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ON THE BETWEENNESS CONDITION OF ROLLE'S THEOREM

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1. Introduction. Rolle's theorem, in its simplest form, when applied to real polynomials $f(x)$, states that (1) between any two consecutive real zeros of $f(x)$ there is an odd number of zeros of the derivative, $Df(x)$; and consequently, (2) the polynomial $Df(x)$ has no more nonreal zeros than $f(x)$ has. Generalizations of this second property have been explored in [**1**, **2**]. Let $\Gamma = \{\gamma_k\}$ be a sequence of real numbers and for an arbitrary real polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$ define

$$\Gamma[f(x)] = \sum_{k=0}^{n} a_k \gamma_k x^k. \tag{1.1}$$

We recall that a sequence $\Gamma = \{\gamma_k\}$ of real numbers is called a multiplier sequence of the first kind if Γ takes every real polynomial $f(x)$ which has only real zeros into a polynomial $\Gamma[f(x)]$ (defined by (1.1)) of the same class. (For the various properties of multiplier sequences of the first kind we refer the reader to Pólya and Schur [**7**], Obreschkoff [**6**] and Craven and Csordas [**1**, **2**]). The relationship between these sequences and Rolle's theorem is suggested by the fact that for the multiplier sequence $\Gamma = \{0, 1, 2, \ldots\}$, we have $\Gamma[f(x)] = xf'(x)$.

The purpose of this paper is to generalize the betweenness condition of Rolle's theorem. We shall show (Corollary 2.4) that the class of linear transformations Γ, defined by (1.1), which satisfy the betweeness property (see Definition 2.1) is precisely the class of nonconstant arithmetic sequences, all of whose terms have the same sign. Our main theorem is a quantitative result on the location of the zero of $\Gamma[f(x)]$ between two consecutive real zeros a and b of f. This result gives the best possible bounds for the zero of $\Gamma[f(x)]$ depending only on Γ, a, b and the degree of f. This generalizes an old theorem of Laguerre for derivatives [**6**, p. 121] and its subsequent extension to a larger class of polynomials by Nagy [**5**]. We conclude the paper with some open problems.

2. The betweenness property. The precise formulation of the betweenness property is as follows.

Definition 2.1. Let $f(x)$ be an arbitrary real polynomial. A real se-

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quence $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ is said to possess the betweenness property if the polynomial $\Gamma[f(x)]$ has at least one real zero between any two real zeros a and b, $a < b$, of $f(x)$.

REMARK 2.2. If the real sequence $\Gamma = \{\gamma_k\}$ possesses the betweenness property, then for any real polynomial $f(x)$, we have $\deg \Gamma[f(x)] = \deg f(x)$. In particular, $\gamma_k \neq 0$, for $k = 1, 2, 3, \ldots$. This assertion follows from the observation that the polynomial

$$\Gamma[x^k(1 + x)] = x^k[\gamma_k + \gamma_{k+1}\, x],\ k = 1, 2, 3, \ldots,$$

must have a root between -1 and 0. Thus, $\gamma_k\gamma_{k+1} > 0$ for $k = 1, 2, 3, \ldots$. In addition, the polynomial

$$\Gamma[(x^2 - 1)] = [\gamma_2 x^2 - \gamma_0]$$

has a root between -1 and 1, so that $\gamma_2\gamma_0 \geqq 0$. This inequality, when combined with $\gamma_k\gamma_{k+1} > 0$, $k = 1, 2, 3, \ldots$, implies that the terms of the sequence $\Gamma = \{\gamma_k\}$ all have the same sign, with γ_0 possibly equal to zero.

Our first result shows that if the sequence $\Gamma = \{\gamma_k\}$ possesses the betweenness property, then Γ is a multiplier sequence of the first kind. Moreover, the following theorem provides a simple representation for the Jensen polynomials, that is for the polynomials $g_n(x) \equiv \Gamma[(1 + x)^n]$, $n = 0, 1, 2, \ldots$, associated with Γ.

THEOREM 2.3. *Suppose that the real sequence $\Gamma = \{\gamma_k\}$ possesses the betweenness property. Then Γ is a multiplier sequence of the first kind. Moreover, for each n, $n = 1, 2, 3, \ldots$, the Jensen polynomial $g_n(x)$ associated with Γ is given by*

$$g_n(x) \equiv \Gamma[(1 + x)^n] = (1 + x)^{n-1}[\gamma_0 + (n\gamma_1 - (n - 1)\gamma_0)x].$$

PROOF. Let n be a fixed, but arbitrary, positive integer. Let $\{a_{1,k}\}, \ldots, \{a_{n,k}\}$ be n real sequences such that (1) for each fixed k, $a_{1,k} < a_{2,k} < \cdots < a_{n,k} < -1$; and (2) $\lim_{k\to\infty} a_{j,k} = -1$, for $j = 1, \ldots, n$. Let

$$f_k(x) = \prod_{j=1}^{n} (x + a_{j,k}), \qquad k = 1, 2, 3, \ldots.$$

Since Γ possesses the betweenness property, the polynomial $\Gamma[f_k(x)]$, $k = 1, 2, 3, \ldots$, has at least $n - 1$ real zeros. But by the above Remark 2.2, $\deg \Gamma[f_k(x)] = \deg f_k(x)$. Consequently, $\Gamma[f_k(x)]$ has precisely n real zeros. Since

$$(*) \qquad \lim_{k\to\infty} \Gamma[f_k(x)] = \Gamma[(1 + x)^n],$$

and the convergence is uniform on compact subsets of the plane, we conclude that $\Gamma[(1 + x)^n]$ has only real zeros for each positive integer n.

Moreover, since the γ_k's all have the same sign (see Remark 2.2), the zeros of $\Gamma[(1+x)^n]$ are all real and negative. Hence by a well-known theorem of Pólya and Schur [7, p. 100], Γ is a multiplier sequence of the first kind.

The above argument thus shows that the polynomial $\Gamma[f_k(x)]$ has n real negative zeros for each $k = 1, 2, 3, \ldots$. Hence, the betweenness property of Γ together with (*) imply that $\Gamma[(1+x)^n]$ has $x = -1$ as a zero of order at least $n-1$. Since the constant sequence $\{c, c, \ldots\}$ does not enjoy the betweenness property, we conclude that the order of the zero $x = -1$ of $\Gamma[(1+x)^n]$ is precisely $n-1$. Thus we have shown that

$$g_n(x) = \Gamma[(1+x)^n] = (1+x)^{n-1}(\alpha_n + \beta_n x), \qquad n = 1, 2, 3, \ldots,$$

where α_n and β_n are real constants. If $n = 0$, then $g_0(x) = \gamma_0$. Since $g_n(0) = \gamma_0$, we have $\alpha_n = \gamma_0$ for $n = 0, 1, 2, \ldots$.

It remains for us to show that $\beta_n = n\gamma_1 - (n-1)\gamma_0$ for $n = 1, 2, 3, \ldots$. To this end we consider the polynomial $g_n^*(x) \equiv x^n g_n(1/x)$ and note that $d\, g_n^*(x)/dx = n g_{n-1}^*(x)$. Thus, with the aid of the above representation for $g_n(x)$ and $g_{n-1}(x)$, we obtain

$$\frac{d}{dx}(1+x)^{n-1}(\gamma_0 x + \beta_n) = n(1+x)^{n-2}(\gamma_0 x + \beta_{n-1}).$$

Simplifying this expression yields the simple difference equation $(n-1)\beta_n = n\beta_{n-1} - \gamma_0$, $n = 2, 3, \ldots$, $\beta_1 = \gamma_1$, whose solution is $\beta_n = n\gamma_1 - (n-1)\gamma_0$, $n = 1, 2, 3, \ldots$. This completes the proof of the theorem.

With the aid of this theorem we are now in position to present several conditions which are equivalent to the betweenness property for a sequence Γ of real numbers.

COROLLARY 2.4. *Let $\Gamma = \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers. Then the following statements are equivalent.*

(i) *Γ possesses the betweenness property.*

(ii) *$\Phi(x) = \sum_{k=0}^{\infty}(\gamma_k/k!)x^k = [\gamma_0 + (\gamma_1 - \gamma_0)x]e^x$, where $\gamma_0(\gamma_1 - \gamma_0) \geqq 0$ and $\gamma_1 \neq \gamma_0$.*

(iii) *Γ is a nonconstant arithmetic sequence all of whose terms have the same sign.*

(iv) *For any real polynomial $f(x)$ with two consecutive real zeros a and b, $a < b$, $\Gamma[f(x)]$ has an odd (even) number of zeros between a and b if $ab \geqq 0$ ($ab < 0$).*

PROOF. We will show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). If Γ possesses the betweenness property, then by Theorem 2.3 and Remark 2.2, we know that Γ is a multiplier sequence of the first kind all of whose terms have the same sign and $\gamma_1 \neq \gamma_0$. Hence, by the transcendental characterization of these sequences ([7,] [6, Chapter 2]),

$\Phi(x) = \sum(\gamma_k/k!)x^k$ is a real entire function of order at most one with only real, nonpositive zeros. Furthermore, $\Phi(x) = \lim_{n\to\infty} g_n(x/n)$, where $g_n(x) = \sum_{k=0}^{n}\binom{n}{k}\gamma_k x^k$ and the convergence is uniform on compact subsets of the plane. Thus, by Theorem 2.3,

$$\begin{aligned}\lim_{n\to\infty} g_n\left(\frac{x}{n}\right) &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n-1}\left[\gamma_0 + \left(\gamma_1 - \left(1-\frac{1}{n}\right)\gamma_0\right)x\right] \\ &= [\gamma_0 + (\gamma_1-\gamma_0)x]e^x, \qquad \gamma_1 \neq \gamma_0, \\ &= \Phi(x).\end{aligned}$$

Since all the zeros of $\Phi(x)$ are nonpositive, we also have that $\gamma_0(\gamma_1-\gamma_0) \geqq 0$.

(ii) $\Rightarrow$ (iii). This implication is obvious since

$$\textstyle\sum_{k=0}^{\infty}\gamma_k x^k/k! = [\gamma_0 + (\gamma_1-\gamma_0)x]e^x$$

says that $\gamma_k = \gamma_0 + k(\gamma_1 - \gamma_0)$, $k = 0, 1, 2, \ldots$.

(iii) $\Rightarrow$ (iv). Suppose that $\Gamma = \{\gamma_k\}$ is a nonconstant arithmetic sequence all of whose terms $\gamma_k = \alpha + k\beta$, $\beta \neq 0$, $k = 0, 1, 2, \ldots$, have the same sign. Thus, we may assume without loss of generality, that $\beta = 1$ and $\Gamma = \{\alpha + k\}_{k=0}^{\infty}$, where $\alpha \geqq 0$. Now, let $f(x)$ be an arbitrary real polynomial with two consecutive zeros a and b, $a < b$. Then a straightforward calculation shows that

$$\begin{aligned}\Gamma[f(x)] &= \alpha f(x) + xf'(x), \qquad \alpha \geqq 0, \\ &= x^{-\alpha+1}\frac{d}{dx}[x^\alpha f(x)].\end{aligned}$$

Thus, by Rolle's theorem $\Gamma[f]$ has an odd number of zeros between a and b if $ab \geqq 0$; and $\Gamma[f]$ has an even number of zeros between a and b if $ab < 0$.

Since the implication (iv) $\Rightarrow$ (i) is clear, the proof of the corollary is complete.

Let Γ be a real sequence as in Corollary 2.4 and let a and b be two consecutive real zeros of a polynomial $f(x)$. Let c be a zero of $\Gamma[f]$ between a and b. By way of generalization of a theorem of Laguerre [**6**, p. 121], we will provide a sharp estimate of the relative position of c in the interval between a and b. The original theorem of Laguerre was valid only for polynomials with only real zeros. Our theorem generalizes a subsequent extension by Nagy [**5**].

THEOREM 2.5. *Let a and b denote two consecutive real zeros of the nth degree polynomial $f(x)$ with real coefficients and no zeros in the interior of the circle with diameter $[a, b]$. Let $\alpha \geqq 0$. If c is a zero of $F(x) = \alpha f(x) + xf'(x)$ between a and b $(a < c < b)$, then c satisfies the inequalities*

(2.6) $$a + \psi \leqq c \leqq b - \varphi,$$

where

$$\varphi \equiv \frac{b(\alpha+n+1) - a(\alpha+1) - sgn(c)([\alpha(b-a) + (n-1)b - a]^2 + 4(n-1)ab)^{1/2}}{2(\alpha + n)}$$

and

$$\psi \equiv \frac{b(\alpha+1) - a(\alpha+n+1) + sgn(c)([\alpha(b-a) + b - (n-1)a]^2 + 4(n-1)ab)^{1/2}}{2(\alpha + n)}.$$

Proof. We begin by assuming $c > 0$ and consider the upper limit. We also assume, for the moment, that $f(x)$ has only real zeros. We shall estimate the logarithmic derivative of $f(x)$; that is

$$\frac{f'(x)}{f(x)} = \sum_{j=1}^{n} \frac{1}{x - x_j}.$$

Let s_j ($j = 1, \ldots, m$; $m \leqq n - 1$) denote the zeros of $f(x)$ such that $s_j \leqq a$ and let t_j ($j = 1, \ldots, k$; $k \leqq n - 1$) denote the zeros of $f(x)$ such that $b \leqq t_j$. Assume that, contrary to the conclusion, we have

(2.7) $$b - \varphi < c < b.$$

Now a lengthy, but tedious calculation shows that $b - a - \varphi > 0$ (cf. Remark 2.10(i) below). Then the first inequality of (2.7) implies that $(c - a)^{-1} < (b - a - \varphi)^{-1}$. Moreover, for $j = 1, \ldots, m$, we have the estimate

(2.8) $$\frac{1}{c - s_j} \leqq \frac{1}{c - a} < \frac{1}{b - a - \varphi}.$$

Inequality (2.7) also yields

(2.9) $$\frac{1}{c - b} < -\frac{1}{\varphi}.$$

Thus, if $F(c) = 0$, then we have

$$\begin{aligned}
0 &= \alpha + c\frac{f'(c)}{f(c)} \\
&= \alpha + c\left[\sum_{j=1}^{m} \frac{1}{c - s_j} + \sum_{j=1}^{k} \frac{1}{c - t_j}\right] \\
&< \alpha + c\left[\frac{m}{b - a - \varphi} + \sum_{j=1}^{k} \frac{1}{c - t_j}\right] \qquad \text{(using (2.8))} \\
(2.10) \quad &\leqq \alpha + c\left[\frac{n-1}{b - a - \varphi} + \frac{1}{c - b}\right] \qquad (m \leqq n - 1; c - t_j < 0)
\end{aligned}$$

$$
\begin{aligned}
&< \alpha + c\left[\frac{n-1}{b-a-\varphi} - \frac{1}{\varphi}\right] \qquad \text{(using (2.9))} \\
&= \alpha - \frac{c(b-a-n\varphi)}{\varphi(b-a-\varphi)} \\
&\leqq \alpha - \frac{(b-\varphi)(b-a-n\varphi)}{\varphi(b-a-\varphi)}.
\end{aligned}
$$

The last inequality uses (2.7) and the fact that $(b - a - n\varphi)\varphi^{-1}(b - a - \varphi)^{-1}$ is nonnegative. This latter assertion is a consequence of an involved, but elementary, computation which shows that

$$\frac{(b-\varphi)(b-a-n\varphi)}{\varphi(b-a-\varphi)} = \alpha,$$

where $\alpha \geqq 0$ by assumption. Continuing the chain of inequalities, we thus have

$$0 < \alpha - \frac{(b-\varphi)(b-a-n\varphi)}{\varphi(b-a-\varphi)} = \alpha - \alpha = 0.$$

This is the desired contradiction.

The proof for the lower limit, $c \geqq a + \psi$, is accomplished in a similar manner. Assume, on the contrary that $a < c < a + \psi$, so that

$$\frac{1}{c - t_j} \geqq \frac{1}{c - b} > \frac{1}{a - b - \psi}$$

and also $\psi^{-1} < (c - a)^{-1}$, where we have used the fact that $a - b + \psi < 0$. Computing as before, we obtain, for $F(c) = 0$, the inequalities

$$
\begin{aligned}
0 &= \alpha + c\frac{f'(c)}{f(c)} \\
&= \alpha + c\left[\textstyle\sum_{j=1}^{m} \frac{1}{c - s_j} + \sum_{j=1}^{k} \frac{1}{c - t_j}\right] \\
&> \alpha + c\left[\frac{1}{c-a} + \frac{n-1}{a-b+\psi}\right] \\
&> \alpha + c\left[\frac{1}{\psi} + \frac{n-1}{a-b+\psi}\right] \\
&= \alpha + c\frac{a-b+n\psi}{\psi(a-b+\psi)} \\
&\geqq \alpha + \frac{(a+\psi)(a-b+n\psi)}{\psi(a-b+\psi)} = 0,
\end{aligned}
\tag{2.11}
$$

a contradiction as before.

Next, we allow f to have nonreal zeros on or outside the circle C with diameter $[a, b]$. Suppose $\gamma = u + iv$ and $\bar{\gamma} = u - iv$ are a conjugate pair of such zeros. In the expression for the logarithmic derivative in (2.10) we have the sum $(c - \gamma)^{-1} + (c - \bar{\gamma})^{-1}$. Since the imaginary parts cancel, we need only be concerned with the real parts, each of which equals $(c - u)/[(c - u)^2 + v^2]$. One easily checks that the absolute value of the reciprocal of this expression is the diameter of the circle C_γ passing through c, γ and $\bar{\gamma}$. Now assume that $u \leqq c$. Since γ is outside C and c is inside C, the circle C_γ contains the real point a, and hence the diameter of C_γ is greater than $c - a$. That is, $\operatorname{Re}(c - \gamma)^{-1} < (c - a)^{-1} < (b - a - \varphi)^{-1}$ and inequality (2.10) continues to hold under the assumption (2.7). If $u > c$, the circle C_γ contains the point b. Similar considerations then show that $\operatorname{Re}(c - \gamma)^{-1} < (c - b)^{-1}$ and again (2.10) holds. This same argument can be used in (2.11) to establish the lower bound for c when f has nonreal zeros.

Finally, if $c < 0$, we may apply the case already proved to $g(x) = f(-x)$ with root $-c > 0$ between $-b$ and $-a$. This yields the desired conclusion and ends the proof of the theorem.

Remark 2.10. (i). The limits given in Theorem 2.5 are best possible. The upper limit is taken on when $f(x) = (x - a)^{n-1}(x - b)$ and the lower limit is taken on when $f(x) = (x - a)(x - b)^{n-1}$.

(ii). Setting $\alpha = 0$, we recover the theorem of Laguerre [**6**, p. 121] with $\varphi = \psi = (b - a)/n$.

(iii). When $\alpha \neq 0$, the formulas for φ and ψ are complicated mainly because, unlike differentiation, our operators Γ are, in general, not translation invariant. The formulas do simplify significantly if either a or b is zero. For example, if $a = 0$, then $\varphi = b/(\alpha + n)$ and $\psi = b(\alpha + 1)/(\alpha + n)$.

Remark 2.11. In the case that the polynomial $f(z)$ has complex coefficients, the problem of obtaining sharp estimates on the location of zeros of $f'(z)$ or $\Gamma[f]$ seems to be very difficult (cf. [**4**]). On the other hand, we have recently shown that the Gauss-Lucas Theorem remains valid if in that theorem $f'(z)$ is replaced by $\Gamma[f]$, where Γ is an increasing multiplier sequence [**3**].

3. Open problems. The foregoing results raise several problems in the theory of distribution of zeros of polynomials. We will now present a few of these questions together with some comments.

Problem 3.1. Let $\mathbf{R}[x]$ denote the vector space of all real polynomials. Characterize all linear transformations $L: \mathbf{R}[x] \to \mathbf{R}[x]$ which possess the betweenness property.

Comment. Corollary 2.4 shows that if $h(x) \in \mathbf{R}[x]$ and if $\theta = xD$, then the linear transformation $h(\theta)$ has the betweenness property if and only if $h(x)$ has the form $h(x) = \alpha + \beta x$, $\alpha\beta \geqq 0$, $\beta \neq 0$. On the other hand, if we do not restrict ourselves to linear transformations defined by (1.1), then it is clear that there are many other linear transformations which enjoy the betweenness property.

PROBLEM 3.2. Let Γ be a multiplier sequence of the first kind. Describe the location of the real zeros of the polynomial $\Gamma[f(x)]$ in terms of the real zeros of $f(x)$.

COMMENT. In Theorem 2.5 we have only provided a solution to this problem in the special case when Γ is an arithmetic sequence.

While the linear transformations $D = d/dx$ and Γ, where Γ is a multiplier sequence of the first kind, have many properties in common (see, for example, [**1**]), there are some significant dissimilarities between these operators. Indeed, the differential operator D is translation invariant, that is, $Df(x + \alpha) = (Df)(x + \alpha)$, for any scalar α. Translation leads to many complications for the operator Γ as we have seen in Theorem 2.5.

PROBLEM 3.3. For a fixed, but arbitrary, multiplier sequence Γ of the first kind and a real polynomial $f(x)$, determine the number of real zeros of $\Gamma[f(x + \alpha)]$ as a function of α.

COMMENT. If $\Gamma = \{\gamma_k\}$ is a multiplier sequence of the first kind such that the entire function $\Gamma[e^x] = \sum_{k=0}^{\infty} \gamma_k x^k/k!$ has an infinite number of real zeros, then it is known [**1**, Theorem 6] that there is a constant K depending on Γ and f such that, for all real α with $|\alpha| > K$, the polynomial $\Gamma[f(x + \alpha)]$ has only real zeros. For related results see also [**1**, Corollary 15] and [**2**, Theorem 4.6].

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