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# BOUNDARY BEHAVIOR AND MONOTONICITY ESTIMATES FOR SOLUTIONS TO NONLINEAR DIFFUSION EQUATIONS

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ABSTRACT. The purpose of this article is to develop a number of estimates bearing on the boundary behavior of solutions to certain nonlinear diffusion equations. These estimates are then applied to show that the boundary behavior of solutions is related in a monotone way to the diffusion coefficient in the equation. This monotonicity may be applied in various ways to the analysis of inverse problems for nonlinear parabolic equations.

**1. Compatibility of Overspecified Data** Suppose $a(s)$ satisfies

$$\begin{aligned} &\text{i) } a(s) \in C^1[0, \infty) \\ (1.1) \qquad &\text{ii) } 0 < A_0 \leqq a(s) \leqq A_1 < \infty \text{ for } s \geqq 0 \\ &\text{iii) } 0 \leqq a'(s) \leqq A_2 \quad \text{for} \quad s \geqq 0 \end{aligned}$$

for a given set of constants, $A_0$, $A_1$, $A_2$. Then we may consider a nonlinear diffusion equation in which $a(s)$ plays the role of a coefficient.

$$(1.2) \qquad \partial_t u(x, t) = \partial_x(a(u)\, \partial_x u), \qquad 0 < x < 1, 0 < t < T.$$

Among the auxiliary conditions that $u(x, t)$ might be expected to satisfy as part of a well posed initial boundary value problem (IBVP) are the following

$$\begin{aligned} &u(x, 0) = u_0, 0 < x < 1, \\ (1.3) \qquad &-a(u)\, \partial_x u(0, t) = g_0(t), \quad u(0, t) = h_0(t), 0 < t < T, \\ &a(u)\, \partial_x u(1, t) = g_1(t), \quad u(1, t) = h_1(t), 0 < t < T. \end{aligned}$$

For $u_0$ a given non-negative constant, define

$$(1.4) \qquad \alpha(s) = \int_{u_0}^{s} a(\tau)\, d\tau, \qquad s \geqq u_0$$

for $a(t)$ satisfying (1.1). Then

$$(1.5) \qquad \alpha'(s) = a(s) \geqq A_0 \quad \text{for } s \geqq u_0$$

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and it follows that the function $\alpha(s)$ is invertible. Then we may define a transformation,

$$(1.6) \qquad v(x,t) = \alpha(u(x,t)),\ u(x,t) = \alpha^{-1}(v(x,t)),\ 0 \leqq x \leqq 1,\ 0 \leqq t \leqq T,$$

and it is not hard to show that if $u(x, t)$ satisfies (1.2) then $v(x, t)$ given by (1.6) must satisfy

$$(1.7) \qquad \partial_t v(x,t) = A(v)\,\partial_{xx} v(x,t),\ 0 < x < 1, \quad 0 < t < T,$$

where

$$(1.8) \qquad A(v) = a(\alpha^{-1}(v)), \quad v \geqq 0.$$

If $a(s)$ satisfies (1.1) then $A(s)$ also satisfies (1.1), possibly with a different $A_2$. Furthermore, for $u(x, t)$ satisfying (1.3), $v(x, t)$ must satisfy

$$(1.9) \qquad \begin{aligned} &v(x,0) = 0, && 0 < x < 1, \\ -\partial_x v(0,t) = g_0(t), \quad & v(0,t) = \alpha(h_0(t)) = f_0(t), && 0 < t < T, \\ \partial_x v(1,t) = g_1(t), \quad & v(1,t) = \alpha(h_1(t)) = f_1(t), && 0 < t < T. \end{aligned}$$

An IBVP comprised of (1.7) together with conditions selected from (1.9) is more easily analysed than the corresponding problem for $u(x, t)$. We will therefore concentrate on the equation (1.7) and the conditions (1.9).

Not all the conditions in (1.9) could be simultaneously imposed for arbitrary data functions $f_0$, $f_1$, $g_0$, $g_1$ if the IBVP is expected to have a solution. However, under certain conditions of compatibility on the data functions, the conditions are not inconsistent. It is our aim in this section to discover conditions of compatibility on the data in (1.9).

LEMMA 1.1. *Suppose $A(s)$ satisfies* (1.1) *and that for some $T > 0$, $g_0(t)$ satisfies*

$$(1.10) \qquad g_0 \in C^1[0,T], \quad g_0(0) = 0, \quad g_0'(t) \geqq 0 \quad \text{for } 0 < t < T.$$

*Then if $v(x, t)$ satisfies*

$$(1.11) \qquad \begin{aligned} \partial_t v(x,t) &= A(v)\,\partial_{xx} v(x,t), \quad 0 < x < 1, \quad 0 < t < T, \\ v(x,0) &= 0, \quad 0 < x < 1, \\ -\partial_x v(0,t) &= g_0(t), \quad v(1,t) = 0, \quad 0 < t < T, \end{aligned}$$

*it follows that $f_0(t) = v(0, t)$ must satisfy,*

$$(1.12) \qquad f_0 \in C^1[0,T], \quad f_0(0) = 0, \quad f_0'(t) \geqq 0 \quad \text{for} \quad 0 \leqq t \leqq T.$$

COROLLARY. *Under the assumptions of this lemma, the solution $v(x, t)$ of* (1.11) *must satisfy,*

$$(1.13) \quad \partial_t v(x,t) \geqq 0, \quad \partial_{xx} v(x,t) \geqq 0, \text{ for } 0 \leqq x \leqq 1,\ 0 \leqq t \leqq T.$$

PROOF. The assumptions on $A(v)$, $g_0(t)$ are sufficient to imply the existence of a solution $v = v(x, t)$ for (1.11) which is twice continuously differentiable in $x$ and once continuously differentiable in $t$ for $0 \leqq x \leqq 1$, $0 \leqq t \leqq T$; i.e., $v(x, t)$ belongs to $C^{2,1}\{[0, 1], [0, T]\}$. It follows that $f_0(t) = v(0, t)$ belongs to $C^1[0, T]$ and $f_0(0) = 0$.

The hypotheses, together with the maximum principle (MP) imply $v(x, t) \geqq 0$ for $0 \leqq x \leqq 1$, $0 \leqq t \leqq T$. Now let $w(x, t) = \partial_t v(x, t)$ and note that $w(x, t)$ then satisfies,

$$(1.14)\qquad \begin{aligned} &\partial_t w(x, t) = A(v)\partial_{xx} w(x, t) + A'(v)/A(v) w^2(x, t),\\ &\quad 0 < x < 1, \quad 0 < t < T, \quad w(x, 0) \geqq 0, 0 < x < 1,\\ &-\partial_x w(0, t) = g_0'(t), \quad w(1, t) = 0, \quad 0 < t < T. \end{aligned}$$

Under the current assumptions, the coefficient $A'(v)/A(v)$ is continuous (and hence bounded) on $[0, 1] \times [0, T]$. It follows then from the MP applied to (1.14) that $w(x, t) \geqq 0$ for $0 \leqq x \leqq 1$, $0 \leqq t \leqq T$. That is, $\partial_t v(x, t) \geqq 0$ and in particular, $\partial_t v(0, t) = f_0'(t) \geqq 0$ for $0 \leqq t \leqq T$. Finally, $\partial_{xx} v(x, t) = (1/A_0)\partial_t v(x, t) \geqq 0$.

LEMMA 1.2. *Suppose the hypotheses of lemma* 1.1 *are satisfied with the single exception that the condition* $v(1, t) = 0$ *in* (1.11) *is replaced by the condition* $\partial_x v(1, t) = 0$. *Then the conclusions* (1.12), (1.13) *hold.*

PROOF. Extend $v(x, t)$ to the strip $1 \leqq x \leqq 2$, $0 \leqq t \leqq T$ as follows:

$$\begin{aligned} v^*(x, t) &= v(x, t), 0 \leqq x < 1, 0 \leqq t \leqq T,\\ &= v(2 - x, t), \quad 1 \leqq x \leqq 2, 0 \leqq t \leqq T. \end{aligned}$$

Then the extension $v^*(x, t)$ satisfies

$$\begin{aligned} \partial_t v^*(x, t) &= A(v^*)\partial_{xx} v^*(x, t), \quad 0 < x < 2, \quad 0 < t < T,\\ v^*(x, 0) &= 0, \quad 0 < x < 2,\\ -\partial_x v^*(0, t), &= g_0(t), \quad 0 < t < T,\\ \partial_x v^*(2, t) &= -\partial_x v(0, t) = g_0(t), \quad 0 < t < T. \end{aligned}$$

Arguing in much the same way we did in the previous lemma, we can infer that (1.12) and (1.13) must hold.

We have shown that for homogeneous boundary conditions at the end $x = 1$, if $g_0(t) = -\partial_x v(0, t)$ satisfies (1.10) then $f_0(t) = v(0, t)$ will necessarily conform to (1.12). We can show that the converse also holds.

LEMMA 1.3. *Suppose* $A(s)$ *satisfies* (1.1) *and that for some* $T > 0$, $f_0(t)$ *satisfies* (1.12). *If* $v(x, t)$ *satisfies*

$$\begin{aligned}
&\partial_t v(x, t) = A(v)\partial_{xx}v(x, t), \quad 0 < x < 1, 0 < t < T,\\
(1.15) \qquad &v(x, 0) = 0, \quad 0 < x < 1,\\
&v(0, t) = f_0(t), \quad v(1, t) = 0, \quad 0 < t < T,
\end{aligned}$$

*then* $g_0(t) = -\partial_x v(0, t)$ *must satisfy* (1.10) *and* (1.13) *holds.*

PROOF. The assumptions on the data $f_0(t)$ and the coefficient $A(s)$ are sufficient to imply the existence of a solution $v(x, t)$ for (1.15) satisfying $v \in C^{2,1}[0, 1] \times [0, T]$ Then $g_0(t) = -\partial_x v(0, t)$ belongs to $C^1[0, T]$ and $g_0(0) = 0$. The MP implies that $v(x, t) \geqq 0$ in $[0, 1] \times [0, T]$. If we let $w(x, t) = \partial_t v(x, t)$ then $w(x, t)$ satisfies

$$\begin{aligned}
&\partial_t w(x, t) = A(v)\partial_{xx}w(x, t) + A'(v)/A(v)\, w^2(x, t),\\
(1.16) \qquad &w(x, 0) \geqq 0,\\
&w(0, t) = f_0'(t), \quad w(1, t) = 0.
\end{aligned}$$

Under the prevailing assumptions, the coefficient $C(x, t) = A'(v) / A(v)$ is continuous and hence bounded on $[0, 1] \times [0, T]$. Therefore we may apply the MP to conclude that for each $t > 0$

$$(1.17) \qquad 0 \leqq w(x, t) \leqq f_0'(t) \qquad \text{for } 0 \leqq x \leqq 1.$$

Then $w_{nax}$ occurs at $x = 0$ which implies that $\partial_x w(0, t) \leqq 0$ for $t > 0$. But,

$$\partial_x w(0, t) = \partial_x[\partial_t v(0, t)] = \partial_t[\partial_x v(0, t)] = -g_0'(t)$$

and hence $g_0'(t) \geqq 0$ for $0 \leqq t \leqq T$.

(1.13) follows, as before, from (1.17) and (1.7).

COROLLARY. *If the condition* $v(1, t) = 0$ *in* (1.15) *is replaced by the condition* $\partial_x v(1, t) = 0$, *then the conclusions of lemma* 1.3 *continue to hold.*

PROOF. Combine the extension procedure used in proving lemma 1.2 with the arguments of lemma 1.3.

We have proved now that the overspecified problem,

$$\begin{aligned}
&\partial_t v(x, t) = A(v)\, \partial_{xx}v(x, t), \quad 0 < x < 1, \quad 0 < t < T,\\
&v(x, 0) = 0, \quad 0 < x < 1,\\
(1.18) \qquad &-\partial_x v(0, t) = g_0(t), \text{ and } v(0, t) = f_0(t), \ 0 < t < T,\\
&v(1, t) = 0, \quad \text{or} \quad \partial_x v(1, t) = 0, \quad 0 < t < T
\end{aligned}$$

is not inconsistent provided that $f_0$ and $g_0$ satisfy (1.12) and (1.10) respectively.

Examining the arguments used here shows that the problem (1.18) is not inconsistent when $f_0(t), g_0(t)$ are each monotone decreasing instead of increasing. In this case, the direction of the inequalities in (1.13) must be reversed.

The change of variable $x \mid \to 1 - x$ in (1.18) shows that results analogous to lemmas 1.1, 1.2, 1.3 are true when the homogeneous conditions are at the end $x = 0$ and the conditions (1.10), (1.12) bear on the functions $g_1(t) = \partial_x v(1, t)$ and $f_1(t) = v(1, t)$.

**2. Bounds On $v(x, t)$ and Derivatives: Temperature Controlled Case** Consider the following problem

$$(2.1)\qquad \begin{aligned} &\partial_t z(x, t) = A\, \partial_{xx} x(x, t), \quad 0 < x < 1, \quad 0 < t < T,\\ &z(x, 0) = 0, \quad 0 < x < 1,\\ &z(0, t) = f(t), \quad z(1, t) = 0, \quad 0 < t < T, \end{aligned}$$

where $A$ denotes a positive constant and $f(t)$ satisfies (1.12). We will refer to (2.1) as a "temperature controlled" problem.

We have

$$(2.2)\qquad z(x, t) = -A \int_0^t \partial_x M(x, A(t-\tau))\, f(\tau)\, d\tau, \quad 0 < x < 1, 0 < t < T,$$

where

$$(2.3)\qquad M(x, t) = \frac{1}{\sqrt{\pi t}} \sum_{n=-\infty}^{\infty} \exp[-(x - 2n)^2/4t].$$

It is not difficult to show that for any $t_0 > 0$ and $x_0$, $0 \leqq x_0 < 1$, there is a positive constant $N_0 = N_0(x_0, t_0)$ such that

$$(2.4)\qquad -A \int_0^t \partial_x M(x, A(t - \tau))\, d\tau > N_0 \text{ for } 0 \leqq x < x_0,\ t_0 \leqq t \leqq T.$$

Then it follows that

$$(2.5)\qquad z(x, t) \geqq N_0 \min_{t_0 \leqq \tau \leqq T} f(\tau), \quad 0 \leqq x \leqq x_0,\ t_0 \leqq t \leqq T.$$

In addition, a simple MP argument leads to the result, for each $t > 0$,

$$(2.6)\qquad z(x, t) \leqq f(t), \qquad \text{for } 0 \leqq x \leqq 1.$$

It can be further inferred from an MP argument that if $w(x, t)$ satisfies

$$(2.7)\qquad \begin{aligned} &\partial_t w(x, t) = A\, \partial_{xx} w(x, t), \quad 0 < x < 1, \quad 0 < t < T,\\ &w(x, 0) = 0, \quad 0 < x < 1,\\ &w(0, t) = f(t), \quad \partial_x w(1, t) = 0, \quad 0 < t < T, \end{aligned}$$

and if $z(x, t)$ satisfies (2.1) then for each $t > 0$,

$$(2.8)\qquad 0 \leqq z(x, t) \leqq w(x, t) \leqq f(t), \quad 0 \leqq x \leqq 1.$$

It follows from (2.2) that

$$\partial_x z(x, t) = -A \int_0^t \partial_{xx} M(x, A(t - \tau)) f(\tau) \, d\tau. \tag{2.9}$$

Moreover, it is not difficult to check that

$$-A\partial_{xx}M(x, A(t - \tau)) = \partial_\tau M(x, A(t - \tau)).$$

Then for $f(t)$ satisfying (1.12), we may integrate by parts to obtain

$$\partial_x z(x, t) = -\int_0^t M(x, A(t - \tau)) f'(\tau) \, d\tau. \tag{2.10}$$

Again, it is possible to show that for each $t_0 > 0$ and all $x_0$, $0 \leqq x_0 \leqq 1$, there is a constant $D_0 = D_0(x_0, t_0) > 0$ such that,

$$\int_0^t M(x, A(t - \tau)) \, d\tau > D_0, \quad 0 \leqq x \leqq x_0, \quad t_0 \leqq t \leqq T. \tag{2.11}$$

This leads to,

$$\partial_x z(x, t) \leqq -D_0 \min_{t_0 \leqq \tau \leqq T} f_0'(\tau) < 0, \quad 0 \leqq x \leqq x_0, \; t_0 \leqq t \leqq T. \tag{2.12}$$

Now consider the following non-linear temperature controlled problem,

$$\begin{aligned} &\partial_t v(x, t) = A(v) \, \partial_{xx} v(x, t), \quad 0 < x < 1, \quad 0 < t < T, \\ &v(x, 0) = 0, \quad 0 < x < 1, \\ &v(0, t) = f_0(t), \quad v(1, t) = 0, \quad 0 < t < T. \end{aligned} \tag{2.13}$$

Then we have:

LEMMA 2.1. *Suppose $A(s)$ satisfies* (1.1) *and $f_0(t)$ satisfies* (1.12). *Let $z_0(x, t)$ denote the solution of* (2.1) *in the case $A = A_0$, $f(t) = f_0(t)$, and let $z_1(x, t)$ denote the solution in the case $A = A_1$ and $f(t) = f_0(t)$. Then if $v(x, t)$ satisfies* (2.13), *we have*

$$z_0(x, t) \leqq v(x, t) \leqq z_1(x, t), \quad 0 \leqq x \leqq 1, \quad 0 \leqq t \leqq T. \tag{2.14}$$

PROOF. If $w(x, t) = v(x, t) - z_0(x, t)$ then $w(x, t)$ satisfies,

$$\begin{aligned} &\partial_t w(x, t) - A(v) \, \partial_{xx} w(x, t) = [A(v) - A_0] \, \partial_{xx} z_0(x, t), \\ &w(x, 0) = 0, \quad 0 < x < 1, \\ &w(0, t) = w(1, t) = 0, \quad 0 < t < T. \end{aligned}$$

Lemma 2.1 of [1] implies $\partial_{xx} z_0(x, t) \geqq 0$, for $0 \leqq x \leqq 1$, $0 \leqq t \leqq T$, and this together with (1.1) imply that $[A(v) - A_0]\partial_{xx} z_0 \geqq 0$. Then the MP implies

$$w(x, t) = v(x, t) - z_0(x, t) \geqq 0, \quad 0 \leqq x \leqq 1, \quad 0 \leqq t \leqq T.$$

The other half of the estimate (2.14) follows from a similar argument applies to $w(x, t) = z_1(x, t) - v(x, t)$.

Nearly identical arguments lead to the following corollary of this lemma. Suppose $v(x, t)$ satisfies (2.13) with the single exception that the condition $v(1, t) = 0$, is replaced by the condition $\partial_x v(1, t) = 0$. Then we have,

$$(2.15) \qquad w_0(x, t) \leqq v(x, t) \leqq w_1(x, t), \quad 0 \leqq x \leqq 1, 0 \leqq t \leqq T,$$

where $w_0(x, t)$, $w_1(x, t)$ denote the solution of (2.7) in the case $A = A_0$, $A = A_1$, respectively and $f(t) = f_0(t)$.

We have also upper and lower estimates for the derivatives of $v(x, t)$.

LEMMA 2.2. *Let $A(s)$, $f_0(t)$ and $v(x, t)$ be as in the previous lemma. Let $S_0(x, t)$ denote the solution of* (2.1) *in the case $A = A_0$ and $f(t) = f_0'(t)$. If $f_0''(t) > 0$ for $t > 0$, then for each $t$, $0 \leqq t \leqq T$,*

$$(2.16) \qquad 0 \leqq S_0(x, t) \leqq \partial_t v(x, t) \leqq f_0'(t), \quad 0 \leqq x \leqq 1.$$

PROOF. The hypotheses of this lemma contain those of lemma 1.3 and hence $w(x, t) = \partial_t v(x, t)$ satisfies (1.16). Then the MP applies and it follows that for each $t$, $0 \leqq t \leqq T$, (1.17) holds. Moreover, it follows from (1.16) that

$$\partial_t(w - S_0) - A(v)\partial_{xx}(w - S_0) = [A(v) - A_0]\partial_{xx}S_0 + A'(v)/A(v)w^2 \geqq 0$$

and,

$$(w - S_0)(x, 0) = 0,$$
$$(w - S_0)(0, t) = (w - S_0)(1, t) = 0.$$

Then the MP implies

$$(w - S_0)(x, t) \geqq 0, \quad \text{for } 0 \leqq x \leqq 1, 0 \leqq t \leqq T,$$

and (2.16) is proved. Here, $f_0''(t) > 0$ implies $\partial_{xx}S_0 > 0$ in $Q_T$. (If $f_0''(t) <$ 0 then (2.16) holds with $S_0(x, t)$ replaced by $S_1(x, t)$.

As a corollary to (2.16) we have that the result continues to hold if $v(x, t)$ satisfies the condition $\partial_x v(1, t) = 0$ with the exception that $S_0(x, t)$ now denotes the solution of (2.7) in the case $A = A_0$ and $f(t) = f_0'(t)$.

LEMMA 2.3. *Let $A(s)$, $f_0(t)$, $z_0(x, t)$, $z_1(x, t)$ and $v(x, t)$ be as in lemma* 2.1. *Then*

$$(2.17) \qquad 0 \geqq \partial_x z_1(0, t) \geqq \partial_x v(0, t) \geqq \partial_x z_0(0, t), \quad 0 \leqq t \leqq T.$$

*That is,*

$$(2.18)\qquad \begin{aligned} -\int_0^t M(0, A_1)(t-\tau))f_0'(\tau)\,d\tau &\geqq \partial_x v(0, t) \\ \geqq -\int_0^t M(0, A_0)(t-\tau))\,f_0'(\tau)\,d\tau. \end{aligned}$$

PROOF. Since $z_0(0, t) = z_1(0, t) = v(0, t) = f_0(t), 0 \leqq t \leqq T$, it follows from (2.14) that for $x > 0$,

$$\frac{z_1(x, t) - z_1(0, t)}{x - 0} \geqq \frac{v(x, t) - v(0, t)}{x - 0} \geqq \frac{z_0(x, t) - z_0(0, t)}{x - 0}.$$

Letting $x > 0$, decrease to zero, we get in the limit,

$$\partial_x z_1(0, t) \geqq \partial_x v(0, t) \geqq \partial_x z_0(0, t), \quad 0 \leqq t \leqq T.$$

(2.10) together with the hypotheses on $f_0(t)$ imply that $\partial_x z_1(0, t) \geqq 0$ and then (2.18) follows.

We make note here of the fact that (1.17) implies

$$(2.19)\qquad A_0\partial_{xx}v(x, t) \leqq \partial_t v(x, t) \leqq F_0^* = \max_{0\leqq t\leqq T} f_0'(t), \quad 0 \leqq x \leqq 1, \quad 0 \leqq t \leqq T.$$

**3. Bounds on $v(x, t)$ and Derivatives: Flux Controlled Case** Consider the problem

$$(3.1)\qquad \begin{aligned} \partial_t z(x, t) &= A\,\partial_{xx}z(x, t), \quad 0 < x < 1, \quad 0 < t < T, \\ z(x, 0) &= 0, \quad 0 < x < 1, \\ -\partial_x z(0, t) &= g(t), \partial_x z(1, t) = 0, \quad 0 < t < T, \end{aligned}$$

where $A$ denotes a positive constant and $g(t)$ satisfies (1.10). We will refer to (3.1) as a "flux controlled" problem. For $0 \leqq x \leqq 1, 0 \leqq t \leqq T$, we have

$$(3.2)\qquad z(x, t) = A\int_0^t M(x, A(t-\tau))\,g(\tau)\,d\tau,$$

for $M(x, t)$ given by (2.3). Moreover, from (2.11) then

$$(3.3)\qquad z(x, t) \geqq D_0 \min_{t_0\leqq\tau\leqq T} g(\tau) \text{ for } 0 \leqq x \leqq x_0,\ t_0 \leqq t \leqq T.$$

Using the MP we can show that for $g(t)$ satisfying (1.10),

$$(3.4)\qquad 0 \leqq z(x, t) \leqq z(0, t) \quad \text{for } 0 \leqq x \leqq 1,$$

for each $t > 0$. Since

$$(3.5)\qquad M(0, t) = \frac{1}{\sqrt{\pi t}}\left[1 + 2\sum_{n=1}^{\infty} e^{-n^2/t}\right] \leqq \frac{1}{\sqrt{\pi t}}[1 + K_0 t]$$

for some $K_0 > 0$, it follows that

$$(3.6) \qquad z(x, t) \leqq \sqrt{\frac{4AT}{\pi}}(1 + K_0T)g(T), \quad 0 \leqq x \leqq 1, 0 \leqq t \leqq T.$$

Note that for $g(t)$ satisfying (1.10),

$$g(T) = \max_{0 \leqq t \leqq T} g(t) \text{ and } g(t_0) = \min_{t_0 \leqq t \leqq T} g(t).$$

Now consider the non-linear flux controlled problem:

$$(3.7) \qquad \begin{aligned} &\partial_t v(x, t) = A(v)\, \partial_{xx} v(x, t), \quad 0 < x < 1, 0 < t < T, \\ &v(x, 0) = 0, \quad 0 < x < 1, \\ &-\partial_x v(0, t) = g_0(t), \quad \partial_x v(1, t) = 0, \quad 0 < t < T. \end{aligned}$$

We have then,

LEMMA 3.1. *Suppose $A(s)$ satisfies (1.1) and $g_0(t)$ satisfies (1.10). Let $z_0(x, t)$ denote the solution of (3.1) in the case $A = A_0$, $g(t) = g_0(t)$ and let $z_1(x, t)$ denote the solution of (3.1) in the case $A = A_1$ and $g(t) = g_0(t)$. Then if $v(x, t)$ satisfies (3.7), we have*

$$(3.8) \qquad z_0(x, t) \leqq v(x, t) \leqq z_1(x, t), \quad 0 \leqq x \leqq 1, \quad 0 \leqq t \leqq T.$$

The proof is similar to the proof of lemma 2.1 and is omitted.

LEMMA 3.2. *Let $A(s)$, $g_0(t)$, and $v(x, t)$ be as in the previous lemma. Let $S_0(x, t)$ denote the solution of (3.1) in the case $A = A_0$ and $g(t) = g_0'(t)$. Then*

$$(3.9) \qquad \partial_t v(x, t) \geqq S_0(x, t), \quad 0 \leqq x \leqq 1, 0 \leqq t \leqq T.$$

The proof of this lemma is similar to the proof of lemma 2.2 and is omitted.

Note that $\partial_t v(x, t) = w(x, t)$ satisfies (1.14). Then it follows that for each $t > 0$,

$$0 \leqq \partial_t v(x, t) \leqq \partial_t v(0, t), \quad 0 \leqq x \leqq 1.$$

Since $\partial_t v(0, t)$ is continuous for $0 \leqq t \leqq T$, if we let $F_T = \max\{0 \leqq t \leqq T: \partial_t v(0, t)\}$ then,

$$(3.10) \qquad 0 \leqq \partial_t v(x, t) \leqq F_T, \quad 0 \leqq x \leqq 1, 0 \leqq t \leqq T.$$

Here, $F_T$ is a positive constant depending on $T$ and on $g_0(t)$.

**4. Monotonicity Estimates: the Flux Controlled Case** We are going to consider IBVP's comprised of the equation (1.7) together with auxiliary conditions selected from (1.9). We plan to examine the dependence on the coefficient $A(v)$ of the solution $v(x, t)$ for the IBVP. To examine this dependence we shall suppose $A_1(s)$, $A_2(s)$ denote two coefficient functions, each of which satisfies (1.1). We wish to consider coefficients $A_1(s)$, $A_2(s)$ which are distinct and for this purpose it will be convenient to suppose

$$A_1(0) = A_2(0), \quad \text{and} \quad A_1'(0) > A_2'(0). \tag{4.1}$$

The condition (4.1) implies that the graphs of $A_1(s)$, $A_2(s)$ have a transversal intersection at $s = 0$ and that the graph of $A_1(s)$ remains above that of $A_2(s)$ on an interval $(0, \sigma)$ for some $\sigma > 0$.

Consider the flux controlled problem,

$$\begin{aligned} \partial_t\, v(x, t) &= A(v)\, \partial_{xx} v(x, t), \quad 0 < x < 1, 0 < t < T, \\ v(x, 0) &= 0, \quad 0 < x < 1, \\ -\partial_x v(0, t) &= g_0(t), \partial_x v(1, t) = 0, \quad 0 < t < T, \end{aligned} \tag{4.2}$$

and denote the solution by $v = v(x, t; A)$. In particular, let $v_i = v(x, t; A_i)$, $i = 1, 2$ denote the solution of (4.2) for $A = A_1, A_2$ satisfying (4.1).

THEOREM 4.1. *Suppose $A_1, A_2$ satisfy* (1.1) *and* (4.1) *and that $g_0(t)$ satisfies* (1.10). *Let $v_i(x, t) = v(x, t; A_i)$, $i = 1, 2$ denote the solution of* (4.2) *corresponding to the coefficient $A = A_i$. Then there exists a constant $T_1 > 0$, such that*

$$v_1(x, t) \geqq v_2(x, t) \geqq 0, \quad 0 \leqq x \leqq 1, 0 \leqq t \leqq T_1. \tag{4.3}$$

*Moreover, there exists a $t_1 > 0$, $0 < t_1 < T_1$, and a constant $C_1 > 0$, such that*

$$v_1(0, t) - v_2(0, t) \geqq C_1(t - t_1), \quad t_1 \leqq t \leqq T_1. \tag{4.4}$$

PROOF. Let $w(x, t) = v_1(x, t) - v_2(x, t), 0 \leqq x \leqq 1, 0 \leqq t \leqq T$. Then since $v_1$ and $v_2$ each satisfy (4.2), it follows that

$$\begin{aligned} \partial_t w(x, t) - A_1(v_1)\partial_{xx} w(x, t) &- [A_1(v_1) - A_1(v_2)]\partial_{xx} v_2 \\ &= \partial_{xx} v_2[A_1(v_2) - A_2(v_2)], \quad 0 < x < 1, 0 < t < T. \end{aligned}$$

The mean value theorem implies that for some $\xi = \xi(x, t)$ between $v_1(x, t)$ and $v_2(x, t)$ we have

$$A_1(v_1(x, t)) - A_1(v_2(x, t)) = A_1'(\xi(x, t))\, w(x, t).$$

Thus $w(x, t)$ satisfies the following IBVP:

$$\begin{aligned} \partial_t w(x,t) - A_1(v_1)\partial_{xx} w(x,t) &- A_1'(\xi)\partial_{xx} v_2(x,t)\, w(x,t) \\ &= \partial_{xx} v_2[A_1 v_2) - A_2(v_2)], \\ w(x, 0) &= 0, \quad 0 < x < 1, \\ \partial_x w(0, t) &= \partial_x w(1, t) = 0, \quad 0 < t < T. \end{aligned} \tag{4.5}$$

Now let $V(x, t)$ denote the solution of the following related IBVP:

$$\begin{aligned} &\partial_t V(x, t) - A_1(v_1)\partial_{xx} V(x, t) = \partial_{xx} v_2[A_1(v_2) - A_2(v_2)], \\ (4.6) \qquad &V(x, 0) = 0, \quad 0 < x < 1, \\ &\partial_x V(0, t) = \partial_x V(1, t) = 0, \quad 0 < t < T. \end{aligned}$$

From (4.1) it follows that there exists $\sigma_2 > 0$ such that

$$(4.7) \qquad A_1(s) - A_2(s) \geqq 0, \quad \text{for } 0 \leqq s \leqq \sigma_2.$$

Let $T_1 > 0$ be chosen such that $z_1(0, T_1) = \sigma_2$, where $z_1(x, t)$ denotes the solution of (3.1) in the case $A = A_1$, $g(t) = g_0(t)$. It follows from (3.8) and (3.4) that

$$(4.8) \qquad v_2(x, t) \leqq \sigma_2 \text{ for } 0 \leqq x \leqq 1, 0 \leqq t \leqq T_1,$$

and hence (4.7) implies

$$A_1(v_2(x, t) - A_2(v_2(x, t)) \geqq 0 \text{ for } 0 \leqq x \leqq 1, 0 \leqq t \leqq T_1.$$

From (3.9) and (4.2) we have,

$$(4.9) \qquad \partial_{xx} v_2(x, t) \geqq \frac{S_0(x, t)}{A_1} \geqq 0 \quad \text{for } 0 \leqq x \leqq 1, 0 \leqq t \leqq T$$

where $S_0(x, t)$ denotes the solution of (3.1) in the case $A = A_0$, $g(t) = g_0'(t)$.

We are now in a position to apply the MP to (4.6) in order to conclude,

$$V(x, t) \geqq 0, \quad \text{for } 0 \leqq x \leqq 1, 0 \leqq t \leqq T_1.$$

It follows from lemma 3.1 of [**2**] that

$$(4.10) \qquad w(x, t) \geqq V(x, t) \geqq 0, \quad 0 \leqq x \leqq 1, \quad 0 \leqq t \leqq T_1,$$

proving (4.3).

Next, fix $t_0$ such that $0 < t_0 < T_1$ and let $\sigma_1 = z_0(0, t_0) > 0$ where $z_0(x, t)$ denotes the solution of (3.1) in the case $A = A_0$, $g(t) = g_0(t)$. Then

$$\sigma_1 = z_0(,0\ t_0) < z_1(0, t_0) < z_1(0, T_1) = \sigma_2.$$

Now we may choose $t_1$ such that $t_0 < t_1 < T_1$ and $\sigma_1 < z_0(0, t_1) < \sigma_2$. Then there exists $x_1$, $0 < x_1 \leqq 1$, such that

$$(4.11) \qquad z_0(x, t) \geqq \sigma_1 \quad \text{for } 0 \leqq x \leqq x_1, \quad t_1 \leqq t \leqq T.$$

Now (3.8), (4.11) and (4.8) together imply

$$(4.12) \qquad \sigma_1 \leqq v_2(x, t) \leqq \sigma_2 \text{ for } 0 \leqq x \leqq x_1, \quad t_1 \leqq t \leqq T_1.$$

From (4.7) and (4.1) we have that there is a constant $\lambda = \lambda(\sigma_1) > 0$ such that

$$(4.13) \qquad A_1(s) - A_2(s) \geqq \lambda \quad \text{for } \sigma_1 \leqq s \leqq \sigma_2.$$

Then (4.12), (4.13) lead to

(4.14) $A_1(v_2(x,t)) - A_2(v_2(x,t)) \geqq \lambda > 0 \quad \text{for } 0 \leqq x \leqq x_1, \quad t_1 \leqq t \leqq T_1.$

From (4.9) and (3.3) we have

$$\partial_{xx} v_2(x, t) \geqq \frac{D_0 G_0^*}{A_1} \text{ for } 0 \leqq x \leqq x_1, t_1 \leqq t \leqq T_1, \tag{4.15}$$

where

$$D_0 = \min_{\substack{t_1 \leqq t \leqq T_1 \\ 0 \leqq x \leqq x_1}} \int_0^t M(x, A_0(t - \tau))\, d\tau > 0,$$

$$G_0^* = \min_{t_1 \leqq t \leqq T_1} g_0'(t) > 0.$$

Now let

(4.16) $\psi(x, t) = \eta(t - t_1)(x_1^2 - x^2), \quad 0 \leqq x \leqq x_1, t_1 \leqq t \leqq T_1,$

where

$$\eta = \frac{D_0 G_0^*}{A_1} \frac{\lambda}{x_1^2 + 2A_1(T_1 - t_1)}. \tag{4.17}$$

Then (4.6), (4.14), (4.15) together imply,

$$\partial_t(V - \psi) - A_1(v_1)\partial_{xx}(V - \psi) \geqq \frac{D_0 G_0^*}{A_1} \lambda - (x_1^2 + 2A_1(t - t_1)) \geqq 0,$$

$$(V - \psi)(x, t_1) \geqq 0, \quad 0 \leqq x \leqq x_1,$$

$$\partial_x(V - \psi)(0, t) = 0, \quad t_1 \leqq t \leqq T_1,$$

$$(V - \psi)(x_1, t) \geqq 0, \quad t_1 \leqq t \leqq T_1.$$

Then the MP implies

$$(V - \psi)(x, t) \geqq 0 \quad \text{for } 0 \leqq x \leqq x_1, \quad t_1 \leqq t \leqq T_1,$$

and from (4, 10) it follows finally that

$$w(0, t) = v_1(0, t) - v_2(0, t) \geqq \psi(0, t) = \eta x_1^2(t - t_1), \; t_1 \leqq t \leqq T_1,$$

that is, (4.4) holds with $C_1 = \eta x_1^2 > 0$.

We point out that this proof goes through with little change if we assume that $A_1$ and $A_2$ are distinct in the sense that on any interval of finite length the graphs of $A_1$, $A_2$ are separated by a positive distance $\lambda$.

THEOREM 4.2. *Suppose $A_1(s)$, $A_2(s)$, $g_0(t)$, and $v_i(x, t) = v(x, t; A_i)$, $i = 1$, 2, are as in the previous theorem. Then for each $\sigma > 0$ there exist positive constants $\tau = \tau(\sigma)$, $K = K(\sigma)$ such that*

$$(4.18)\qquad \begin{aligned} &a)\ 0 \leqq v_i(x, t) \leqq \sigma \text{ for } 0 \leqq x \leqq 1,\ 0 \leqq t \leqq \tau(\sigma),\\ &b)\ |v_1(x, t) - v_2(x, t)| \leqq \mu K,\ 0 \leqq x \leqq 1,\ 0 \leqq t \leqq \tau \end{aligned}$$

*where*

$$\mu = \max \{0 \leqq s \leqq \sigma: |A_1(s) - A_2(s)|\}.$$

PROOF. Let $z_1(x, t)$ denote the solution of (3.1) in the case $A = A_1$, $g(t) = g_0(t)$. Then for $\sigma > 0$ given, let $\tau = \tau(\sigma)$ be chosen such that $z_1(0, \tau) = \sigma$. Then (4.18)a follows from (3.4) and (3.8).

Now it follows from (3.10) and (4.2) that

$$\partial_{xx} v_2(x, t) \leqq \frac{F_\tau}{A_0} \text{ for } 0 \leqq x \leqq 1,\ 0 \leqq t \leqq \tau.$$

Then

$$Z(x, t) = \frac{F_\tau}{A_0}\, \mu\, t \exp[tF_\tau A_2/A_0]$$

satisfies

$$\begin{aligned} \partial_t Z(x, t) - A_1 \partial_{xx} Z(x, t) &= A_2 Z(x, t) + \mu\, F_\tau/A_0,\ 0 < x < 1, 0 < t < \tau,\\ Z(x, 0) &= 0,\\ \partial_x Z(0, t) &= \partial_x Z(1, t) = 0. \end{aligned}$$

Since $w(x, t) = v_1(x, t) - v_2(x, t)$ satisfies (4.5), it follows that

$$\begin{aligned} \partial_t(Z - w) &- A_1(v_1)\partial_{xx}(Z - w) - A_1'(\xi)\,(Z - w)\\ &= [A_1 - A_1(v_1)]\partial_{xx} Z + (A_2 - A_1'(\xi))Z\\ &\qquad + \mu\, F_\tau/A_0 - \partial_{xx} v_2[A_1(v_2) - A_2(v_2)]. \end{aligned}$$

There is no loss in generality here in assuming $A_1(s) - A_2(s) \geqq 0$, for $0 \leqq s \leqq \sigma$. Then,

$$\begin{aligned} \partial_t(Z - w) - A_1(v_1)\partial_{xx}(Z - w) - A_1'(\xi)(Z - w) &\geqq 0,\ \ 0 < x < 1,\ 0 < t < \tau\\ (Z - w)\,(x, 0) = 0,&\qquad 0 < x < 1,\\ \partial_x(Z - w)\,(0, t) = \partial_x(Z - w)\,(1, t) &= 0,\ 0 < t < \tau, \end{aligned}$$

and it follows from the MP that $(Z - w)\,(x, t) \geqq 0$, for $0 \leqq x \leqq 1$, $0 \leqq t \leqq \tau$. Then (4.18) follows for

$$K(\tau) = \mu\tau F_\tau/A_0 \exp[\tau F_\tau A_2/A_0].$$

The significance of theorems 4.1 and 4.2 is the following. Let $g_0(t)$ denote a fixed function in $C^1[0, T]$ satisfying (1.10). Then for each coefficient A$(s)$ which satisfies (1.1), we can solve (4.2) for $v = v(x, t; A)$ in

$C^{2,1}[0, 1] \times [0, T]$. It follows from lemma 1.1 that $f_0(t) = v(0, t; A)$ then satisfies (1.12). We may view this as defining a mapping $\Phi$ from the coefficient class $C_A = \{A$ satisfies (1.1)$\}$ into the data class $C_f = \{f$ satisfies (1.12)$\}$. The mapping $\Phi$ has the following properties:

a) for $0 < \sigma_1 < \sigma_2$ there exist $t_1 = t_1(\sigma_1)$, $T_1 = T_1(\sigma_2)$ such that $0 < t_1 < T_1$ and

$$\Phi[A_1](t) - \Phi[A_2](t) \geqq \tilde{C}(t - t_1)\lambda \; t_1 \leqq t \leqq T_1,$$

where $\tilde{C} > 0$, and $\lambda = \min\{\sigma_1 \leqq s \leqq \sigma_2: A_1(s) - A_2(s)\}$.

b) for $\sigma > 0$ there exists $\tau = \tau(\sigma) > 0$ such that

$$\Phi[A_1](t) - \Phi[A_2](t) \leqq K(\tau)\mu \; 0 \leqq t \leqq \tau,$$

where $K(\tau) > 0$, and $\mu = \max\{0 \leqq s \leqq \sigma: A_1(s) - A_2(s)\}$.

Property a) implies that identical flux controlled experiments involving distinct coefficients $A_1$, $A_2$ from $C_A$ cannot produce identical data $f_0(t)$. Property b) then implies that identical flux controlled experiments with identical coefficients $A_1$, $A_2$ must produce identical data $f_0(t)$.

**5. Monotonicity Estimates: Temperature Controlled Case** We want to derive for the temperature controlled problem estimates of the sort found in the previous section for the flux controlled problem. Consider then the following temperature controlled problem,

$$\begin{aligned} &\partial_t v(x, t) = A(v)\, \partial_{xx} v(x, t), && 0 < x < 1, 0 < t < T, \\ &v(x, 0) = 0, && 0 < x < 1, \\ &v(0, t) = f_0(t), v(1, t) = 0, && 0 < t < T. \end{aligned} \tag{5.1}$$

THEOREM 5.1 *Suppose* $A_1(s)$, $A_2(s)$ *satisfy* (1.1) *and* (4.1) *and that* $f_0(t)$ *satisfies* (1.12). *Let* $v_i(x, t) = v(x, t; A_i)$, $i = 1, 2$ *denote the solution of* (5.1) *corresponding to coefficient* $A = A_i(v)$. *Then there exists* $T_1 > 0$, *such that*

$$0 \geqq \partial_x v_1(0, t) \geqq \partial_x v_2 0(, t), \quad 0 \leqq t \leqq T_1. \tag{5.2}$$

*Moreover, there exist* $t_1$, $0 < t_1 < T_1$ *and* $C = C(t_1) > 0$ *such that*

$$\partial_x v_1(0, t) - \partial_x v_2(0, t) \geqq C(t - t_1), \quad t_1 \leqq t \leqq T_1. \tag{5.3}$$

PROOF. Let $w(x, t) = v_1(x, t) - v_2(x, t)$ and proceed as in the proof of theorem 4.1 to show that $w(x, t)$ must satisfy

$$\begin{aligned} &\partial_t w(x, t) - A_1(v_1)\partial_{xx} w(x, t) - A_1'(\xi) w(x, t) = \partial_{xx} v_2[A_1(v_2) - A_2(v_2)] \\ &w(x, 0) = 0, 0 < x < 1, \\ &w(0, t) = w(1, t) = 0, 0 < t < T. \end{aligned} \tag{5.4}$$

Consider now the related IBVP,

$$\begin{aligned} &\partial_t V(x,t) - A_1(v_1)\partial_{xx}V(x,t) = \partial_{xx}v_2[A_1(v_2) - A_2(v_2)], \\ (5.5) \qquad &V(x,0) = 0, \quad 0 < x < 1, \\ &V(0,t) = V(1,t) = 0, \quad 0 < t < T. \end{aligned}$$

As in the proof of theroem 4.1 we show that there exists a $T_1 > 0$, such that

$$(5.6) \qquad A_1(v_2(x,t)) - A_2(v_2(x,t)) \geqq 0 \text{ for } 0 \leqq x \leqq 1, 0 \leqq t \leqq T_1.$$

In addition, from (2.16) and (5.1) we have

$$(5.7) \qquad \partial_{xx}v_2(x,t) \geqq S_0(x,t)/A_1 \geqq 0, \quad 0 \leqq x \leqq 1, 0 \leqq t \leqq T,$$

where $S_0(x, t)$ denotes the solution of (2.1) in the case $A = A_0$ and $f(t) = f_0'(t)$. Apply the MP then to (5.5) to infer that

$$(5.8) \qquad V(x,t) \geqq 0, \quad 0 \leqq x \leqq 1, 0 \leqq t \leqq T_1,$$

and then by lemma 3.1 of [2] it follows that

$$(5.9) \qquad w(x,t) \geqq V(x,t) \geqq 0, \quad 0 \leqq x \leqq 1, 0 \leqq t \leqq T_1.$$

Then (5.9) together with lemma 1.3 leads to (5.2).

Now, arguing as we did to establish (4.12), we can show that there exist constants $x_1$, $t_1$ such that $0 < x_1 \leqq 1$, $0 < t_1 < T_1$ and

$$(5.10) \qquad \sigma_1 \leqq v_2(x,t) \leqq \sigma_2, \text{ for } 0 \leqq x \leqq x_1, t_1 \leqq t \leqq T_1,$$

where $0 < \sigma_1 < \sigma_2$. It follows then from (4.13) that

$$(5.11) \quad A_1(v_2(x,t)) - A_2(v_2(x,t)) \geqq \lambda > 0, 0 \leqq x \leqq x_1, t_1 \leqq t \leqq T_1.$$

In addition, it follows from (5.7) and (2.5) that

$$(5.12) \qquad \partial_{xx}v_2(x,t) \geqq \frac{N_0 F_0^*}{A_1} \text{ for } 0 \leqq x \leqq x_1, t_1 \leqq t \leqq T_1$$

where

$$N_0 = \min_{\substack{t_1 \leq t \leq T_1 \\ 0 \leq x \leq x_1}} \int_0^t M_x(x, A_0(t-\tau))\, d\tau > 0$$

$$F_0^* = \min_{t_1 \leq t \leq T_1} f_0'(t) > 0.$$

Now let

$$\psi(x,t) = \eta(t - t_1)x(x_1 - x), \quad 0 \leqq x \leqq x_1, t_1 \leqq t \leqq T_1,$$

where

$$\eta = \frac{N_0F_0^*}{A_1}\,\frac{2\lambda}{x_1^2 + 4A_1(T_1 - t_1)}.$$

Then it is easy to show that

$$\begin{aligned} &\partial_t(V-\psi) - A_1(v_1)\partial_{xx}(V-\psi) \geqq 0, \text{ for } 0 < x < x_1,\ t_1 < t < T_1,\\ (5.13)\qquad &(V-\psi)(x, t_1) \geqq 0,\ 0 < x < x_1,\\ &(V-\psi)(0, t) = 0,\ (V-\psi)(x_1, t) \geqq 0,\ t_1 < t < T_1, \end{aligned}$$

and then the MP implies $(V-\psi)(x, t) \geqq 0$ for $0 \leqq x \leqq x_1$, $t_1 \leqq t \leqq T_1$. From (5.9) we have then

$$\begin{aligned} (5.14)\qquad & v_1(x,t) - v_2(x,t) \geqq \frac{N_0F_0^*}{A_1}\,\frac{2\lambda(t-t_1)\,x(x_1-x)}{A_1x_1^2 + 4A_1(T_1-t_1)}\\ & 0 \leqq x \leqq x_1,\ t_1 \leqq t \leqq T_1. \end{aligned}$$

From (5.14) we have

$$\begin{aligned} \partial_x v_1(0,t) &= \lim_{x\to 0} \frac{v_1(x,t) - v_1(0,t)}{x-0}\\ &\geqq \lim_{x\to 0}\left[\frac{v_2(x,t) - v_2(0,t)}{x-0} + \frac{N_0F_0^*}{A_1}\,\frac{(t-t_1)\lambda(x_1-x)}{x_1^2 + 4A_1(T_1-t_1)}\right], \end{aligned}$$

and this, together with lemma 1.3 and the fact that $v_1(0, t) = v_2(0, t) = f_0(t)$, implies (5.3) for

$$(5.15)\qquad C = \frac{N_0F_0^*}{A_1}\,\frac{\lambda\, x_1}{x_1^2 + 4A_1(T_1-t_1)} > 0.$$

THEOREM 5.2. *Let $A_1(s)$, $A_2(s)$, $f_0(t)$ and $v_i(x, t) = v(x, t; A_i)$, $i = 1, 2$ be as in theorem 5.1. Then for each $\sigma > 0$ there exist positive constants $\tau = \tau(\sigma)$, $K = K(\tau)$ such that*

$$\begin{aligned} (5.16)\qquad &\text{a) } 0 \leqq v_i(x,t) \leqq \sigma,\ 0 \leqq x \leqq 1,\ 0 \leqq t \leqq \tau,\ i = 1, 2,\\ &\text{b) } \partial_x v_1(0,t) - \partial_x v_2(0,t) \leqq K\mu,\quad 0 \leqq t \leqq \tau, \end{aligned}$$

where

$$\mu = \max\{0 \leqq s \leqq \sigma\colon |A_1(s) - A_2(s)|\}.$$

PROOF. The hypotheses of this theorem include those of leamm 2.1. Then (5.16)a follows from (2.14) and (2.6) with $\tau = \tau(\sigma)$ chosen such that $f_0(\tau) = \sigma$.

We have that $w(x, t) = v_1(x, t) - v_2(x, t)$ satisfies (5.4). From (2.19) and (5.16)a it follows that

$$\partial_{xx}v_2(x,t)[A_1(v_2) - A_2(v_2)] \leqq \frac{F_0^*\mu}{A_0},\ 0 \leqq x \leqq 1,\ 0 \leqq t \leqq \tau,$$

The function

$$Z(x, t) = \frac{F_0^* \mu t}{A_0} \exp[F_0^* t A_2/A_0] \frac{4}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - e^{-(2n-1)^2\pi^2 t}}{(2n-1)^3} \sin(2n-1)\pi x$$

satisfies

$$\partial_t Z(x, t) - A_1 \partial_{xx} Z(x, t) + \frac{A_2 F_0^*}{A_0} Z(x, t) = F_0^* \mu / A_0,$$

$$Z(x, 0) = 0, \quad 0 < x < 1,$$

$$Z(0, t) = Z(1, t) = 0, \quad 0 < t < \tau.$$

An MP argument leads to the result $(Z - w)(x, t) \geqq 0$, for $0 \leqq x \leqq 1$, $0 \leqq t \leqq \tau$, and it follows that

$$\frac{Z(x, t) - Z(0, t)}{x - 0} \geqq \frac{v_1(x, t) - v_1(0, t)}{x - 0} - \frac{v_2(x, t) - v_2(0, t)}{x - 0}$$

for $x > 0$, $0 \leqq t \leqq \tau$. Then and since

$$\partial_x Z(0, t) \geqq \partial_x v_1(0, t) - \partial_x v_2(0, t), \quad 0 \leqq t \leqq \tau,$$

$$\partial_x Z(0, t) = \frac{F_0^* \mu t}{A_0} \exp[t F_0^* A_2/A_0] \sum_{n=1}^{\infty} \frac{4}{\pi^2} \frac{1 - e^{-(2n-1)^2\pi^2 t}}{(2n-1)^2},$$

(5.16)b follows for

$$K = \frac{F_0^* \tau}{A_0} \exp[\tau F_0^* A_2/A_0] \sum_{n=1}^{\infty} \frac{4}{\pi^2} (2n-1)^{-2} > 0.$$

The importance of theroems 5.1, 5.2 lies in the following observation. Let $f_0(t)$ denote a fixed function in $C^1[0, T]$ satisfying (1.12). Then for each $A(s)$ satisfying (1.1) we can solve (5.1) for $v = v(x, t; A)$ in $C^{2,1}[0, 1] \times [0, T]$. Then it follows from lemma 1.3 that $g_0(t) = -\partial_x v(0, t; A)$ must satisfy (1.10). In this way we define a mapping $\Psi$ from $C_A = \{A \text{ satisfies } (1.1)\}$ into the class of boundary flux data $C_g = \{g \text{ satisfies } (1.10)\}$. From theorems 5.1, 5.2 it foll ows that the mapping $\Psi$ has the following properties:

a)for $0 < \sigma_1 < \sigma_2$ there exist $t_1 = t_1(\sigma_1)$ and $T_1 = T_1(\sigma_2)$ such that $0 < t_1 < T_1$ and

$$\Psi[A_1](t) - \Psi[A_2](t) \geqq C(t - t_1)\lambda, \; t_1 \leqq t \leqq T_1,$$

where $C > 0$, and $\lambda = \min\{\sigma_1 \leqq s \leqq \sigma_2 \colon A_1(s) - A_2(s)\}$.

b) for $\sigma > 0$ there exists $\tau = \tau(\sigma) > 0$ such that

$$\Psi[A_1](t) - \Psi[A_2](t) \leqq K(\tau)\mu, \; 0 \leqq t \leqq \tau,$$

where $K(\tau) > 0$ and $\mu = \max\{0 \leqq s \leqq \sigma \colon A_1(s) - A_2(s)\}$.

It follows from property a) that identical temperature controlled experiments with distinct coefficients $A_1$, $A_2$ from $C_A$ cannot produce identical flux data $g_0(t)$. Then property b) implies that identical temperature controlled experiments with identical coefficients $A_1$, $A_2$ must produce identical flux data on the boundary.

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## References

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