \int_{C_n}^{z_n+1} (t - C_n)q(t)dt = 0.$$

Conclusion (iv) follows easily, i.e.,

(16) $$\frac{1}{2}\left\{\int_{C_n}^{z_n+1} q^{1/2}(t)dt\right\}^2 = \int_{C_n}^{z_n+1} q^{1/2}(t)\left[\int_t^{z_n+1} q^{1/2}(s)ds\right]dt \leqq \int_{C_n}^{z_n+1}\left[\int_t^{z_n+1} q(s)ds\right]dt = \int_{C_n}^{z_n+1}(t - C_n)q(t)dt.$$

It remains to prove (v). We will show that

(17) $$\int_{B_n}^{b_n} (b_n - x)q(x)dx = \frac{1}{2}(\varepsilon - \delta)^2 + o(1), \qquad \text{as } n \to \infty,$$

and

(18) $$\lim_{\delta\to 0} \lim\sup_{n\to\infty} \int_{z_n}^{B_n} (b_n - x)q(x)dx = 0.$$

From earlier arguments, we have $\int_{B_n}^{b_n} q^{1/2}(x)dx = \varepsilon - \delta + o(1)$, and $q(x) = q(B_n)\,[1 + o(1)]$ on $[B_n, C_n]$, as $n \to \infty$. Thus

(19) $$(b_n - B_n)q^{1/2}(B_n) = (\varepsilon - \delta)[1 + o(1)].$$

(20) $$\int_{B_n}^{b_n} (b_n - x)q(x)dx = \frac{1}{2}(B_n - b_n)^2 q(B_n)[1 + o(1)].$$

Combining (19) and (20) gives (17).

Turning to (18), we note that, on $[z_n, B_n]$, we have $|y(x)| \geqq R\cos\delta$, so we can write

(21) $$\begin{aligned}\int_{z_n}^{B_n} (b_n - x)(x)dx &\leqq \frac{1}{R\cos\delta}\int_{z_n}^{B_n}(b_n - x)q(x)y(x)dx \\ &= \frac{1}{R\cos\delta}\{-(b_n - B_n)y'(B_n) - y(B_n) + y(z_n)\},\end{aligned}$$

where we have used the fact that $y'' = -qy$, combined with an integration by parts. Now,

$$y(z_n) - y(B_n) = y(z_n)(1 - \cos\delta) \to R(1 - \cos\delta),$$

so this expression satisfies $\lim_{\delta\to 0}\lim_{n\to\infty}\sup = 0$. We estimate the remaining term, using (19),

$$\begin{aligned}(b_n - B_n)y'(B_n) &= q^{1/2}(B_n(b_n - B_n)][y'(B_n)/q^{1/2}(B_n)] \\ &= (\varepsilon - \delta)[1 + p(1)][r^2(B_n) - y^2(B_n]^{1/2} \\ &= 0(1)[r^2(B_n) - r^2(z_n)\cos^2\delta]^{1/2}.\end{aligned}$$

Thus, the $\lim_{n\to\infty}\sup$ of this expression is $O(R\sin\delta)$. This completes the proof of the Theorem.

We now present a more restrictive definition of "irregular growth", so as to provide a stronger theorem on global asymptotic stability than the definitions of either McShane or Sansone.

DEFINITION. *Let $q(x)$, $0 \leqq x < X \leqq \infty$, be positive, non-decreasing, right-continuous and unbounded. If $X < \infty$, assume also that solutions of* (1) *are oscillatory at X. We then say that $q(x)$ is of irregular growth if there exist sequences $\{x_n\}$, $x_n \to X$, $\{z_n\}$, $\{b_n\}$, $\{c_n\}$ satisfying* (i) – (v) *of the above Theorem, plus:*

(vi) *The sequences $\{x_n\}$, $\{z_n\}$ are independent of ε;*

(vii) $\pi q^{-1/2}(x_{n+1} - 0) \leqq x_{n+1} - x_n \leqq \pi q^{-1/2}(x_n)$.

It is immediate that, as desired, this notion of irregular growth is stricter than that of McShane [**1, 2**]). For one thing, this involves inequalities in the analogues of (iii)-(v) rather than limiting equalities; we have also made the additional requirements (vi), (vii).

For the purposes of comparison with the Sansone condition [**1, 3**] we confine attention to the case $X = \infty$, and make minor modifications to allow $q(x)$ to have jump discontinuities. The above requirement (vii) is one of Sansone's; it may be seen as a consequence of the Sturm comparison theorem for the situation that (1) has a solution with zeros at the x_n. A further Sansone condition is, modified to allow for discontinuities,

$$\limsup_{n\to\infty} q(x_n - 0)/q(x_{n+1}) = 1.$$

This is a consequence of (vii) and the requirement that $x_n \to \infty$. For suppose if possible, that, for all large n and some ℓ with $\ell < 1$, we have

$$q(x_n - 0) < \ell^2\, q(x_{n+1})$$

and so

$$q^{-1/2}(x_{n+1}) < \ell\, q^{-1/2}(x_n - 0).$$

Using (vii), we deduce that

$$(x_{n+2} - x_{n+1}) < \ell(x_n - x_{n-1}),$$

contradicting the hypothesis that $x_n \to \infty$.

It remains to discuss Sansone's analogue of (ii) which, as modified slightly by Macki, takes the form

$$\sum_{n=1}^{\infty}\left[\frac{q(x_n + \ell\Delta_n}{q(x_n - \ell\Delta_n} - 1\right] < \infty, \qquad \text{for all } 0 < \ell < 1/2,$$

where $\Delta_n = \pi q^{-1/2}(x_n)$. Since $q(x)$ is non-decreasing, it will follow that our requirement is more stringent if we can show that

$$\text{(22)} \quad b_n < x_n - \ell\pi q^{-1/2}(x_n) < x_n + \ell\pi q^{-1/2}(x_n) < c_n, \qquad 0 < \ell < 1/2,$$

for n large and ε sufficiently small in the specification (3) of b_n and c_n.

We recall now that, by (7) and (ii),

$$\theta'(x) = q^{1/2}(x)\,\{1 + o(1)\} = q^{1/2}(x_n)\,\{1 + o(1)\}$$

uniformly on $[b_n, c_n]$, as $n \to \infty$. It thus follows from (9) that

$$\pi/2 - \varepsilon = (x_n - b_n)q^{1/2}(x_n)\,(1 + o(1))$$

so that, for large n,

$$b_n < x_n - q^{-1/2}(x_n)\,(\pi/2 - 2\varepsilon),$$

say. This implies the first of (22), for a given $\ell \in (0, 1/2)$ and large n if $\varepsilon > 0$ is suitably chosen. The last inequality in (22) is discussed similarly.

This completes the proof that the irregular growth requirements (1) – (vii) in the above Definition are indeed stricter than those of Sansone, so that the regularity criteria are broader.

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