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PARALLEL MAPS THAT PRESERVE GEOMETRIC OBJECTS OF HYPERSURFACES

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ABSTRACT. It is known that parallel maps of hypersurfaces in R^{n+1} preserve principal directions, umbilics and the third fundamental form [4]. We study the conditions under which the parallel map $f_t^\#$ of a parallel Σ_{-t} of a hypersurface Σ into the parallel Σ_t preserves other geometric objects besides the three mentioned above and show, in particular, that when the determinant of the Jacobian matrix of $f_t^\#$ is 1 and n is even, Σ is a certain non-trivial minimal hypersurface and $f_t^\#$ preserves the element of area and all the even order elementary symmetric functions of principal curvatures.

Introduction. Let Σ_t and Σ_{-t} denote parallel hypersurfaces of an immersed hypersurface Σ in R^{n+1} for a sufficiently small parameter t. The parallel maps of Σ into Σ_{-t} and Σ_t, which we can assume to be local diffeomorphisms, define a parallel map $f_t^\#$ of Σ_{-t} into Σ_t. As a parallel map $f_t^\#$ preserves principal directions, umbilics, and the third fundamental form. In this paper we investigate the conditions under which other geometric objects of the hypersurfaces besides the three mentioned above are preserved by $f_t^\#$ and show that they occur in the form of restrictions on the non-singular Jacobian matrix of $f_t^\#$. We illustrate the use of such conditions in the proof of our main results stated in Proposition 2.1.

1. Parallel immersions. Let M be a connected, orientable smooth manifold of dimension n. Let $X: M \to R^{n+1}$ be an immersion. For sufficiently small values of t, the mappings $X_t, X_{-t}: M \to R^{n+1}$, defined by

$$(1.1)\quad X_t(p) = X(p) + t\,N(X(p)), \qquad X_{-t}(p) = X(p) - t\,N(X(p)),$$

where $p \in M$ and N is a unit normal vector field on $X(M)$, are also imimersions. Let $X(M) = \Sigma$, $X_t(M) = \Sigma^t$ and $X_{-t}(M) = \Sigma_{-t}$. Define f_t: $\Sigma \to \Sigma_t$ and f_{-t}: $\Sigma \to \Sigma_{-t}$ by

$$(1.2)\qquad f_t \circ X(p) = X_t(p), \qquad f_{-t} \circ X(p) = X_{-t}(p),$$

for all $p \in M$. We assume f_t and f_{-t} are local diffeomorphisms.

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Observing that quantitites for Σ_{-t} can be obtained from those for Σ_t by changing t to $-t$, we usually write results for Σ_t and write those for Σ_{-t} only when necessary.

We write the first equation of (1.1) as $f_t \circ X = X + t\, N \circ X$, where N is viewed as the Gauss map of Σ into the unit sphere S^n. The derivative map, when we identify $X_* Z$ with $Z \in T_p(M)$, gives

$$f_{t^*}Z = Z + t\, L(Z), \tag{1.3}$$

where L is the Weingarten map for Σ. Since N is normal to Σ_t also, we have $N \circ X = N \circ X_t$ which yields

$$L(Z) = L_t(f_{t^*}Z), \tag{1.4}$$

where L_t is the Weingarten map for Σ_t. From (1.4), we get the known result [4] that parallel maps preserve principal directions, umbilics and the third fundamental form.

Choose an orthonormal frame $e_1, \ldots, e_n$ at $X(p)$ such that $\det(e_1, \ldots, e_n, N) = 1$. Since the tangent planes at $X(p)$ and $X_t(p)$ are parallel, e_i can be chosen as an orthonormal frame at $X(p)$ also. Let τ^i and τ_t^k denote 1-forms dual to e_i at $X(p)$ and $X_t(p)$, respectively. Then $dX = \sum \tau^i e_i$ and $dX_t = \sum \tau_t^i e_i$. But, from (1.1), we have $dX_t = dX + t dN$. So

$$\sum f_t^* \tau_t^i e_i = \sum (\delta_j^i + t a_j^i) \tau^j e_i, \tag{1.5}$$

where we have set $dN = \sum a_j^i \tau^j e_i$. In (1.5) we need the pull back symbol because the τ_t^i live in Σ_t. Clearly, (a_j^i) is the symmetric matrix of the Weingarten map L for Σ. From (1.5), we have

$$f_t^* \tau_t^i = \sum (\delta_j^i + t a_j^i) \tau^j. \tag{1.6}$$

Similarly,

$$f_{-t}^* \tau_{-t}^i = \sum (\delta_j^i - t a_j^i) \tau^j, \tag{1.7}$$

Let

$$A(t) = (\delta_j^i + t a_j^i) \text{ and } A(-t) = (\delta_j^i - t a_j^i). \tag{1.8}$$

In matrix notation (1.6) and (1.7) take the form

$$f_t^* \tau_t = A(t)\tau, \qquad f_{-t}^* \tau_{-t} = A(-t)\tau, \tag{1.9}$$

where we regard τ, τ_t and τ_{-t} as column vectors of 1-forms. Since f_t^* and f_{-t}^* are isomorphisms, their matrices $A(t)$ and $A(-t)$ are non-singular. Solving the second equation of (1.9) for τ and substituting in the first gives

$$f_t^* \tau_t = C(t) f_{-t}^* \tau_{-t}, \tag{1.10}$$

where

$$C(t) = A(t) A(-t)^{-1}. \tag{1.11}$$

We set

$$f_t^{\#} = f_t \circ f_{-t}^{-1} \tag{1.12}$$

and observe that $f_t^{\#}$ is locally a diffeomorphism of $\sum_{-t}$ into $\sum_t$. If Z is a tangent vector to $\sum_{-t}$, then

$$(f_t^{\#})_* Z = \sum C(t)^i_j\, Z^j e_i, \tag{1.13}$$

where Z^j are components of Z with respect to e_i.

LEMMA 1.1. *$f^{\#}$ preserves principal directions, umbilics, and the third fundamental form.*

PROOF. By our construction in (1.12), $f_t^{\#}$ is a parallel map and the result follows immediately.

The following Lemma is due to Gardner [2].

LEMMA 1.2. *Let $dA_t = f_t^*(\tau_t^1 \wedge \ldots \wedge \tau_t^n)$ and $dA = \tau^1 \wedge \ldots \wedge \tau^n$. Then*

$$dA_t = \sum_{i=0}^{n} \binom{n}{i} \sigma_i t^i dA = \det A(t) dA, \tag{1.14}$$

where $\sigma_0 = 1$ and the σ_i are the elementary symmetric functions of principal curvatures of $\sum$. Further, if $\sigma_i^{(t)}$ denote elementary symmetric functions of principal curvatures of $\sum_t$, then

$$\binom{n}{i} \sigma_i^{(t)}\, dA_t = \sum_{j=i} \binom{j}{i}\binom{n}{j} \sigma_j t^{j-i}\, dA, \qquad \text{for } 0 \leqq i \leqq n. \tag{1.15}$$

PROOF. On using (1.6) in $dA_t = f_t^*\tau_t^1 \wedge \ldots \wedge f_t^*\tau_t^n$, we get (1.14).

If (a_{tj}^i) denote the matrix of the Weingarten map L_t for $\sum_t$, then since $N = N_t$, we have $dN = dN_t$, from which we obtain

$$\sum a_j^i \tau^j = \sum a_{tj}^i\, f_t^* \tau_t^j, \qquad i = 1, 2, \ldots, n. \tag{1.16}$$

Use of (1.6) and (1.16) gives

$$\begin{aligned} &\sum_{i=0}^{n} \binom{n}{i} \sigma_i^{(t)} s^i dA_t \\ &= (f_t^*\tau^1 + \textstyle\sum s a_{tk}^1 f_t^* \tau_t^k) \wedge \cdots \wedge (f_t^*\tau^n + \textstyle\sum s a_{tk}^n \tau_t^k) \\ &= (\tau^1 + \textstyle\sum (s+t) a_k^1 \tau^k) \wedge \cdots \wedge (\tau^n + \textstyle\sum (s+t)\, a_k^n \tau^k) \\ &= \sum_{i=0}^{n} \binom{n}{i} \sigma_i (t+s)^i\, dA. \end{aligned} \tag{1.17}$$

Differentiating (1.17) with respect to s and equating the coefficients of powers of s we obtain (1.15).

LEMMA 1.3. *Let σ_ℓ, for $\ell = 1, 2, \ldots, 2m$, be elementary symmetric functions of real numbers k_i, for $i = 1, 2, \ldots 2m$. If $\sigma_1 = 0$ and, in con-*

sistency with this, $k_1 + k_2 = 0$, $k_3 + k_4 = 0, \ldots, k_{2m-1} + k_{2m} = 0$ *and* $k_i \neq 0$, *for all i, then*

$$\sigma_i = 0, \qquad i = 3, 5, \ldots, 2m - 1,$$
$$\sigma_i \neq 0, \qquad i = 2, 4, \ldots, 2m.$$

PROOF. Let

$$P_\ell = \sum_{i=1}^{2m} (k_i)^\ell,\ s_\ell = \binom{2m}{\ell}\sigma_\ell, \qquad \ell = 1, 2, \ldots, 2m.$$

The p_ℓ's and s_ℓ's are related by Newton's formulas

(1.18) $$p_r - s_1p_{r-1} - \cdots - (-1)^r s_{r-1}p_1 + (-1)^r s_r = 0,$$

where $1 \leqq r \leqq 2m$. It is easy to see from (1.18) and the hypothesis that $s_1 = 0$, $s_2 \neq 0$. We prove the Lemma by induction.

Suppose $s_3 = 0, s_5 = 0, \ldots, s_{2j-3} = 0$ and $s_4 \neq 0, \ldots, s_{2j-2} \neq 0$, $j < m$. We need to show that $s_{2j-1} = 0$ and $s_{2j} \neq 0$, $1 \leqq j \leqq m$.

By the induction hypothesis, (1.18), for $r = 2j - 1$, reduces to

$$(2j - 1)\, s_{2j-1} = p_{2j-1} = \sum_{\ell=1}^{m} (k_{2\ell-1}^{2j-1} + k_{2\ell}^{2j-1})$$
$$= \sum_{\ell=1}^{m} (k_{2\ell-1} + k_{2\ell})\,(k_{2\ell-1}^{2j-2} - \cdots + k_{2\ell}^{2j-2}) = 0.$$

Thus $s_{2j-1} = 0$.

It is known [3, p.51] that, for all real values of k_i, $\sigma_\ell^2 \geqq \sigma_{\ell-1}\sigma_{\ell+1}$, $1 \leqq \ell \leqq 2m - 1$, and the equality holds if and only if $k_1 = k_2 = \cdots = k_{2m}$. Suppose $\ell = 2j - 1$. Then $\sigma_{2j-1} = 0$, since $s_{2j-1} = 0$. Since, by hypothesis of the lemma, k_i cannot all be equal, and by the induction hypothesis $s_{2j-2} \neq 0$, the above inequality reduces to $0 > s_{2j-2}s_{2j}$ which implies $s_{2j} \neq 0$. This completes the proof of the lemma.

2. Geometric objects preserved by parallel maps. The matrix $C(t)$ defined in (1.11) relates a coframe of Σ_t to that of Σ_{-t} and (1.13) shows that it can be viewed as the Jacobian matrix of $f_t^\#$. Clearly, $C(t)$ is an element of $GL(n, R)$; restriction on $C(t)$ that it be an element of a particular subgroup of $GL(n, R)$ requires Σ to be a special surface and forces $f_t^\#$ to preserve additional geometric objects besides those mentioned in Lemma 1.1. The following Proposition illustrates this in two cases.

PROPOSITION 2.1. *Suppose Σ is a connected, orientable smooth n-manifold and X_t, X_{-t}: $M \to R^{n+1}$, are parallel immersions of an immersion X: $M \to R^{n+1}$ for all sufficiently small t. Let $C(t)$ be the Jacobian matrix of the parallel map $f_t^\#$ of $\Sigma_{-t} = X_{-t}(M)$ into $\Sigma_t = X_t(M)$.*

(a) *If the principle curvature k_i of M are all non-zero*, det $C(t) = 1$ *for* *all such t, that is, if $C(t) \in SL(n, R)$, and $n = 2m$, then $\Sigma = X(M)$ is a non-trivial immersed minimal hypersurface of R^{2m+1} and $f_t^\#$ preserves:*

(i) *the element of area;*

(ii) *all even order elementary symmetric functions of principal curvatures; and*

(iii) *the absolute value of each odd order elementary symmetric function of principal curvatures.*

(b) *If $C(t) = \lambda I$, for some $\lambda > 0$, then Σ is an umbilical hypersurface or a hyperplane in R^{n+1} and $f_t^\#$ is conformal. In particular,*

(i) *if $\lambda \neq 1$ and Σ is compact, then Σ is an Euclidean sphere;*

(ii) *if $\lambda = 1$, then Σ is a hyperplane in R^{n+1} and $f_t^\#$ is an isometry.*

PROOF. (a). If $\det C(t) = 1$, then, by the definition (1.11) of $C(t)$, it follows that $\det A(t) = \det A(-t)$. Hence

$$dA_t = \det A(t)\, dA = \det A(-t)\, dA = dA_{-t}. \tag{2.1}$$

By using the formula (1.14) in (2.1) and writing $n = 2m$, we have

$$t\binom{2m}{1}\sigma_1 + t^3\binom{2m}{3}\sigma_3 + \cdots\cdots + t^{2m-1}\binom{2m}{2m-1}\sigma_{2m-1} = 0,$$

for all sufficiently small t, which implies

$$\sigma_1 = 0, \sigma_3 = 0, \ldots, \sigma_{2m-1} = 0. \tag{2.2}$$

Thus, if $\det C(t) = 1$, then $\sum_t$ and $\sum_{-t}$ have the same element of area at corresponding points and the principal curvatures $k_1, \ldots, k_{2m}$ satisfy (2.2). In view of Lemma 1.3 we may choose $k_1, \ldots, k_{2m}$ such that $k_1 + k_2 = 0, \ldots, k_{2m-1} + k_{2m} = 0$ and $k_i \neq 0$, for all $i = 1, 2, \ldots, 2m$. With this choice, $\sum$ is a non-trival minimal hypersurface in R^{2m+1} and we may write (2.1) as

$$dA_t = \left(1 + \binom{2m}{2}t^2\sigma_2 + \cdots\cdots + t^{2m}\sigma_{2m}\right)dA = dA_{-t}. \tag{2.3}$$

Now consider the formula in (1.15) for the elementary symmetric functions $\sigma_i^{(t)}$ of principal curvatures of $\sum_t$. When i is even, the right hand member of (1.15) is a polynomial in even powers of t. This is so because odd powers of t multiply odd order elementary symmetric functions which vanish. Thus, using (1.15) and (2.3), we obtain

$$\sigma_i^{(t)} = \sigma_i^{(-t)}, \qquad i = 2, 4, \ldots\ldots, 2m. \tag{2.4}$$

When i is odd, the right hand side of (1.15) is a polynomial in odd powers of t and so again in view of (2.3), we get

$$\sigma_i^{(t)} = -\sigma_i^{(-t)}, \qquad i = 1, 3, \ldots\ldots, 2m - 1. \tag{2.5}$$

The formulas (2.3), (2.4), and (2.5) prove the statement in part (a).

(b). If $C(t) = \lambda I, \lambda > 0$, from (1.11), we have $A(t) = \lambda A(-t)$. Using (1.8), we get

$$a^i_j = \frac{1}{t}\frac{\lambda - 1}{\lambda + 1}\delta^i_j \tag{2.6}$$

which shows that the Weingarten map L of Σ is a constant multiple of the identity map. Hence we conclude that Σ is umbilical or flat, depending on $\lambda \neq 1$ or $\lambda = 1$.

Let U, V be tangent vectors to Σ_{-t} at $X_{-t}(p)$. Then (1.13), together with $C(t) = \lambda I$, gives

$$\langle f^\sharp_t U, f^\sharp_t V\rangle = \lambda^2\langle U, V\rangle,$$

which shows that $f^\sharp_t$ is conformal.

(i) If $\lambda \neq 1$, L is a non-zero constant multiple of the identity map and, further, compactness of Σ implies that it is an Euclidean sphere.

(ii) If $\lambda = 1$, then $L = 0$, and hence Σ is a hyperplane. In this case, $f^\sharp_t$ is clearly an isometry.

When Σ is a surface in 3-dimensional Euclidean space $\mathbf{R}^3$, (1.14) and (1.15) give the well known formulals

$$\begin{aligned} dA_t &= (1 + 2\sigma_1 t + \sigma_2 t^2)dA, \qquad \sigma_1^{(t)} = \frac{\sigma_1 + \sigma_2 t}{1 + 2\sigma_1 + t^2\sigma_2} \\ \sigma_2^{(t)} &= \frac{\sigma_2}{1 + 2t\sigma_1 + t^2\sigma_2}. \end{aligned} \tag{2.7}$$

If Σ is minimal, we have $\sigma_1^{(t)}/\sigma_2^{(t)} = t = -\ \sigma_1^{(-t)}/\sigma_2^{(-t)}$, which imply that the sum of the principal radii of curvature is constant on each of the parallel surfaces of the minimal surface Σ. We prove the converse.

PROPOSITION 2.2. *If S is a surface in $\mathbf{R}^3$ with Gaussian curvature $\sigma_2 < 0$ and at each point of which the sum of the principal radii of curvature is constant, then S is a parallel of a minimal surface* (*which may be degenerate*).

PROOF. Let $\sigma_1/\sigma_2 = t =$ constant. Consider a parallel surface S_{-t} of S defined by

$$X_{-t} = X - tN \tag{2.8}$$

Exterior differentiation gives $dX_{-t} = dX - tdN$. Since $N = N_{-t}$, we have

$$dN_{-t} \times dX_{-t} = dN \times dX - tdN \times dN. \tag{2.9}$$

It is well known [1] that $dN \times dX = 2\sigma_1 dAN$ and $dN \times dN = 2\sigma_2 dAN$, so (2.9) reduces to

$$2\sigma_1^{(-t)}\ dA_{-t}N_{-t} = 2\sigma_1 dAN - \left(\frac{\sigma_1}{\sigma_2}\right)2\sigma_2\ dAN = 0.$$

This implies $\sigma_1^{(-t)} = 0$, that is, S_{-t} is a minimal surface. Now, we may write (2.8) as $X = X_{-t} + tN_{-t}$, which shows that S is a parallel of the minimal surface S_{-t}.

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