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DERIVATIVE AND APPROXIMATION THEOREMS ON LOCAL FIELDS

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ABSTRACT. The concept of a derivative of functions on local fields K plays a key role in approximation theory. In this note such a concept is given. The formula $\chi_\lambda^{\langle 1\rangle}(x) = |\lambda|\chi_\lambda(x)$ for characters χ_λ, $\lambda \in K$ is obtained. With some modification it is applicable to more cases; *e.g.*, to the a-adic group Ωa. Let $f \in L^r(K)$, $1 \leqq r < \infty$, and consider the linear operator

$$L(f, x, \lambda) = \int_K f(t)\, |\lambda|\, w(\lambda(x - t))dt, \qquad \lambda \in K,$$

where the kernel w is generated by some $\omega \in L^1(K)$, $w = \omega$. Then, by means of the above derivative, we prove several lemmas including the Bernstein inequality and establish some inverse approximation theorems for the class $W[L^r, |x|^\alpha]$ and $\mathrm{Lip}_r\alpha$. An application to the kernel G^α for the Bessel potential introduced by M. Taibleson is also included.

1. We use the notation in M. Taibleson's book [**4**]. Let K be a local field. It is well-known that K is locally compact, nondiscrete, complete and totally disconnected, and that the p-adic fields, p-series fields (p: prine) as well as their finite algebraic extensions are the only examples of such fields. Denote by $\mathcal{O}$ the ring of integers, $\mathcal{O} = \{x \in K: |x| \leqq 1\}$. $\mathscr{P} = \{x \in K: |x| < 1\}$ is its prime ideal, then $\mathcal{O}/\mathscr{P}$ is isomorphic with a finite field $GF(q)$, where $q = p^c$ for some prime p and positive integer c. There is a prime element $\mathfrak{p}$ of K such that $\mathscr{P} = (\mathfrak{p}) = \mathfrak{p}\mathcal{O}$. The spheres with center 0 (the center is not unique) in K are $\mathscr{P}^{-k} = \{x \in K: |x| \leqq q^k\}$, their Haar measures are $|\mathscr{P}^{-k}| = q^k$, $k \in Z$. In the sequel we state the concepts and theorems in one-dimensional form, even though most of them remain valid in the n-dimensional case.

Let $\chi_1(x)$ be any fixed nontrivial character of K^+ which is trivial on $\mathcal{O}$. As usual, denote by $\hat{f}$ the Fourier transform of f, and by $f{*}g$ the convolution of f and g.

C.W. Onneweer has given a formula for derivatives for p-adic fields and p series fields [**3**], i.e.,

(1) $$\lim_{N\to\infty}\sum_{\ell=-N}^{N-1} p(\ell, N) \sum_{(\ell, N)} (f(x) - f(x + Z_{qN}),$$

where $p(\ell, N) = p^{-N+1}(p + 1)^{-1}(p^{2\ell+1} + p^{-2N})$, and $\sum (\ell, N)$ denotes the summation over all $Z_{qN} \in \mathscr{P}^{\ell} \sim \mathscr{P}^{\ell+1}$ such that the jth coordinates of Z_{qN} are zero as $j \geqq N$. We will give another formula for a derivative which seems to be applicable to more cases; e.g., to the a-adic group Ω_a [2]. Our definition, as one will see, has the advantage that, for $f(x) = \chi_y(x)$, $y \in \mathscr{P}^s \sim \mathscr{P}^{s+1}$, one can almost catch the differential coefficient $|y| = q^{-s}$ by using only one term for the p-adic fields case. Furthermore, the formula

(2) $$[f(\mathfrak{p}^k x)]^{\langle 1\rangle} = |\mathfrak{p}^k| f^{\langle 1\rangle}(\mathfrak{p}^k x), \qquad k \in Z$$

is easy to deduce.

Let $W [L^r, \psi]$ denote the class of functions f such that there exists $g \in L^r(K)$, $\psi\hat{f} = \hat{g}$, $1 \leqq r \leqq \infty$ (where, for $2 < r \leqq \infty$, $\hat{f}$ is defined by distribution; see §3). Introducing the convolution integral

(3) $$\int_K f(t)\rho w(\lambda(x - t))dt,$$

where w is an integrable kernel, $\|w\|_1 = 1$ and $\rho = |\lambda| \to \infty$ is a parameter, we will establish some theorems characterizing $f \in W[L^r, |x|^\alpha]$ or Lipschitz class according to the degree of approximation by the operators (3). As is expected, the higher the approximation degree one has, the better properties of f one obtains. Some simple applications are also included.

2. Let $\chi_1(x)$ be a nontrivial character of K^+. There is a $k \in Z$, such that χ_1 is trivial on $\mathscr{P}^k$ but is nontrivial on $\mathscr{P}^{k-1}$. Without loss of generality we may assume $k = 0$ (otherwise use $\chi(x) = \chi_1(\mathfrak{p}^{-k}x)$ in place of $\chi_1(x)$). Note that any character $\chi_y(x)$ can be expressed as $\chi_y(x) = \chi(yx)$; this is due to the isomorphism $\hat{K} \cong K$.

Recall that $\mathscr{P}^{-1}/\mathscr{P}^0$ is a finite field $GF(p^c)$. If we let $q = p^c$, it is isomorphic with the set $\{\varepsilon_0\mathfrak{p}^{-1}, \varepsilon_1\mathfrak{p}^{-1}, \ldots, \varepsilon_{q-1}\mathfrak{p}^{-1}\}$, where $\varepsilon_0 = 0$, $|\varepsilon_1| = \cdots = |\varepsilon_{q-1}| = 1$. The set forms the entire set of representatives of $\mathscr{P}^0$ in $\mathscr{P}^{-1}$, and as a subgroup it is isomorphic with the cyclic group $Z(q)$. χ_1 is also a character when restricted to this set. In fact, we have

$$\chi_1(x + y) = \chi_1(x), \quad x \in \{\varepsilon_0\mathfrak{p}^{-1}, \ldots, \varepsilon_{q-1}\mathfrak{p}^{-1}\}, \qquad y \in \mathscr{P}^0.$$

Since $\hat{Z}(q) \cong Z(q)$, for every $x \in \{\varepsilon_1\mathfrak{p}^{-1}, \ldots, \varepsilon_{q-1}\mathfrak{p}^{-1}\}$, there is a $k \in \{1, 2, \ldots, q - 1\}$, depending only on j, such that

(4) $$\chi_1(\varepsilon_j\mathfrak{p}^{-1}) = \exp(2\pi ikq^{-1}), \qquad j = 1, \ldots, q - 1.$$

It is clear that $\chi_1(\varepsilon_j\mathfrak{p}^{-1}) \neq 1$, for $j \in \{1, \ldots, q - 1\}$. In the following we always make the assumption on χ_1 that it is nontrivial on $\mathscr{P}^{-1}$ but is trivial on $\mathscr{P}^0 = \mathcal{O}$.

The definition of a derivative is given as follows. Let $f\colon K \to \mathbf{C}$ be given, $N \in \mathbf{N}$, $t \in \mathbf{Z}$. Set

$$\Delta_N f(x) = \sum_{j=-N-t}^{N+t} q^{-N-j+1} \sum_{\nu=1}^{q-1} \sum_{\ell=0}^{q^N-1} \exp(-2\pi i \ell \nu q^{-1}) f(x+\ell \mathfrak{p}^j). \tag{5}$$

If for any fixed t, $\lim_{N\to\infty} \Delta_N f(x)$ exists, denoted by $f^{\langle 1\rangle}(x)$ (not depending on t), we call it the derivative of $f(x)$ (in the pointwise case). Similarly, one may define the derivative in the $L^r(K)$ sense, higher order derivatives, partial derivatives and weak derivatives in the usual way.

We begin with two simple lemmas.

LEMMA 1. *If a bounded function f has derivative a at each x, then so does $f(\mathfrak{p}^s x)$, moreover,*

$$[f(\mathfrak{p}^s x)]^{\langle 1\rangle} = |\mathfrak{p}^s| f^{\langle 1\rangle}(\mathfrak{p}^s x), \qquad s \in Z. \tag{6}$$

PROOF. By definition (5), if $s \geqq 0$ and $t \in \mathbf{Z}$, then

$$\Delta_N f(\mathfrak{p}^s x) = \sum_{k=-N-s-t}^{N-s+t} q^{-N-s-k+1} \sum_{\nu=1}^{q-1} \sum_{\ell=0}^{q^N-1} \exp(-2\pi i \ell \nu q^{-1}) f(\mathfrak{p}^s x + \ell \mathfrak{p}^{-k})$$

$$= q^{-s} \sum_{k=-N-s-t}^{N+s+t} q^{-N-k+1} \sum_{\nu=1}^{q-1} \sum_{\ell=0}^{q^N-1} \exp(-2\pi i \ell \nu q^{-1}) f(\mathfrak{p}^s x + \ell p^{-k}) - q^{-s} \sum_{k=N-s+t+1}^{N+s+t} .$$

The first sum in the right side tends to $|p^s| f^{\langle 1\rangle}(p^s x)$ as $N \to \infty$, and the other sum tends to zero because there are only $2s$ terms of $O(1)$.

The case $s < 0$ can be treated similarly. The lemma is proved.

We define $\mathscr{E}_s$ to be the function class on K, where $f \in \mathscr{E}_s$ means $f \in L^r(K)$, for some r, $1 \leqq r \leqq \infty$, f is constant on each coset of $\mathscr{P}^{s_1-1}$, for some $s_1 \in Z$, and s is the infimum of such s_1 ($\mathscr{E}_s$ corresponds to the locally constant function class $\mathscr{O}_M$(see [**4**, p.123]) and, in a Walsh system, to the class W_N [**5**]). Obviously $f =$ const. implies $s = -\infty$. Otherwise $s > -\infty$. The following lemma is a counterpart of S. Bernstein's inequality in approximation theory.

LEMMA 2. *Let $f \in \mathscr{E}_s$. Then $f^{\langle k\rangle} \in \mathscr{E}_s$, where $f^{\langle k\rangle}$ is the k^{th} derivative of f defined by induction, i.e., $f^{\langle k\rangle} = (f^{\langle k-1\rangle})^{\langle 1\rangle}$, $k \in \mathbf{N}$. Moreover,*

$$\|f^{\langle k\rangle}\|_r \leqq q^{sk} \|f\|_r, \qquad k \in \mathbf{N},\ s \in \mathbf{Z}. \tag{7}$$

PROOF. For $j \leqq -s+1$, we have $\mathfrak{p}^{-j} \in \mathscr{P}^{s-1}$. Consequently, $\ell \mathfrak{p}^{-j} \in \mathscr{P}^{s-1}$ as $\ell \in \mathbf{N}$. It follows by definition (5) that

$$\Delta_N f(x) = \sum_{j=-s+2}^{N} q^{-N-j+1} \sum_{\nu=1}^{q-1} \sum_{\ell=0}^{q^N-1} \exp(-2\pi i \ell \nu q^{-1}) f(x + \ell \mathfrak{p}^{-j}).$$

Therefore

$$(8)\qquad \|\Delta_N f\|_r \leqq \sum_{j=-s+2}^{N} q^{-N-j+1}(q-1)q^N \|f\|_r \leqq q^s \|f\|_r.$$

The estimate (8) also tells us $\{\Delta_N f\}$ is a Cauchy sequence in $L^r(K)$, so the limit of $\Delta_N f$ as $N \to \infty$ exists in $L^r(K)$. Letting $N \to \infty$ in (8), we obtain (7), for $k = 1$. We then are done by induction (obviously $f^{\langle k\rangle} \in \mathscr{E}_s$).

Let us examine the derivative of a character. We may find that, for both q-adic fields and q-series fields (q prime) the derivative of $\chi_{-\lambda_s}$ $\lambda \in \hat{K}$ exists, but the numerical results are somewhat different. It depends on the topological structure of $\hat{K}$. We will deal with them separately.

LEMMA 3. *Let χ_1 satisfy the following assumptions.*

(i) χ_1 *is trivial on* $\mathscr{P}^0$, *but* $\chi_1(x) \neq 1$ *if* $x \neq 0$ *and is in* $\mathscr{P}^{-j} \sim \mathscr{P}^0$, $j \in \mathbf{N}$;

(ii) *For* $j \in \mathbf{N}$, $|x| = 1$,

$$(2\pi i)^{-1} q^j \log \chi_1(x\mathfrak{p}^{-j}) \equiv 0 \pmod 1.$$

Then $\chi_\lambda(x) \equiv \chi_1(\lambda x)$ *has a derivative for every* $\lambda \in \hat{K}$ *(regarding* $\hat{K}$ *as the same of* K*). Moreover,*

$$(9)\qquad \lim_{N\to\infty} \Delta_N \chi_\lambda(x) = |\lambda| \chi_\lambda(x).$$

PROOF. We may assume $t = 0$. Then we have

$$(10)\qquad \Delta_N \chi_\lambda(x) = \sum_{j=-N}^{N} q^{-N-j+1} \sum_{\nu=1}^{q-1} \sum_{\ell=0}^{q^N-1} \exp(-2\pi i \ell \nu q^{-1}) \chi_\lambda(\ell \mathfrak{p}^{-j}) \chi_\lambda(x).$$

Since $\lambda = 0$, (9) is obvious. So we assume $\lambda \neq 0$, $\lambda \in \mathscr{P}^s \sim \mathscr{P}^{s+1}$ and $|\lambda| = q^{-s}$, and suppose $s \geqq 0$. If $j \leqq s$ and $\lambda\mathfrak{p}^{-j} \in \mathscr{P}^0$, by (i), we have

$$(11)\quad \chi_\lambda(\ell\mathfrak{p}^{-j}) - 1 = [\chi_1(\lambda\mathfrak{p}^{-j})]^\ell - 1 = 0, \qquad \ell = 0, 1, \ldots, q^N - 1.$$

If $j = s + 1$, the corresponding term in (10) is

$$(12)\qquad q^{-N-s} \sum_{\nu=1}^{q-1} \sum_{\ell=0}^{q^N-1} \exp(-2\pi i \ell \nu q^{-1})\, \chi_\lambda\,(\ell\mathfrak{p}^{-s-1})\, \chi_\lambda\,(x).$$

Since $\chi_\lambda(\ell\mathfrak{p}^{-s-1}) = [\chi_1(\lambda'\mathfrak{p}^{-1})]^\ell$ for some λ', $|\lambda'| = 1$, there is a $j_1 \in \{1, \ldots, q-1\}$ such that $\lambda'\mathfrak{p}^{-1} \in \varepsilon_{j_1}\mathfrak{p}^{-1} + \mathscr{P}^0$; by (4) there is a $k_1 \in \{1, \ldots, q-1\}$ such that $\chi_1(\lambda'\mathfrak{p}^{-1}) = \chi_1(\varepsilon_{j_1}\mathfrak{p}^{-1}) = \exp(2\pi i k_1{}^{-1} q)$.
Therefore

$$(13)\qquad \chi_\lambda(\ell\mathfrak{p}^{-s-1}) = \exp(2\pi i k_1 \ell q^{-1}), \qquad k_1 \not\equiv 0, \bmod q.$$

Clearly (12) is equal to (corresponding to the index $\nu \equiv k_1$, mod q)

$$(14)\qquad q^{-N-s} \sum_{\ell=0}^{q^N-1} e^0\, \chi_\lambda(x) = q^{-s}\, \chi_\lambda(x) = |\lambda| \chi_\lambda(x).$$

If $j > s + 1$, it follows by (i) that

$$\chi_\lambda(\mathfrak{p}^{-j}) = \chi_1(\lambda'\mathfrak{p}^{s-j}) \neq 1, \qquad |\lambda'| = 1. \tag{15}$$

Meanwhile we have, by (ii),

$$(2\pi i)^{-1}q^{j-s}\log\chi_\lambda(\mathfrak{p}^{-j}) = (2\pi i)^{-1}q^{j-s}\log\chi_1(\lambda'\mathfrak{p}^{s-j}) \equiv 0 \pmod 1.$$

Therefore there is an m $\in \mathbf{Z}$, depending only on λ, j such that

$$\chi_\lambda(\mathfrak{p}^{-j}) = \exp(2\pi i m q^{s-j}), \qquad s - j < -1.$$

Thus

$$\sum_{\ell=0}^{q^N-1}\exp(-2\pi i\ell\nu q^{-1})\chi_\lambda(\ell\mathfrak{p}^{-j}) = \sum_{\ell=0}^{q^N-1}\exp(-2\pi i\ell q^{-1}(\nu - mq^{s-j+1})) = 0. \tag{16}$$

This is true since mq^{s-j+1} is not an integer, by (15) (consequently exp $(-(2\pi i)/q(\nu - m/(qj - s - 1)))$ does not equal 1 for any $\nu = 1, \ldots, q-1$), and since $N \geqq j - s$ ($j \leqq N$, $s \geqq 0$). From (12), (14), and (16) we obtain (9) for such λ, $|\lambda| = q^s$, $s \geqq 0$.

Assume now $\lambda \in \mathscr{P}^s \sim \mathscr{P}^{s+1}$, $s < 0$. Then, for $j \leqq s + 1$, the argument above on the summation (11) applies and gives the same result. For $j > s + 1$, one should note that all the corresponding terms in (11) for $s + 1 < j \leqq N + s$ vanish. For $N + s + 1 \leqq j \leqq N$, we have

$$\left|\sum_{j=N+s+1}^{N} q^{-N-j+1}\sum_{\nu=1}^{q-1}\sum_{\ell=0}^{q^N-1}\exp(-2\pi i\ell\nu q^{-1})\chi_\lambda(\ell\mathfrak{p}^{-j})\chi_\lambda(x)\right| < q^{-N-s+1},$$

which tends to zero for fixed s as $N \to \infty$. Therefore, the case of $s < 0$(9) is also valid. The proof is complete.

To apply the above concept of derivative to the q-series field K_q, we use a different but equivalent topology for the dual of K_q. That is, if $\lambda \in \hat{K}_q$(regarding it as the same as K_q), $\lambda = (\cdots 0, \lambda_s, \lambda_{s+1}, \cdots)$, $\lambda_s \neq 0$, we set $\|\lambda\| = \sum\{q^{-s_k}\colon \lambda_{s_k} \neq 0\}$ and $\|0\| = 0$. It is easy to verify $\|\cdot\|$ and $|\cdot|$ are equivalent since

$$|\lambda| \leqq \|\lambda\| \leqq q\,|\lambda|, \quad \text{for any } \lambda \in K_q.$$

Furthermore, $\|\cdot\|$ satisfies the following properties:

(i) $\|\lambda\| \geqq 0$ for all $\lambda \in K_q$ and $\|\lambda\| = 0$ if and only if $\lambda = 0$;

(ii) $\|\mathfrak{p}^k\lambda\| = |\mathfrak{p}^k|\,\|\lambda\|$, $k \in \mathbf{Z}$, $\lambda \in K_q$; and

(iii) $\|\lambda + \mu\| \leqq \|\lambda\| + \|\mu\|$, $\lambda, \mu \in K_q$.

LEMMA 4. *Let $K = K_q$ be the q-series field, χ_1 be a character of K which is nontrivial on $\mathscr{P}^{-1}$ and is trivial on $\mathscr{P}^0$. Let $\{\chi_\lambda(x)\colon \lambda \in K\}$ be the dual of K with the "norm" $\|\lambda\|$. Then the derivative of χ_λ exists and we have the following formula.*

$$\lim_{N\to\infty} \Delta_N\chi_\lambda(x) = \|\lambda\|\chi_\lambda(x). \tag{17}$$

PROOF. In this case we have $\chi_\lambda(\mathfrak{p}^{-j}) = \exp(2\pi i \lambda_{j-1} q^{-1})$. Thus, from (5), for $\lambda = (\cdots 0, \lambda_s \lambda_{s+1})$, $\lambda_s = 0$, $t \in \mathbf{Z}$,

$$\Delta_N \chi_\lambda(x) = \sum_{j=-N-t}^{N+t} q^{-N-j+1} \sum_{\nu=1}^{q-1} \sum_{\ell=0}^{q^N-1} \exp(-2\pi i \ell q^{-1}(\nu - \lambda_{j-1}))\chi_\lambda(x)$$
$$\to \sum \{q^{-s_k}: \lambda_{s_k} \neq 0\} \chi_\lambda(x) = \|\lambda\| \chi_\lambda(x), \qquad N \to \infty.$$

For $\lambda = 0$, the result to be proved is obvious.

We close this section with a few remarks.

REMARK 1. In comparing with the definition of derivative of Onneweer, it is basic that both provide the formula (6). The coefficients $f(0)$ and $f(\mathfrak{p}^{-N})$ in question (assume $x = 0$) are

$$\frac{N(2N+1)(p^2-1)^2 + p^{4N+3} - (2N+1)p^3 + 2Np}{p^{3N-1}(p-1)^2(p+1)} \text{ and } -p^{-3N+1}$$

for Onneweer's, respectively, while in (5) they are (in the special case $K = Q_p$)

$$(p^{2N+1} - 1)\, p^{-2N-1} \text{ and } -p^{-2N+1}.$$

REMARK 2. Our definition (5) with a slight modification could be applied to define a derivative of functions on an a-adic group Ω_a([**2**, p. 106]). That is, use

$$\Delta_N f(x) = \sum_{j=-N}^{N} A_{-j} \sum_{\nu=1}^{a_{-j}-1} \sum_{\ell=0}^{a_{-j}\cdots a_N - 1} \exp(-2\pi i \ell \nu a_{-j}^{-1}) f(x + \ell e_{-j}) \tag{18}$$

in place of (5), where $A_{-j} = q^{-j}(a_{-j}a_{-j+1} \cdots a_N)^{-1}$, $e_{-j} = (\cdots, (-j)\ 0, 1, 0, \cdots)$. In this case, the assumption (ii) on χ_1 in Lemma 3 with necessary modification is automatically satisfied.

3. In this section we shall establish some approximation theorems. Let $\omega(t)\ L \in (K)$ such that the following conditions are satisfied:

(i) ω is radial;

(ii) There exists $w \in L(K)$ such that $\hat{w} = \omega$;

(iii) $\lim_{t\to 0}(\omega(t) - 1)/|t|^\alpha = C \neq 0$ for some $\alpha > 0$, whence $\omega(0) = 1$.

Note that from (ii) and (iii) we obtain

(iv) $|\omega(t) - \omega(\mathfrak{p}t)| \leqq M|t|^\alpha$, M is a const.

Consider the approximation operator of convolution type

$$L(f, x, \rho) = \int_K f(t)\, \rho\, w(\lambda(x - t))dt, \tag{19}$$

where $\lambda \in K$, $\rho = |\lambda|$ is a parameter, $\rho \to \infty$, and the kernel w is generated by ω. By (i) and $\omega(0) = 1$, $L(f, \cdot, \rho)$ provides a strong approximation process in $L^r(K)$, $1 \leqq r < \infty$.

The following functions are examples for ω:

$$\exp(-|t|^\alpha),\ (1+|t|^\alpha)^{-1},\ \min(1+|t|^\alpha, |t|^{-\beta}), \qquad \alpha, \beta > 0.$$

LEMMA 5. (see [3] *For any $k \in \mathbf{Z}$, there is a function $V_k \in L(K)$ such that*

$$(V_k)^\wedge(t) = |t|^{-1}\{1 - \Phi_k(t)\}, \qquad t \in K, \tag{20}$$

where Φ_k is the characteristic function of $\mathscr{P}^k$. Moreover,

$$\|V_k\|_1 = 0\,(q^k), \qquad k \to -\infty. \tag{21}$$

The first part of the lemma follows easily from [4; p. 138 Lemma (5.2)]. For (21), see the remark after Lemma 8.

LEMMA 6. *If $w^{\langle 1\rangle}$ exists in $L(K)$, then, for $f \in L^r(K)$, $1 \leqq r \leqq \infty$, the operator* (19) *with $\lambda = \mathfrak{p}^{-k}$ has a derivative in $L^r(K)$. Moreover,*

$$\|L^{\langle 1\rangle}(f, \cdot, \rho)\|_r \leqq \rho\|f\|_r\|w^{\langle 1\rangle}\|_1, \qquad \rho = q^k, k \in \mathbf{Z}. \tag{22}$$

PROOF. Suppose $\lambda = \mathfrak{p}^{-k}$. By Lemma 1, $\Delta_N w(\lambda(x-t)) \to \rho w^{\langle 1\rangle}(\lambda(x-t))$ in $L(K)$ as $N \to \infty$, where t is regarded as a parameter. Thus we have

$$\begin{aligned}\|\Delta_N L(f, \cdot, \rho) &- \int_K f(t)\,\rho^2 w^{\langle 1\rangle}(\lambda(x-t))\,dt\|_r \\ &\leqq \|f\|_r\,\|\rho\Delta_N w(\lambda(\cdot\ -t)) - \rho^2 w^{\langle 1\rangle}(\lambda(\cdot\ -t))\|_1 \\ &= \rho\,\|f\|_r\|\Delta_N\, w(\lambda(\cdot)) - \rho w^{\langle 1\rangle}(\lambda(\cdot))\|_1 \to 0\end{aligned}$$

as $N \to \infty$. It follows that

$$L^{\langle 1\rangle}(f, \cdot, \rho) = \rho\int_K f(t)\rho w^{\langle 1\rangle}(\lambda(x-t))dt, \qquad \lambda = \mathfrak{p}^{-k},$$

and, consequently, the estimate (22) holds.

LEMMA 7. *Let ω satisfy the conditions* (i), (ii), *and* (iv), *with $\alpha = \beta$ $\omega \in \hat{L}(K)$ if $\beta > 0$ and $\omega \in \hat{L^2}(K)$ if $\beta = 0$, where $\hat{A}$ denotes the class of Fourier transforms of the class A.*

PROOF. That ω is radial implies w is radial and vice versa. Moreover, any one of them is the Fourier transform of the other. Let $\omega_\ell = \omega(x)$, where $|x| = q^\ell$, $\ell \in \mathbf{Z}$. It is not hard to prove that

$$w_{\ell+1} - w_\ell = q^{-\ell}(\omega_{\ell+1} - \omega_{-\ell}), \qquad \ell \in \mathbf{Z}. \tag{23}$$

(For K^n, one should replace the factor $q^{-\ell}$ in the right side by $q^{-\ell n}$.) Obviously w is continuous, hence is locally integrable. Thus we need only to consider the asymptotic property of w_ℓ as $\ell \to \infty$. By (iv) we may write

$$|\omega_{-\ell+1} - \omega_{-\ell}| \leqq M\,q^{(-\ell+1)\beta}, \qquad \ell > \ell\,0. \tag{24}$$

It follows from (23) that

$$|w_{\ell+1} - w_\ell| \leqq Mq^\beta q^{-\ell(1+\beta)}. \tag{25}$$

By induction we have

$$|w_{\ell+s} - w_\ell| \leqq Mq^\beta \sum_{k=0}^{s-1} q^{-(\ell+k)(1+\beta)} \leqq Mq^\beta(1-q^{-1-\beta})^{-1} q^{-\ell(1+\beta)}.$$

We see $\lim_{s\to\infty} w_{\ell+s} = 0$ by the Riemann-Lebesgue Lemma. Accordingly,

$$w_\ell = 0(q^{-\ell(1+\beta)}). \tag{26}$$

If $\beta > 0$, then w is integrable in a neighbourhood of ∞. And if $\beta = 0$, w is square integrable in that domain. This completes the proof.

LEMMA 8. *Let $\mathcal{O}(t)$ be defined by*

$$\mathcal{O}(t) = \min(|t|^\alpha, |t|^{-\beta}), \qquad t \in K, \tag{27}$$

where $\alpha \geqq 0$, $\beta > 0$. Then, for $\alpha > 0$ and $\beta > 0$, we have $\mathcal{O} \in \hat{L}(K)$, *while, for $\alpha = 0$ and $\beta > 0$*, we have $\mathcal{O} \in L^{\hat{2}}(K)$.

PROOF. The result follows from [4, Lemma (5.2)]. If we set $\mathcal{O}(t) = \mathcal{O}_1(t) + \mathcal{O}_2(t)$, where $\mathcal{O}_1(t) = |t|^{\alpha\Phi_0}(t)$ and $\mathcal{O}_2 = \mathcal{O} - \mathcal{O}_1$, we can verify directly that the functions

$$\varphi(x) = \left\{\frac{1 - q^{-1}}{q^{\beta-1} - 1} - \frac{1 - q^{-\beta}}{q^{\beta-1} - 1}|x|^{\beta-1}\right\}\Phi_0(x) \tag{28}$$

and

$$\psi(x) = \left\{\frac{q^{-1} - 1}{\log q}\log|x| - q^{-1}\right\}\Phi_0(x), \tag{29}$$

being in $L(K)$, satisfy $\hat{\varphi} = \mathcal{O}_2$ since $\beta \neq 1$ and $\hat{\psi} = \mathcal{O}_2$ since $\beta = 1$.

REMARK. From (29) we can deduce Lemma 5. In fact, let $\beta = 1$. If we set $\lambda(t) = |t|^{-1}\{1 - \Phi_k(x)\}$, then $q^{-k}\lambda(\mathfrak{p}^k u) = \mathcal{O}_2(u)$, since $\hat{\psi}(u) = \mathcal{O}_2(u)$. Thus $\hat{\psi}(\mathfrak{p}^k\cdot)(t) = q^k\mathcal{O}_2(\mathfrak{p}^{-k}t) = \lambda(t)$. Moreover,

$$\|\psi(\mathfrak{p}^k\cdot)\|_1 = q^k\|\psi(\cdot)\|_1 = 0(q^k), \qquad k \to \infty.$$

For $L^r(K)$, $1 \leqq r \leqq 2$, the Fourier transform is defined in the usual way. For $2 < r < \infty$, we would like to define the Fourier transform on $L^r(K)$ by the dual method. Thus, let $f \in L^r(K)$, $2 < r < \infty$, and let r' be the conjugate index to r. If there is a continuous linear functional g on $L^r(K)$ such that, for any $\varphi \in L^{r'}(K)$,

$$(g, \hat{\varphi}) = (f, \varphi), \tag{30}$$

we say that g is the Fourier transform of f and denote it by $\hat{f}$. Since the

Fourier transform is a homeomorphism of $\mathscr{S}$ onto $\mathscr{S}$([4], p. 37, 122]) and $\mathscr{S}$ is dense in any $L^r(K)$, $1 \leqq r < \infty$ or in C_0, it is convenient to use $\mathscr{S}$ instead of $L^{r'}(K)$ in (30) to define the Fourier transform of f:

$$(31) \qquad (g, \hat{\varphi}) = (f, \varphi), \qquad \varphi \in \mathscr{S}.$$

The convolution of $f \in L^r(K)$ $(1 \leqq r \leqq \infty)$ and $w \in L^1(K)$ satisfies the relation

$$(32) \qquad (f * w, \varphi) = (f, \tilde{w} * \varphi), \qquad \varphi \in \mathscr{S},$$

where $\tilde{w}$ is the reflection of w, and obviously $\tilde{w} * \varphi \in L^r(K)$, for any $r \in [1, \infty]$. It follows easily that

$$(33) \qquad ((f * w)^\wedge, \varphi) = (\hat{f}\hat{w}, \varphi), \qquad \varphi \in \mathscr{P}.$$

By virtue of $(|\lambda|\delta_\lambda w)^\wedge = \delta_{\lambda^{-1}}(\hat{w})$, applying (33) to f and $|\lambda|\delta_\lambda w$, where $f \in L^r(K)$ $(2 < r < \infty)$, $w \in L^1(K)$, we obtain

$$(34) \qquad ((f * |\lambda|\, \delta_\lambda w)^\wedge, \varphi) = (\hat{f}\delta_{\lambda^{-1}}(\hat{w}), \varphi), \varphi \in \mathscr{S}.$$

Let $h \in K$, $k \in Z$. One can verify that the inverse Fourier transform of $|v|^\alpha \chi_h(v)\, \Phi_{-k}(v)$ is

$$(35) \qquad \psi_0(v) = \begin{cases} q^{k(1+\alpha)}(1-q^{-1})\,(1-q^{-1-\alpha})^{-1}, & \text{if } |v-h| \leqq q^{-k}, \\ q^{-\ell(1+\alpha)}(1-q^{\alpha})\,(1-q^{-1-\alpha})^{-1}, & \text{if } q^{\ell} = |v-h| > q^{-k}. \end{cases}$$

From this we assert that, for any $\varphi \in \mathscr{S}$(φ is some finite linear combination of $\chi_h \Phi_{-k}$), there is a $\psi \in L^r(K)$ $(1 \leqq r \leqq \infty)$ such that

$$(36) \qquad \psi^\vee(v) = |v|^\alpha\, \varphi(v), \qquad \alpha > 0.$$

The class $W[L^r, |x|^\alpha]$, for $1 \leqq r \leqq 2$, is defined as usual; i.e., $f \in W[L^r, |x|^\alpha]$ means that there is a function $g \in L^r(K)$ such that $|v|^\alpha \hat{f}(v) = \hat{g}(v)$ a.e. But, for $2 < r < \infty$, we mean that there is a $g \in L^r(K)$; its Fourier transform (in the distribution sense) $\hat{g}$ satisfies the relation

$$(37) \qquad (|v|^\alpha \hat{f}(v), \varphi(v)) = (g(v), \varphi(v)), \qquad \varphi \in \mathscr{S}.$$

Now we may state and prove the following

THEOREM 1. *Let $L(f, \cdot, \rho)$ be the operator* (19) *with a generating function ω satisfying* (i), (ii) *and*

$$(38) \qquad \text{(v)}\ \frac{\omega(v) - 1}{|v|^\alpha} = \hat{\mu}(v),$$

where $\alpha > 0$, $\mu \in L(K)$, $\|\mu\|_1 = 1$, and $f \in L^r(K)$, $1 \leqq r < \infty$. If there is a $g \in L^r(K)$ such that, for $\lambda = \mathfrak{p}^{-k}(\rho = q^k)$ and $k \in \mathbf{N}$,

$$(39) \qquad \|\rho^\alpha[L(f, \cdot, \rho) - f(\cdot)] - g(\cdot)\|_r = 0\,(1), \qquad \rho \to \infty,$$

then $f \in W[L^r, |x|^\alpha]$.

PROOF. Suppose $1 \leqq r \leqq 2$. Then we have

$$\{\rho^\alpha[L(f, \cdot, \rho) - f(\cdot)] - g(\cdot)\}^\wedge(v) = \frac{\omega(\lambda^{-1}v)-1}{|\lambda^{-1}v|^\alpha} |v|^\alpha \hat{f}(v) - \hat{g}(v).$$

By the Hausdorff—Young inequality,

$$\left\| \frac{\omega(\lambda^{-1}v)-1}{|\lambda^{-1}v|^\alpha} |v|^\alpha \hat{f}(v) - \hat{g}(v) \right\|_{r'} \leqq \|\rho^\alpha[L(f, \cdot, \rho) - f(\cdot)] - g(\cdot)\|_r,$$

where r' is the conjugate index to r. From (38) and $\|\mu\|_1 = 1$ we see that $\lim_{v\to 0} |v|^{-\alpha}(\omega(v) - 1) = 1$. Hence, by Fatou's Lemma, we have a.e. $v \in K$, $|v|^\alpha \hat{f}(v) = \hat{g}(v)$. That is, $f \in W[L^r, |v|^\alpha]$, $1 \leqq r \leqq 2$.

Suppose now, that $2 < r < \infty$. From (39) it is easy to see that, for any $\varphi \in \mathscr{S}$,

$$(40) \qquad (\rho^\alpha[f * |\lambda|\, \delta_\lambda\, w]^\wedge, \varphi) - (\hat{f}, \varphi) \to (\hat{g}, \varphi), \qquad \rho \to \infty.$$

By (34), (38) the left hand side of (40) equals $(|v|^\alpha \hat{f}(v)\, \hat{\mu}(\lambda^{-1} v), \varphi(v))$. By virtue of (36) it then equals (ω is radial)

$$\begin{aligned}
&(\hat{f}(v)\hat{\mu}(\lambda^{-1}v), |v|^\alpha \varphi(v)) = (\hat{f}(v)\hat{\mu}(\lambda^{-1}v), \phi^\vee(v)) \\
&= (\hat{f}(v), \check{\mu}(\lambda^{-1} v)\, \phi^\vee(v)) = (\hat{f}(v)), [|\lambda|\mu(\lambda\cdot) * \phi(\cdot)]^\vee(v)) \\
&= (f(v), |\lambda|(\mu(\lambda\cdot) * \phi(\cdot))\,(v)).
\end{aligned}$$

Since $|\lambda|\mu(\lambda x)$ is an approximate identity kernel, $|\lambda|\mu(\lambda\cdot) * \phi(\cdot)$ tends to $* \phi(\cdot)$ in $L^{r'}(K)$ as $\lambda \to \infty$. We have $(f(v), |\lambda|\,(\mu(\lambda\cdot) * \phi(\cdot))\,(v)) \to (f(v), \phi(v))$ as $\lambda \to \infty$; by (36) it equals $(\hat{f}(v), |v|^\alpha\varphi(v)) = (|v|^\alpha\hat{f}(v), \varphi(v))$. Comparing this result with (40), we get (37).

THEOREM 2. *Let* $f \in L^r(K)$, $1 \leqq r \leqq 2$. *If* ω *satisfies the conditions* (i), (ii) *and*

$$\text{(vi)} \qquad \frac{\omega(v) - 1}{|v|^\alpha} = \check{\mu}(v),$$

where $\alpha > 0$, μ *is a finite Borel measure with total variation* $\|\mu\|_{BV}$, *and* $\hat{\mu}(x) = \int_K \chi_\lambda(t)d\mu(t)$, $\hat{\mu}(o) = 1$. *Then the operator* $L(f, \cdot, \rho)$ *in* (19) *is saturated in* $L^r(K)$ *with the order* $0(\rho^{-\alpha})$.

PROOF. By Theorem 1, the usual argument in approximation theory [1] will offer a proof of the theorem if we provide a nonzero function $\lambda_0 \in L^r(K)$ such that $L(\lambda_0, x, \rho) - \lambda_0(x)$ has the exact degree $0(\rho^{-\alpha})$.

To that end, we take the function $\lambda_0(x) = \Phi_0(x)$. Plainly, $\hat{\lambda_0} = \lambda_0$, and, by Lemma 8, there is a function $H \in L^r(K)$ such that $\hat{H}(t) = |t|^\alpha \lambda_0(t)$. It is easy to find that

$$\rho^\alpha[L(\lambda_0, x, \rho) - \lambda_0(x)] = \int_K \frac{\omega(\lambda^{-1}v) - 1}{(|v|/\rho)^\alpha} |v|^\alpha \lambda_0(v)\chi(xv)\,dv$$

$$= \int_K \hat{\mu}(\lambda^{-1} v)\,\hat{H}(v)\,\chi(xv)\,dv = \int_K H(x - t)d\mu(\lambda t).$$

Therefore, we have

$$\rho^\alpha[L(\lambda_0, x, \rho) - \lambda_0(x)] - H(x) = \int_K [H(x - t) - H(x)]\,d\mu(\lambda t).$$

Since $d\mu(\lambda t)$ is an approximate identitity kernel,

$$\|L(\lambda_0, \cdot, \rho) - \lambda_0(\cdot) - \rho^{-\alpha} H(\cdot)\|_r = o(\rho^{-\alpha}), \qquad \rho \to \infty,$$

which completes the proof of the theorem.

There is another type, the Bernstein type, of inverse approximation theorem.

THEOREM 3. *If the operator $L(f, x, \rho)$ with such a kernel w, $w^{\langle 1\rangle} \in L(K)$, provides a degree of approximation to $f \in L^r(K)$, $1 \leqq r \leqq \infty$,*

(42) $\|L(f, \cdot, \rho) - f(\cdot)\|_r = 0(\rho^{-s-\alpha})$, for some $s = \in \mathbf{P}$, $\alpha > 0$, $\rho \to \infty$,

where, for $r = \infty$, one should replace $L^\infty(K)$ by $C(K)$. Then f has an s^{th} derivative. Moreover, $f^{\langle s\rangle} \in Lip_r\alpha$, where Lip$_r\alpha$ *denotes the class Kipα with the $L^r(K)$ norm.*

PROOF. We only prove the case $s = 0$. For the general case, it is done by induction. As usual, let us select the subsequence $q^k(\lambda = \mathfrak{p}^{-k})$, $k = 1, 2, \ldots$, By (42),

(43) $$\|L(f, \cdot, q^k) - f(\cdot)\|_r \leqq A\, q^{-k\alpha},$$

where A is a constant, not depending on k (the same for A_1, A_2, below).
Let

(44) $$\begin{aligned} U_k(x) &= L(f, x, q^k) - L(f, x, q^{k-1}) \\ &= L(F_{k-1}, x, q^k) - L(F_k, x, q^{k-1}), \end{aligned}$$

where $F_k = f - L(f, \cdot, q^k)$. From (44) we obtain the estimate

(45) $$\|U_k\|_r \leqq \|F_{k-1}\|_r + \|F_k\|_r \leqq A_1 q^{-k\alpha}.$$

Thus the series

(46) $$f(x) = U_2(x) + U_3(x) + U_4(x) + \cdots$$

converges to $f(x)$ in $L^r(K)$.

Suppose $h \in K$, $h \neq 0$. There is an integer $m > 0$ such that $q^{-m} < |h| \leqq q^{-m+1}$. Consequently, by (45),

$$\sum_{k=m+1}^{\infty} \|U_k\|_p \leqq \sum_{k=m+1}^{\infty} A_1 q^{-k\alpha} \leqq A_2 |h|^\alpha,$$

whence

$$\|f(x+h) - f(x)\|_r \leqq \sum_{k=2}^{m} \|U_k(\cdot + h) - U_k(\cdot)\|_r + A_3|h|^\alpha. \tag{47}$$

By the Fourier transform method, it is easy to verify the equality

$$U_k(x+h) - U_k(x) = \int_K V_{-m}(u)\{U_k^{\langle 1\rangle}(x+h-u) - U_k^{\langle 1\rangle}(x-u)\}\,du, \tag{48}$$

where V_m is the function in Lemma 5. Hence, by that lemma, (44), and Lemma 6,

$$\begin{aligned}\|U_k(\cdot + h) - U_k(\cdot)\|_r &\leqq \|V_{-m}\|_1 \|U_k^{\langle 1\rangle}(\cdot + h) - U_k(\cdot)\|_r \\ &\leqq A_4 q^{-m} \|L^{\langle 1\rangle}(F_{k-1}, \cdot + h, q^k) - L^{\langle 1\langle}(F_k, \cdot + h, q^{k-1}) \\ &\quad - L^{\langle 1\rangle}(F_{k-1}, \cdot, q^k) + L^{\langle 1\rangle}(F_k, \cdot, q^{k-1})\|_r \\ &\leqq A_4 q^{-m} q^k \|F_{k-1}\|_r \|w^{\langle 1\rangle}\|_1 \leqq A_5 \|w^{\langle 1\rangle}\|_1 q^{-m+k(1-\alpha)}.\end{aligned}$$

Back to (47), we obtain

$$\|f(\cdot + h) - f(\cdot)\|_r \leqq \sum_{k=2}^{m} A_5 \|w^{\langle 1\rangle}\|_1 q^{-m+k(1-\alpha)} + A_3|h|^\alpha \leqq A_6|h|^\alpha.$$

The theorem is proved.

THEOREM 4. *Let $f \in L^r(K)$, $1 \leqq r < \infty$ and ω satisfies* (i), (ii), *and* (vi). *Then if $f \in W[L^r, c|v|^\alpha]$, we have $\|L(f, \cdot, \rho) - f(\cdot)\|_r = 0(\rho^{-\alpha})$, $\rho \to \infty$. Conversely, for $1 < r < \infty$, $\|L(f, \cdot, \rho) - f(\cdot)\|_r = 0(\rho^{-\alpha})$ implies $f \in W[L^r, c|v|^\alpha]$, where the class W is the same as in the statement preceding* (37) *and c is a nonzero constant.*

PROOF. Suppose $f \in W[L^r, c|v|^\alpha]$. Then there exists $g \in L^r(K)$, $1 \leqq r < \infty$, such that $c|v|^\alpha \hat{f}(v) = \hat{g}(v)$ a.e. for $1 \leqq r \leqq 2$ and that

$$(c|v|^\alpha \hat{f}(v), \varphi(v)) = (\hat{g}(v), \varphi(v)) \qquad (\varphi \in \mathcal{S})$$

for $2 < r < \infty$, respectively. Meanwhile

$$\begin{aligned}(\rho^\alpha[L(f, v, \rho) - f(v)], \hat{\varphi}(v)) &= (\rho^\alpha[w(\lambda^{-1}v) - 1]\hat{f}(v), \varphi(v)) \\ &= (\hat{\mu}(\lambda^{-1}v)\hat{g}(v), \varphi(v)) = (q * d\mu(\lambda\cdot)(v), \hat{\varphi}(v)).\end{aligned}$$

Since $\mathcal{S}$ is dense in $L^{r'}(K)$, we have a.e. $\rho^\alpha[L(f, x, \rho) - f(x)] = (g * d\mu(\lambda\cdot))(x)$ so that

$$\|\rho^{\alpha}[L(f,\ \cdot,\ \rho) - f(\cdot)]\|_r \leqq \|g\|_r\ \|\mu\|_{BV}.$$

Now suppose $\|L(f,\ \cdot, \rho) - f(\cdot)\|_r = 0(\rho^{-\alpha})$, $1 < r < \infty$. At first we consider the case $1 < r \leqq 2$. By the Hausdorff-Young inequality, there is a ρ_0 such that $\rho > \rho_0$,

$$\left\| \frac{\omega(\lambda^{-1}\cdot)-1}{\rho^{-\alpha}} \hat{f}(\cdot)\right\|_{r'} \leqq \|\rho^{\alpha}[L(f,\ \cdot,\ \rho) - f(\cdot)]\|_r \leqq M,$$

where M is a constant. Therefore, according to (vi).

$$\|\hat{\mu}(\lambda^{-1}\ \cdot)\ |\cdot|^{\alpha} \hat{f}(\cdot)\ \|_{r'} \leqq M, \qquad |\lambda| > \rho_0.$$

Since $\hat{\mu}(x)$ is continuous, $\hat{\mu}(0) = 1$, it follows from Fatou's lemma that $|v|^{\alpha}\hat{f}(v) \in L^{r'}(K)$. Let us examine the linear functional on $L^{r'}$,

$$\ell(\hat{\varphi}) = (c|v|^{\alpha}\ \hat{f}\ (v),\ \varphi(v)), \qquad \hat{\varphi} \in \mathscr{S},$$

where ℓ is induced by f and the acting space is $\mathscr{S}$ that is dense in $L^{r'}$. It's easy to see that $|\ell(\hat{\varphi})| \leqq M\ \|\hat{\varphi}\|_{r'}$. Thus, by the Riesz representation theorem, there is a function $g \in L^r(K)$ such that, for every $\hat{\varphi} \in \mathscr{S}$,

$$\ell(\hat{\varphi}) = \int_K g(u)\hat{\varphi}(u)\ du = \int_K \hat{g}g(u)\varphi(u)du.$$

By the uniqueness theorem, we obtain $c|v|^{\alpha}\hat{f}(v) = \hat{g}(v)$ a.e.; that is $f \in W[L^r,\ c|v|^{\alpha}]$, for $1 < r \leqq 2$.

Assume now $2 < r < \infty$. As is mentioned, in this case the definitions of Fourier transform and convolution should be understood in the distribution sense. So, for every $\varphi \in \mathscr{S}$,

$$(\rho^{\alpha}[L(f,\ \cdot,\ \rho) - f(\cdot)]^{\wedge},\ \varphi) = (\rho^{\alpha}[L(f,\ \cdot,\ \rho) - f(\cdot)],\ \hat{\varphi}).$$

Applying Holder's inequality we obtain the estimate

$$|(\hat{\mu}(\lambda^{-1}\ \cdot)\ |\cdot|^{\alpha}\ \hat{f}\ (\cdot),\ \varphi)| \leqq M\ \|\hat{\varphi}\|_{r'}, \qquad \varphi \in \mathscr{S}.$$

Just as in the proof of Theorem 1, we have

$$|(f(v),\ |\lambda|\ (\mu(\lambda\cdot) * \phi(\cdot))\ (v))| \leqq M\ \ \|\hat{\varphi}\|_{r'}, \qquad \hat{\varphi} \in \mathscr{S},$$

as well as

$$|(|v|^{\alpha}\ \hat{f}\ (v),\ \varphi\ (v))| \leqq M\ \|\hat{\varphi}\|_{r'}, \qquad \hat{\varphi} \in \mathscr{S}.$$

The idea of the first paragraph of the proof yields a function $g \in L^r(K)$ such that $(|v|^{\alpha}\hat{f}(v),\ \varphi\ (v)) = (g,\ \hat{\varphi})$, $\hat{\varphi} \in \mathscr{S}$. Obviously this means that g is the Fourier transform of $|v|^{\alpha}\ \hat{f}(v)$ in the distribution sense and, moreover, $\hat{g}\ (v) = |v|^{\alpha}\ \hat{f}\ (v)$. That is, $f \in W\ [L^r,\ |v|^{\alpha}]$, $2 < r < \infty$.

The proof is complete.

As an application we consider the kernel for the Bessel potential. Here we have the functions

$$G^\alpha(x) = \frac{1}{\Gamma_n(\alpha)} (|x|^{\alpha-n} - q^{\alpha-n})\, \Phi_0(x), \quad \text{for } \alpha \neq n,$$

and

$$G^n(x) = (1 - q^{-n}) \log_q (q/|x|)\, \Phi_0(x),$$

where Re $\alpha > 0$, $\Gamma_n(\alpha) = (1 - q^{\alpha-n})\,(1 - q^{-\alpha})^{-1}$ (see [**4**, p. 136–142]). We know

(49) $$G^{\hat{\alpha}}(t) = \min (1, |t|^{-\alpha})$$

and $G^\alpha \in \mathrm{Lip}_r\, \beta$, for Re $\alpha - n/r' = \beta > 0$:

(50) $$\|G^\alpha (\cdot + h) - G^\alpha(\cdot)\|_r \leqq A_{\alpha r}\, |h|^\beta.$$

The case $r = \infty$ is easy to verify. Let us show (50) for the case $1 \leqq r < \infty$ by the approximation theorems above (in n-dimensional fashion).

From Lemma 8, the function $\min(1, |t|^{-\alpha})$ is the Fourier transform of some function (in fact, G^α) in $L^1(K^n)$. For $\beta = \mathrm{Re}\alpha - n/r' > 0$, select ω such that (i), (ii), (v) are satisfied and $\hat{\omega}^{\langle 1 \rangle} \in L\,(K^n)$ for such a β (instead of α there). Now, since $\beta - \alpha < 0$ by Lemma 8, $|t|^\beta \hat{G}^\alpha(t) = \min (|t|^\beta, |t|^{\beta-\alpha})$ is the Fourier transform of some function $g \in L^1(L^n)$, and from (28), (29) we also see that $g \in L^r(K^n)$, $1 \leqq r \leqq \infty$. Therefore $\hat{g}\,(t) = |t|^\beta \hat{G}^\alpha(t)$ in a certain sense. By Theorem 4 we have, for $1 \leqq r < \infty$,

$$\|L(G^\alpha, \cdot, \rho) - f(\cdot)\|_r = 0\,(\rho^{-\beta}), \qquad \rho \to \infty.$$

From this we conclude that $G^\alpha \in \mathrm{Lip}_r\beta$, $1 \leqq r < \infty$, by Theorem 3.

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