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ORTHOGONAL POLYNOMIALS AND MEASURES WITH END POINT MASSES

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ABSTRACT. Let a sequence of orthogonal polynomials with respect to a given measure $d\psi(x)$ be explicitly known and let a new measure $d\psi^*(x)$ be constructed from $d\psi(x)$ by adjoining a positive mass at one point. When $d\psi(x)$ corresponds to one of the classical orthogonal polynomials of Jacobi, Hermite or Laguerre, the orthogonal polynomials relative to $d\psi^*(x)$ have been found by H. L. Krall and others. Here we consider general $d\psi(x)$ and obtain formulas for constructing the polynomials associated with $d\psi^*(x)$. A number of nonclassical examples are explicitly given.

1. Introduction. There has been a renewal of interest in the question of orthogonal polynomial solutions to linear differential equations. Much of the recent work can be considered a continuation of the work of Bochner [3] and his characterization of the classical orthogonal polynomials as the only orthogonal polynomials which are eigensolutions of $Ly = \lambda_n y$, where L is a second order linear differential operator with polynomial coefficients independent of n. H. L. Krall [11], [12] extended Bochner's work to the fourth order case and found three new sequences of orthogonal polynomials. The spectral measures for two of these could be obtained from the measures for the Laguerre and certain special Jacobi polynomials, respectively, by adjoining mass at one end of the spectral intervals. The third measure was obtained from the Legendre measure by adjoining equal masses at each end of the spectral interval. The three sets of polynomials have been studied in some detail by A. M. Krall [10].

Littlejohn [13], [14], [15] has continued this study of $Ly = \lambda_n y$. He has found a 6th order case which has orthogonal polynomial solutions. These polynomials are orthogonal with respect to the measure obtained from the Legendre measure by adjoining unequal masses at each end of the spectral interval. Koornwinder [9] generalized these results by obtaining explicitly the polynomials which are orthogonal with respect to the measure obtained from the most general Jacobi measure by adjoining arbitrary masses at both ends of the spectral interval. Koornwinder also indicates that these polynomials satisfy an 8th order case of $Ly = \lambda_n y$. Hendriksen

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and van Rossum [8] have recently studied polynomials of the above types from the viewpoint of Padé tables and shown they all satisfy certain 2nd order differential equations. For a general theory of such differential equations with orthogonal polynomial solutions, see Hahn [7]. See also [2], [16], [17], [19].

In this paper, we show that if one begins with a known system of orthogonal polynomials (not necessarily a classical one) and if mass is added at one end of the spectral interval, the corresponding orthogonal polynomials frequently can be found explicitly independently of any differential equations. We will obtain general formulas which will permit us to obtain explicitly the orthogonal polynomials for several nonclassical examples. Our methods also lead to formulas which, in theory, would handle the case of masses added at both ends of a spectral interval. However, as we will see, in this case the formulas become so complicated that they hardly seem useable. A rather interesting example will be that involving the generalized Stieltjes-Wigert polynomials. This is due to the fact that the Hamburger moment problem associated with these polynomials is indeterminate.

2. Preliminaries. Let ψ be a distribution function with an infinite spectrum on an interval $[a, b]$. Since we wish to add mass at one end of the spectrum, we take a to be finite and then there is no loss of generality if we further assume that $a \geqq 0$. Let $\{P_n(x)\}$ be the sequence of monic orthogonal polynomials with respect to $d\psi(x)$ on $[a, b]$ and let it satisfy the three term recurrence relation

$$(2.1)\qquad \begin{aligned} &P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \qquad n \geqq 1,\\ &P_{-1}(x) = 0,\ P_0(x) = 1, \qquad c_n \text{ real},\ \lambda_n > 0. \end{aligned}$$

Next let $\{Q_n(x)\}$ denote the corresponding sequence of monic kernel polynomials; that is, the $Q_n(x)$ are the monic polynomials that are orthogonal over $[0, b]$ with respect to the measure $x\,d\psi(x)$. Let the corresponding three term recurrence relation be

$$(2.2)\qquad \begin{aligned} &Q_n(x) = (x - d_n)Q_{n-1}(x) - \nu_n Q_{n-2}(x), \qquad n \geqq 1,\\ &Q_{-1}(x) = 0,\ Q_0(x) = 1, \qquad d_n \text{ real},\ \nu_n > 0. \end{aligned}$$

Referring to [4], [5], we now recall some relations among these polynomials.

Relating the coefficients in the two recurrence relations, there are positive constants γ_n $(n > 1)$ such that

$$(2.3)\qquad c_n = \gamma_{2n-1} + \gamma_{2n},\ \lambda_{n+1} = \gamma_{2n}\gamma_{2n+1}, \qquad (\gamma_1 = 0)$$

$$(2.4)\qquad d_n = \gamma_{2n} + \gamma_{2n+1},\ \nu_{n+1} = \gamma_{2n+1}\gamma_{2n+2}, \qquad n \geqq 1.$$

The two sequences of orthogonal polynomials are additionally related by

(2.5) $$xQ_{n-1}(x) = P_n(x) + \gamma_{2n}P_{n-1}(x)$$

(2.6) $$P_n(x) = Q_n(x) + \gamma_{2n+1}Q_{n-1}(x)$$

and

(2.7) $$\gamma_{2n} = -\frac{P_n(0)}{P_{n-1}(0)}.$$

Let $\|\cdot\|$ denote the L^2 norm relative to the measure $d\psi$. From the recurrence formula (2.1) we have

(2.8) $$\|P_n\|^2 = \int P_n^2(x)d\psi(x) = \lambda_1\lambda_2 \cdots \lambda_{n+1}.$$

For the norm of $Q_n(x)$ relative to $xd\psi(x)$,

$$\|Q_n\|_0^2 = \int Q_n^2(x)\, xd\psi(x) = \nu_1\nu_2 \cdots \nu_{n+1},$$

we use (2.5) to get

(2.9) $$\begin{aligned}\|Q_n\|_0^2 &= \int P_{n+1}(x)x^n d\psi(x) + \gamma_{2n+2}\int P_n(x)x^n d\psi(x)\\ &= \gamma_{2n+2}\,\|P_n\|^2.\end{aligned}$$

Finally, we recall that the corresonding sequence $\{S_n(x)\}$ of symmetric polynomials defined by

(2.10) $$S_{2n}(x) = P_n(x^2), \quad S_{2n+1}(x) = xQ_n(x^2)$$

satisfies the recurrence relation

(2.11) $$S_n(x) = xS_{n-1}(x) - \gamma_n S_{n-2}(x).$$

These polynomials are orthogonal over $[-\sqrt{b}, \sqrt{b}]$ with respect to $d\varphi(x)$, where

(2.12) $$\varphi(x) = \begin{cases} \psi(x^2) - \psi(0^-) & x > 0\\ 0 & x = 0\\ -\psi(x^2) + \psi(0^-) & x < 0.\end{cases}$$

We note also that if we set

(2.13) $$\mu_n = \gamma_{2n+1}, \lambda_n = \gamma_{2n+2}, \qquad n \geqq 0$$

(λ_n has different meaning here than in (2.1) of course), then

(2.14) $$\Phi_n(x) = (-1)^n[\lambda_0\lambda_1 \cdots \lambda_{n-1}]^{-1} P_n(x)$$

satisfies the recurrence relation

$$-x\,\Phi_n(x) = \mu_n\Phi_{n-1}(x) - (\lambda_n + \mu_n)\Phi_n(x) + \lambda_n\Phi_{n+1}(x) \tag{2.15}$$

corresponding to a birth and death process with reflecting barrier at 0.

On the other hand, if we set

$$\mu_n = \gamma_{2n+2},\ \lambda_n = \gamma_{2n+3}, \qquad n \geqq 0, \tag{2.16}$$

then

$$\Phi_n(x) = (-1)^n[\mu_0\mu_1 \cdots \mu_{n-1}]\, Q_n(x) \tag{2.17}$$

satisfies (2.15) which then corresponds to a birth and death process with absorption at -1.

3. Adding mass at 0. Although in some examples, the distribution function will be continuous at the origin, we will consider others where ψ has a jump there. Hence we let $J_0 \geqq 0$ denote the jump of ψ at the origin. We then consider the measure obtained from $d\psi$ by adding a mass $J \geqq -J_0$ at 0.

Let $P_n^*(x) = P_n^*(x;0,J)$ denote the monic polynomials that are orthogonal with respect to the measure

$$d\psi^*(x;0,J) = d\psi(x) + J\delta(x)dx, \qquad J \geqq -J_0. \tag{3.1}$$

Let the three term recurrence relation satisfied by the $P_n^*(x)$ be

$$P_n^*(x) = (x - c_n^*)P_{n-1}^*(x) - \lambda_n^* P_{n-1}^*(x). \tag{3.2}$$

Now, if $Q_n(x)$ still denotes the kernel polynomials of §2, it is clear they are also the kernel polynomials corresponding to the polynomials $P_n^*(x;0,J)$. Thus, in addition to (2.5) and (2.6), we also have

$$xQ_{n-1}(x) = P_n^*(x) + \gamma_{2n}^* P_{n-1}^*(x) \tag{3.3}$$

$$P_n^*(x) = Q_n(x) + \gamma_{2n+1}^* Q_{n-1}(x) \tag{3.4}$$

where the coefficients γ_n^* are determined by $\gamma_1^* = 0$ and

$$c_n^* = \gamma_{2n-1}^* + \gamma_{2n}^*,\ \lambda_{n+1}^* = \gamma_{2n}^*\gamma_{2n+1}^* \tag{3.5}$$

(cf. (2.3)). They are related to the γ_k by means of (2.4) and the relations

$$d_n = \gamma_{2n}^* + \gamma_{2n+1}^*,\ \nu_{n+1} = \gamma_{2n+1}^*\gamma_{2n+2}^*. \tag{3.6}$$

Of course, we also have

$$\gamma_{2n}^* = -\frac{P_n^*(0)}{P_{n-1}^*(0)}.$$

From (3.3) and (2.6) we also obtain

$$P_n^*(x) = P_n(x) + A_n Q_{n-1}(x), \tag{3.7}$$

where

(3.8) $$A_n = \gamma_{2n-1}^* - \gamma_{2n+1}.$$

Using (2.6), we can rewrite (3.7) in the form

(3.9) $$P_n^*(x) = Q_n(x) + \gamma_{2n+1}^* Q_{n-1}(x).$$

Next we notice that

(3.10) $$\int \pi(x)d\psi^*(x) = \int \pi(x)d\psi(x) + J\pi(0).$$

Therefore we obtain from (3.7) and the orthogonality of the $P_n(x)$,

$$\int P_n^*(x)P_{n-1}(x)d\psi^*(x) - JP_n^*(0)P_{n-1}(0) = A_n \int Q_{n-1}(x)P_{n-1}(x)d\psi(x)$$
$$= A_n\|P_{n-1}\|^2$$

(since $Q_{n-1}(x)$ is monic). Thus, by the orthogonality of the $P_n^*(x)$,

$$A_n = \frac{-JP_n^*(0)P_{n-1}(0)}{\|P_{n-1}\|^2}.$$

Setting $x = 0$ in (3.7), we then obtain from the last formula,

(3.11) $$P_n^*(0) = \frac{P_n(0)\|P_{n-1}\|^2}{\|P_{n-1}\|^2 + JP_{n-1}(0)Q_{n-1}(0)}.$$

This then yields, finally,

(3.12) $$A_n = \frac{-JP_n(0)P_{n-1}(0)}{\|P_{n-1}\|^2 + JP_{n-1}(0)Q_{n-1}(0)}.$$

Next using (3.7), we obtain

$$\int P_n^*(x)x^n d\psi^*(x) = \int P_n(x)x^n d\psi^*(x) + A_n \int Q_{n-1}(x)x^{n-1}x d\psi^*(x).$$

Then using (3.10) and (2.9), we can write the appropriate L^2 norm of $P_n^*(x)$ as

(3.13) $$\int [P_n^*(x; 0, J)]^2 d\psi^*(x; 0. J) = \|P_n\|^2 + A_n\gamma_{2n}\|P_{n-1}\|^2.$$

Finally, for the coefficients in the recurrence formula (3.2), we obtain from (3.8), (3.6) and (2.4),

(3.14) $$\gamma_{2n}^* = \gamma_{2n} - A_n, \quad \gamma_{2n+1}^* = \gamma_{2n+1} + A_n.$$

Using (2.7) and (3.12), the first of these can be rewritten

$$\gamma_{2n}^* = \gamma_{2n}\frac{\|P_{n-1}\|^2 + JP_{n-1}(0)[Q_{n-1}(0) - P_{n-1}(0)]}{\|P_{n-1}\|^2 + JP_{n-1}(0)Q_{n-1}(0)}.$$

From (2.8), (2.3) and (2.6), we have

$$\|P_{n-1}\|^2 = \|P_{n-2}\|^2 \gamma_{2n-2}\gamma_{2n-1}, \qquad n \geqq 2,$$
$$Q_{n-1}(0) - P_{n-1}(0) = -\gamma_{2n-1}Q_{n-2}(0).$$

Therefore, for $n \geqq 2$,

$$\gamma_{2n}^* = \gamma_{2n-2}\gamma_{2n-1}\gamma_{2n}\frac{\|P_{n-2}\|^2 + JP_{n-2}(0)Q_{n-2}(0)}{\|P_{n-1}\|^2 + JP_{n-1}(0)Q_{n-1}(0)}.$$

We also have, according to (3.6) and (2.4), $\gamma_{2n+1}^* = \nu_{n+1}/\gamma_{2n+2}^* = \gamma_{2n+1}\gamma_{2n+2}/\gamma_{2n+2}^*$. Thus if we let

(3.15) $$\theta_n = \|P_n\|^2 + JP_n(0)Q_n(0),$$

we can write

(3.16)
$$\gamma_2^* = \gamma_2/(1+J)$$
$$\gamma_{2n+1}^* = -\frac{\theta_n}{\gamma_{2n}\theta_{n-1}}$$
$$\gamma_{2n+2}^* = -\gamma_{2n}\gamma_{2n+1}\gamma_{2n+2}\frac{\theta_{n-1}}{\theta_n}, \qquad n \geqq 1.$$

The recurrence formula for $\{P_n^*(x; 0, J)\}$ can now be written using (3.5).

Also, corresponding to (2.10) – (2.12), (3.16) can be used to construct the related symmetric OPS, while, corresponding to (2.13) – (2.17), the recurrence relations in birth and death process form can be written.

REMARK. The orthogonal polynomials $P_n^*(x; 0, J)$ can also be obtained, in theory, by considering the parameters of the chain sequence associated with the kernel polynomials $Q_n(x)$. According to [**4**], if we are given (2.3) and (2.4) and if we set

(3.18) $$\beta_n = \nu_{n+1}/(d_n d_{n+1}), \qquad n \geqq 1,$$

then $\{\beta_n\}$ is a chain sequence that does not determine its parameters uniquely. That is, there is a maximal initial parameter $M_0 > 0$ such that for each h_0, $0 \leqq h_0 \leqq M_0$, there is a corresponding parameter sequence $h = \{h_n\}$ for which

(3.19) $$\beta_n = (1 - h_{n-1})h_n, \qquad n = 1, 2, 3, \ldots$$

Corresponding to each $h_0 > 0$, the sequence $\{\gamma_n^h\}$ can be defined by

(3.20) $$\gamma_{2n}^h = h_{n-1}d_n, \; \gamma_{2n+1}^h = (1 - h_{n-1})d_n, \qquad n \geqq 1.$$

The coefficients in (3.2) can then be defined by (3.5) with $\gamma_n^* = \gamma_n^h$. If $h = M = \{M_n\}$, then the resulting polynomials $P_n^M(x)$ defined by (3.2) will be orthogonal with respect to a measure $d\psi^M(x)$ where ψ^M is the solution of a determined Hamburger moment problem and is continuous at the origin. For each positive $h_0 < M_0$, the corresponding polynomials

$P_n^h(x)$, defined by (3.2), will then be orthogonal with respect to a measure of the form (3.1) with $\psi = \psi^M$. The jump J which is added to ψ^M to obtain $\psi^h = \psi^*$ is given by

$$J = [\psi(\infty) - \psi(-\infty)]\,(M_0 - h_0)/h_0. \tag{3.21}$$

In practice, however, the above cannot be applied because usually more than one or two different parameter sequences for a given chain sequence can not be found explicitly, hence the polynomials corresponding to different jumps cannot be explicitly found. (An exception, corresponding to Tchebichef polynomials, is given in [**4**, p. 7].) However, when the formulas of this section can be explicitly computed, one can reverse the above viewpoint and use (3.19), (3.20) to explicitly find all parameter sequences for $\{\beta_n\}$. Also, when the Hamburger moment problem associated with the original orthogonal polynomial sequence is indeterminate, these formulas lead to a rather interesting development (see the example of the generalized Stieltjes-Wigert polynomials in §6).

4. Adding mass at $d \geqq b$. The case where the measure is altered by adding mass at a point $d \geqq b$ can be handled by applying the method of §3 after a reflection and translation of the interval of orthogonality. Assume b is finite, $d \geqq b$, and consider the monic polynomials $P_n^*(x; d, H)$ that are orthogonal with respect to the measure

$$d\psi^*(x; d, H) = d\psi(x) + H\delta(x - d)dx. \tag{4.1}$$

Let us now introduce the following notation. We write

$$\pi_n(x) \sim d\varphi(x) \tag{4.2}$$

to mean $\{\pi_n(x)\}$ is the sequence of monic orthogonal polynomials with respect to the distribution $d\varphi(x)$. Thus, for example, with this notation we have

$$\begin{gathered} P_n(x) \sim d\psi(x), \quad P_n^*(x) \sim d\psi(x) + J\delta(x)dx, \quad Q_n(x) \sim xd\psi(x), \\ P_n^*(x; d, H) \sim d\psi(x) + H\delta(x-d)dx \end{gathered} \tag{4.3}$$

Thus $(-1)^n P_n^*(d - x; d, H) \sim -d\psi(d - x) + H\delta(x)dx$ and, therefore, if

$$R_n(x) \sim (d - x)d\psi(x), \tag{4.4}$$

$(-1)^n R_n(d - x) \sim xd\psi(d - x)$. So, by (3.7) and (3.11),

then

$$\begin{gathered} (-1)^n P_n^*(d-x; d, H) = (-1)^n P_n(d-x) + B_n(-1)^{n-1} R_{n-1}(d-x) \\ P_n^*(x; d, H) = P_n(x) - B_n R_{n-1}(x), \end{gathered} \tag{4.5}$$

where

$$B_n = \frac{HP_n(d)P_{n-1}(d)}{\|P_{n-1}\|^2 + HP_{n-1}(d)R_{n-1}(d)}. \tag{4.6}$$

Since $(-1)^n R_n(d - x)$ is related to $(-1)^n P_n(d - x)$ in the same way that $Q_n(x)$ is related to $P_n(x)$, formulae corresponding to (2.1) - (2.4) can easily be written. In particular, we have the recurrence relation

$$R_n(x) = (x - f_n)R_{n-1}(x) - \rho_n R_{n-2}(x), \tag{4.7}$$

where $f_n = c_{n+1} + \delta_{2n} - \delta_{2n+2}$ and $\rho_{n+1} = \lambda_{n+1}\delta_{2n+2}/\delta_{2n}$. Also, corresponding to (2.5), we have

$$(x - d)R_{n-1}(x) = P_n(x) - \delta_{2n}P_{n-1}(x), \tag{4.8}$$

so that

$$\delta_{2n} = \frac{P_n(d)}{P_{n-1}(d)}. \tag{4.9}$$

Using (4.8), we then obtain

$$\int (x - y)R_{n-1}^2(x)d\psi(x) = -\delta_{2n}\int P_{n-1}(x)R_{n-1}(x)d\psi(x)$$

hence

$$\int R_n^2(x)(d - x)d\psi(x) = \|P_n\|^2 \delta_{2n+2}. \tag{4.10}$$

5. Two additional masses. We now suppose that a and b are both finite and adjoin to ψ a jump J at $0 \leqq a$ and a jump H at $d \geqq b$. Thus we consider the distribution

$$d\psi^\#(x; 0, J; d, H) = d\psi(x) + [J\delta(x) + H\delta(x - d)]dx, \tag{5.1}$$

and the corresponding monic orthogonal polynomials

$$P_n^\#(x; 0, J; d, H) \sim d\psi^\#(x; 0, J; d, H). \tag{5.2}$$

Since $d\psi^\#(x; 0, J; d, H) = d\psi^*(x; 0, J) + H\delta(x - d)dx$, we have, by (4.5) and (4.6),

$$P_n^\#(x; 0, J; d, H) = P_n^*(x; 0, J) - C_nG_{n-1}(x), \tag{5.3}$$

where

$$G_n(x) \sim (d - x)d\psi^*(x) = (d - x)d\psi(x) + dJ\,\delta(x)dx \tag{5.4}$$

and

$$\begin{aligned} C_n &= \frac{HP_n^*(d; 0, H)P_{n-1}^*(d; 0, H)}{f_{n-1}^2 + HP_{n-1}^*(d; 0, H)G_{n-1}(d)} \\ f_n^2 &= \int [P_n^*(x; 0, J)]^2 d\psi^*(x; 0, J), \end{aligned} \tag{5.5}$$

and the latter is given by (3.13).

We next let

$$T_n(x) \sim x(d - x)d\psi(x) \tag{5.6}$$

so that the $T_n(x)$ are the kernel polynomials corresponding to the $R_n(x)$ (given by (4.4)). Then, corresponding to (3.7) and (3.12),

$$G_n(x) = R_n^*(x; 0, dJ) = R_n(x) + D_n T_{n-1}(x), \tag{5.7}$$

where

$$\begin{aligned} D_n &= \frac{-dJR_n(0)R_{n-1}(0)}{g_{n-1}^2 + dJR_{n-1}T_{n-1}(0)} \\ g_n^2 &= \int R_n^2(x)(d - x)d\psi(x) = \|P_n\|^2 \delta_{2n+2} \end{aligned} \tag{5.8}$$

(see (4.10)).

The $T_n(x)$ are related to the $R_n(x)$ and the $Q_n(x)$ by formulas corresponding to (2.5), (2.6) (and others) and a number of formulas and recurrence relations could be written. However, since we will not make use of them, we will omit them and simply note in summary that we now have

$$P_n(x; 0, J; d, H) = P_n(x) + A_n Q_n(x) - C_n R_{n-1}(x) - C_n D_{n-1} T_{n-2}(x) \tag{5.9}$$

where A_n and D_n are given by (3.11) and (5.8), respectively, and

$$\begin{aligned} C_n &= \frac{HP_n^*(d; 0, J)P_{n-1}^*(d; 0, J)}{f_{n-1}^2 + HP_{n-1}^*(d; 0, J)\, R_{n-1}^*(d; 0, dJ)} \\ f_n^2 &= \|P_n\|^2 - \|P_{n-1}\|^2 A_n P_n(0)/P_{n-1}(0). \end{aligned} \tag{5.10}$$

In practice, the coefficients A_n, C_n and D_n will be so complicated that it is unlikely (5.9) will be of any use. Also, in theory, coefficients for the recurrence formula for these polynomials could be written but these are even worse and the dearth of specific examples to which such formulas could be applied is motivation enough not to inflict these on the reader.

6. Examples. We will here present a number of examples. We will maintain the generic notation of the preceding sections so that $P_n(x)$ denotes the monic polynomials orthogonal over a subset of $[0, \infty)$ with respect to a measure $d\psi(x)$. (We will renormalize, if necessary, so as to have total mass 1). Also $Q_n(x)$ will denote the corresponding monic kernel polynomials. We will determine the coefficients γ_n^* (using (3.16)) and obtain $P_n^*(x; 0, J)$ in the form (3.9). Because of (3.5), the recurrence formulas for these polynomials are thus known but we will not write them out. In some cases we will also give the parameters of the chain sequence $\{\beta_n\}$ (3.18).

A). Laguerre type polynomials. We take in standard notation [**20**]

$$d\psi(x) = [\Gamma(\alpha + 1)]^{-1}x^{\alpha}e^{-x}dx, \qquad 0 \leqq x < \infty$$

$$P_n(x) = (-1)^n n! L_n^{\alpha}(x), \quad Q_n(x) = (-1)^n n! L_n^{\alpha+1}(x).$$

Using well known formulas for $L_n^{\alpha}(x)$, we find

$$P_n(0) = (-1)^n(\alpha + 1)_n, \quad \|P_n\|^2 = n!(\alpha + 1)_n,$$

and for the coefficients in the recurrence formula (2.1), we have, corresponding to (2.3), $\gamma_{2n} = n + \alpha$ and $\gamma_{2n+1} = n$, for $n \geqq 1$. Therefore, from (3.16), we obtain

$$\begin{aligned} &\gamma_2^* = (1 + \alpha)/(1 + J), \\ &\gamma_{2n+1}^* = b_n/b_{n-1}, \gamma_{2n+2}^* = n(n + \alpha + 1)b_{n-1}/b_n, \qquad n \geqq 1 \end{aligned} \tag{6.1}$$

where $b_n = n! + J(\alpha + 2)_n$. Thus, for the measure

$$d\psi^*(x; 0, J) = \{[\Gamma(\alpha + 1)]^{-1}x^{\alpha}e^{-x} + J\delta(x)\}dx, \tag{6.2}$$

the corresponding orthogonal polynomials are

$$P_n^*(x\colon 0, J) = (-1)^n n! \left[L_n^{\alpha+1}(x) - \frac{n! + J(\alpha + 2)_n}{n! + Jn(\alpha + 2)_{n-1}} L_{n-1}^{\alpha+1}(x) \right] \tag{6.3}$$

which is in agreement with the formula obtained by Koornwinder [9]. The corresponding recurrence formula (3.2) can now be written using the above and (3.5). (Note that we could also now write formulas for the polynomials orthogonal with respect to the weight function obtained from the Hermite weight function by adding mass at the origin.)

Also, referring to the remark at the conclusion of §3, we obtain

$$h_0 = \frac{\alpha + 1}{(\alpha+2)(1+J)}, \quad h_n = \frac{(n+\alpha+1)[n! + Jn(\alpha+2)_{n-1}]}{(2n+\alpha+2)[n! + J(\alpha+2)_n]}, \qquad n \geqq 1 \tag{6.4}$$

as the non-minimal parameters for the chain sequence $\{\beta_n\}$, where

$$\beta_n = \frac{n(n + \alpha + 1)}{(2n + \alpha)(2n + \alpha + 2)}, \tag{6.5}$$

for each positive $h_0 \leqq M_0 = (\alpha + 1)/(\alpha + 2)$.

B). Charlier type polynomials. We take (for notation, see [5])

$$d\psi(x) = e^{-a} \sum_{k=0}^{\infty} \frac{a^k}{k!} \delta(x - k)dx$$

$$P_n(x) = C_n^{(a)}(x) = (-1)^n a^n c_n^{(a)}(x), \quad Q_n(x) = C_n^{(a)}(x - 1).$$

Using known formulas for the Charlier polynomials, we find

$$P_n(0) = (-a)^n, \quad \|P_n\|^2 = a^n n!, \quad Q_n(0) = (-1)^n n! e_n(a),$$

(6.6)
$$e_n = e_n(a) = \sum_{k=0}^{n} a^k/k!$$

$$\gamma_{2n} = a, \ \gamma_{2n+1} = n.$$

Note that the jump of ψ at 0 is e^{-a}. We obtain, from (3.16),

(6.7) $$\gamma_2^* = a/(1 + J), \gamma_{2n+1}^* = nb_n/b_{n-1}, \gamma_{2n+2}^* = ab_{n-1}/b_n,$$

where $b_n = 1 + Je_n$, $J \geqq -e^{-a}$. Thus corresponding to the measure $d\psi^*(x; 0, J)$, we have the orthogonal polynomials

(6.8) $$P_n^*(x; 0, J) = C_n^{(a)}(x - 1) + \frac{n(1 + Je_n)}{1 + Je_{n-1}} C_{n-1}^{(a)}(x - 1).$$

Corresponding to the chain sequence $\{\beta_n\} = \{an/[(n + a)(n + a + 1)]\}$, we have the noniminimal parameters

(6.9) $$h_n = \frac{a(1 + Je_{n-1})}{(n + a + 1)(1 + Je_n)}, \qquad n \geqq 0, e_{-1} = 0, J \geqq -e^{-a}.$$

The h_n become maximal parameters when $J = -e^{-a}$ (i.e., when the corresponding distribution function is continuous at 0).

C). Meixner polynomials of the first kind. With the notation of **[5]**, we take

$$d\psi(x) = (1 - c)^\beta \sum_{k=0}^{\infty} \frac{(\beta)_k c^k}{k!} \delta(x)dx$$

$$P_n(x) = (-a)^n m_n(x; \beta, c), Q_n(x) = (-a)^n m_n(x-1; \beta+1, c), a = c/(1-c).$$

From the known formulas for $m_n(x; \beta, c)$, we find

$$P_n(0) = (-a)^n(\beta)_n, \quad \|P_n\|^2 = a^{2n}c^{-n}n!(\beta)_n, \quad Q_n(0) = (-a)^n c^{-n} n!\, f_n,$$

where

(6.10)
$$f_n = \sum_{k=0}^{n} \frac{(\beta)_k c^k}{k!},$$

$$\gamma_{2n} = a(n + \beta - 1), \gamma_{2n+1} = an/c.$$

For the polynomials corresponding to $d\psi^*(x; 0, J)$, we then obtain

(6.11) $$\gamma_2^* = a\beta/(1 + J), \gamma_{2n+1}^* = anb_n/(cb_{n-1}), \gamma_{2n+2}^* = a(n + \beta)b_{n-1}/b_n$$

where, referring to (6.10), $b_n = 1 + Jf_n$, $J \geqq -(1 - c)^{-\beta}$.

(6.12)
$$P_n^*(x; 0, J) = (-a)^n\left[m_n(x-1; \beta+1, c) - \frac{n(1+Jf_n)}{c(1+Jf_{n-1})} m_{n-1}(x-1; \beta+1, c)\right].$$

Corresponding to

$$(6.13)\qquad \beta_n = \frac{cn(n+\beta)}{[(1+c)n + c(\beta-1)][(1+c)(n+1)+c(\beta-1)]},$$

we have the parameters

$$(6.14)\qquad h_n = \frac{c(n+\beta)}{(1+c)n + c\beta + 1}\cdot\frac{1+Jf_{n-1}}{1+Jf_n},\qquad f_{-1}=0,$$

with the maximal parameters occuring when $J = -(1-c)^{-\beta}$.

D). Generalized Stieltjes-Wigert type polynomials. The generalized Stieltjes-Wigert polynomials [**5**] are also known as q-Laguerre polynomials (see [**1**], [**18**]). This is, in many respects, our most interesting example because the associated Hamburger and Stieltjes moment problems are both indeterminate. Thus there are infinitely many nonequivalent measures on $[0, \infty)$ with respect to which these polynomials are orthogonal and many of these are explicitly known [**1**], [**6**], [**18**]. However, referring to the remarks at the conclusion of §3, we also know that there exists a distribution function ψ^M which is the solution of a determined Hamburger moment problem and such that the generalized Stieltjes-Wigert polynomials are orthogonal with respect to

$$(6.15)\qquad d\psi(x) = d\psi^M(x) + J_0\delta(x)dx,$$

for some positive jump J_0. Although ψ^M is unknown, we will describe the polynomials $P_n^*(x;\,0,\,J)$ relative to the polynomials orthogonal with respect to $d\psi^M(x)$.

With the notation of [**5**, p. 174], we have

$$P_n(x) = S_n(x;p,q),\quad Q_n(x) = q^{-n}S_n(qx;pq,q)$$

$$P_n(0) = (-1)^n q^{-n(2n+1)/2}[p]_n,\quad Q_n(0) = (-1)^n q^{-n(2n+3)/2}[pq]_n$$

$$\|P_n\|^2 = [p]_n[q]_n q^{-2n(n+1)},$$

where $[a]_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$.
Also, from the recurrence formula, we have $\gamma_{2n} = (1-pq^{n-1})q^{-2n+1/2}$, $\gamma_{2n+1} = (1-q^n)q^{-2n/2}$, whence

$$(6.16)\qquad \begin{aligned} &\gamma_2^* = (1-p)q^{-3/2}/(1+J)\\ &\gamma_{2n+1}^* = q^{-2n-1/2}\frac{b_n}{b_{n-1}},\quad \gamma_{2n+2}^* = (1-pq^n)(1-q^n)q^{-2n-3/2}\frac{b_{n-1}}{b_n}\end{aligned}$$

where $b_n = [q]_n + J[pq]_n$. The polynomials which are orthogonal with respect to

$$(6.17)\qquad d\psi^*(x;0,J) = d\psi(x) + J\delta(x)dx = d\psi^M(x) + (J+J_0)\delta(x)dx$$

are thus

(6.18) $P_n^*(x;0,J)=q^{-n}[S_n(qx;pq,q)+(b_n/b_{n-1})q^{-2n+1/2}S_{n-1}(qx;pq,q)]$.

Of course, these polynomials are also orthogonal with respect to the measure obtained by adding mass J at the origin to any of the other measures known for the generalized Stieltjes-Wigert polynomials.

For the corresponding chain sequence $\{\beta_n\}$,

$$\beta_n = \frac{(1-pq^n)(1-q^n)q}{[1+q-(1+p)q^n]\,[1+q-(1+p)q^{n+1}]}, \tag{6.19}$$

we obtain the parameters

$$h_0 = \frac{(1-p)q}{1+q-(1+p)q},\quad h_n = \frac{(1-pq^n)(1-q^n)qb_{n-1}}{[1+q-(1+p)q^{n+1}]b_n},\qquad n \geqq 1, \tag{6.20}$$

where $b_n = [q]_n + J[pq]_n$. Now $\{\beta_n\}$ converges to $q(1+q)^{-2} < 1/4\ (0 < q < 1)$ so all nonmaximal parameter sequences converge to $q/(1+q)$ while the maximal parameter sequence converges to $1/(1+q)$ [**5**, pp. 102, 103]. Thus we will obtain the maximal parameters if we choose

$$J = -J_0 = -[q]_\infty/[pq]_\infty. \tag{6.21}$$

In other words, if J is given by (6.21), then the corresponding polynomials (6.18) are orthogonal with respect to a measure that corresponds to a determined Hamburger moment problem. Since the coefficients in the recurrence formula are known (via (6.16)), one might hope that this measure could be recovered but the complicated nature of these coefficients does not seem to offer much real hope.

E). Jacobi type polynomials. Using the standard notation [**20**], we have

$$\begin{aligned}
d\psi(x) &= 2^{-(\alpha+\beta+1)}\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}(2-x)^\alpha x^\beta\,dx, \qquad 0 \leqq x \leqq 2\\
P_n(x) &= 2^n\binom{2n+\alpha+\beta}{n}^{-1}P_n^{(\alpha,\beta)}(x-1)\\
Q_n(x) &= 2^n\binom{2n+\alpha+\beta+1}{n}^{-1}P_n^{(\alpha,\beta+1)}(x-1)\\
P_n(0) &= \frac{(-2)^n(\beta+1)_n(\alpha+\beta+1)_n}{(\alpha+\beta+1)_{2n}},\quad Q_n(0) = \frac{(-2)^n(\beta+2)_n(\alpha+\beta+2)_n}{(\alpha+\beta+2)_n}\\
\|P_n\|^2 &= \frac{2^{2n}n!(\alpha+1)_n(\beta+1)_n(\alpha+\beta+1)_n}{(\alpha+\beta+1)_{2n}(\alpha+\beta+2)_{2n}}\\
A_n &= \frac{2J(\beta+1)_n(\alpha+\beta+2)_{n-1}}{(\alpha+\beta+2n)b_{n-1}}
\end{aligned} \tag{6.22}$$

where $b_n = (\alpha+1)_n n! + J(\beta+2)_n(\alpha+\beta+2)_n$.

Beginning with the recurrence formula for the monic form of the Jacobi polynomials (e.g., see [5, p. 153]), we find

$$\gamma_{2n} = \frac{2(\beta + n)(\alpha + \beta + n)}{(\alpha+\beta+2n-1)(\alpha+\beta+2n)}, \quad \gamma_{2n+1} = \frac{2n(n + \alpha)}{(\alpha+\beta+2n)\,(\alpha+\beta+2n+1)}.$$

For the $P_n^*(x;\,0,\,J)$ we then obtain

$$\text{(6.23)}\quad \begin{aligned} &\gamma_2^* = \frac{(2\beta + 1)}{(\alpha+\beta+2)(1+J)}, \quad \gamma_{2n+1}^* = \frac{2b_n}{(\alpha+\beta+2n)(\alpha+\beta+2n+1)b_{n-1}} \\ &\gamma_{2n+2}^* = \frac{2n(\alpha + n)(\beta + n + 1)(\alpha + \beta + n + 1)b_{n-1}}{(\alpha + \beta + 2n + 1)(\alpha + \beta + 2n + 2)b_n} \end{aligned}$$

$$\text{(6.24)}\quad \begin{aligned} P_n^*(x;\,0,\,J) = {}& 2^n\binom{2n + \alpha + \beta + 1}{n}^{-1} \\ & \left[P_n^{(\alpha,\beta+1)}(x-1) + \frac{b_n}{n(n+\alpha+\beta+1)b_{n-1}}P_{n-1}^{(\alpha,\beta+1)}(x-1)\right]. \end{aligned}$$

If we next consider adding a jump H at $x = 2$, we find that, for the coefficients in (5.9), A_n is given by (6.22), D_n is obtained from A_n by replacing J by $2J$ and α by $\alpha + 1$. To compute C_n in (5.10), we obtain

$$P_n^*(2;\,0,\,J) = \frac{2^n(\alpha + 1)_{n-1}(\alpha + \beta + 1)_n c_n}{(\alpha + \beta + 1)_{2n} b_{n-1}},$$

where

$$c_n = (\alpha + 1)_n(n - 1)! + J(\beta + 2)_{n-1}(\alpha + \beta + 2)_n,$$

$$f_n^2 = \frac{2^{2n}(\alpha + 1)_{n-1}(\beta + 1)_n(\alpha + \beta + 1)_n(n - 1)!b_n}{(\alpha + \beta + 1)_{2n}(\alpha + \beta + 1)_{2n+1}\, b_{n-1}},$$

and $R_n^*(2;\,0,\,2J)$ is obtained from $P_n^*(2;\,0,\,2J)$ by replacing α by $\alpha + 1$. These are so complicated, it is not worth pursuing the calculation of C_n to the bitter end. It is also pointless in view of the elegent ${}_4F_3$ representation Koornwinder [9] has given for these polynomials.

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