

THE COEFFICIENTS OF THE INVERSE OF AN ODD CONVEX FUNCTION

RICHARD J. LIBERA AND ELIGIUSZ J. ZŁOTKIEWICZ

1. Background information. \mathcal{P} is the class of functions regular and with positive real part in the open unit disk \mathcal{U} , $\mathcal{U} = \{z \in \mathbb{C}: |z| < 1\}$, having a series representation

$$(1.2) \quad P(z) = 1 + c_1z + c_2z^2 + \dots, \quad z \in \mathcal{U}.$$

The family \mathcal{K} of regular convex functions of the form

$$(1.3) \quad f(z) = z + a_2z^2 + a_3z^3 + \dots$$

is defined by the condition

$$(1.4) \quad \frac{zf''(z)}{f'(z)} + 1 \in \mathcal{P}$$

(see [4], for example).

In recent years the peculiar behavior of the coefficients of inverses of functions in \mathcal{K} and in similar classes has attracted attention [1, 2, 7, 8, 10, 11]. If the inverse of $f(z)$ in \mathcal{K} is

$$(1.5) \quad \check{f}(w) = w + A_2w^2 + A_3w^3 \dots,$$

then it has been shown ([1, 10]) that $|A_k| \leq 1$, $k = 2, 3, \dots, 8$, but that there are members of \mathcal{K} for which $|A_{10}| > 1$, [7]. The exact bound for $|A_9|$ appears to be unknown at this time.

The purpose of the present work is to examine the coefficients of (1.5) when $f(z)$ is an odd function in \mathcal{K} . Suppose then that

$$(1.6) \quad f(z) = z + b_3z^3 + b_5z^5 + \dots$$

is an odd member of \mathcal{K} . Then its inverse

$$(1.7) \quad \check{f}(w) = w + B_3w^3 + B_5w^5 + \dots$$

is likewise odd. In this case we may write (1.4) as

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$$(1.8) \quad 1 + \frac{zf''(z)}{f'(z)} = \frac{1}{Q(z)};$$

or, using the relations $f(\check{f}(w)) = w$, $f'(\check{f}(w))\check{f}'(w) = 1$ and $f''(\check{f}(w))\check{f}'(w)^2 + f'(\check{f}(w))\check{f}''(w) = 0$, as

$$(1.9) \quad 1 - \frac{\check{f}(w)\check{f}''(w)}{\check{f}'(w)^2} = \frac{1}{Q(\check{f}(w))},$$

$Q(z)$ being necessarily an even function in \mathcal{D} of the form

$$(1.10) \quad Q(z) = 1 + d_2z^2 + d_4z^4 + \dots$$

The right sides of (1.8) and (1.9) are expressed as reciprocals as an aid in computation, i.e., the representations for the coefficients B_k have more tractable representations when these forms are used.

2. Conclusions and Proofs.

THEOREM 1. *If $f(z)$ is an odd function in K and $\check{f}(w) = w + B_3w^3 + \dots$, then*

$$(2.1) \quad |B_3| \leq \frac{1}{3}, |B_5| \leq \frac{2}{15}, |B_7| \leq \frac{17}{315} \text{ and } |B_9| \leq \frac{17}{360},$$

and these bounds are best possible.

THEOREM 2. *If $f(z)$ is an odd function in K and $\check{f}(w) = w + B_3w^3 + \dots$, then*

$$(2.2) \quad |B_{2k-1}| \leq \frac{1}{2k-1} \frac{\Gamma\left(\frac{2k+1}{2}\right)}{\Gamma\left(\frac{2k+3}{4}\right)^2},$$

for all k .

The proof of the first theorem begins with the substitution of forms (1.7) and (1.10) into (1.9), from which, after considerable computation, one obtains

$$(2.3) \quad \begin{cases} 3!B_3 = d_2, \\ 5!B_5 = 6d_4 + d_2^2, \\ 7!B_7 = 120 d_6 + 6d_2d_4 + d_2^3, \text{ and} \\ 9!B_9 = 5040 d_8 + 960d_2d_6 + 132d_2^2d_4 - 1764 d_4^2 + d_2^4. \end{cases}$$

The bounds for $|B_3|$, $|B_5|$ and $|B_7|$ are obtained directly from the first three of these equations, along with an application of the triangle inequality

and Caratheodory's inequalities, $|d_k| \leq 2$, all k [4]. These bounds are made sharp by the function

$$f_1(z) = \frac{1}{2i} \log \left(\frac{1 + iz}{1 - iz} \right),$$

which is obtained from (1.4) by letting $P(z)$ be $(1 + z)^2/(1 - z)^2$ and for which the inverse is

$$\begin{aligned} \check{f}_1(w) &= \tan w \\ &= \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k} - 1)}{(2k)!} |\beta_{2k}| w^{2k-1} \\ &= w + \frac{w^3}{3} + \frac{2w^5}{15} + \frac{17w^7}{315} + \dots, \end{aligned}$$

with the coefficients being the Bernoulli numbers $\beta_2 = 1/6$, $\beta_4 = -1/30$, $\beta_6 = 1/42$, $\beta_8 = -1/30$, $\beta_{10} = 5/66$, etc.

The derivation of the bound for $|B_9|$ requires a little more effort. If we let $P(z) = Q(\sqrt{z})$, in (1.10), then we obtain a new member of \mathcal{P} , say $P(z) = 1 + C_1z + C_2z^2 + \dots$, where $C_k = d_{2k}$, $k = 1, 2, 3, \dots$. Using this transformation, the last equation in (2.3) may be written

$$(2.4) \quad 9!B_9 = 5040C_4 + 960C_1C_3 + 132C_1^2C_2 - 1764C_2^2 + C_1^4.$$

To maximize the magnitude of B_9 , we now need to consider only the first four coefficients of $P(z)$ over the whole class \mathcal{P} . Because $|C_4| \leq 2$, we may write

$$(2.5) \quad 9!|B_9| \leq (5040) (2) + \max_{\mathcal{P}} |C_1^4 + 132C_1^2C_2 + 960C_1C_3 - 1764C_2^2|.$$

To compute an upper bound on the second term in (2.5) we appeal to another result of Caratheodory, which, stated in a form due to Toeplitz, appears in [5].

LEMMA. *The power series for $P(z)$ given in (1.2) converges in Δ to a function in \mathcal{P} if and only if the Toeplitz determinants*

$$(2.6) \quad D_n = \begin{vmatrix} 2 & C_1 & C_2 & \dots & C_n \\ C_{-1} & 2 & C_1 & \dots & C_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{-n} & C_{-n+1} & C_{-n+2} & \dots & 2 \end{vmatrix}, \quad n = 1, 2, 3, \dots,$$

and $C_{-k} = \bar{C}_k$, are all nonnegative. They are strictly positive except for $P(z) = \sum_{k=1}^m \rho_k P_0(e^{it_k z})$, $\rho_k > 0$, $\sum \rho_k = 1$, $P_0(z) = (1 + z)/(1 - z)$, t_k real and $t_k \neq t_j$ for $k \neq j$; in this case $D_n > 0$ for $n < m - 1$ and $D_n = 0$ for $n \geq m$.

If we assume $C_1 \geq 0$, computation and careful arrangement of terms

reduces the relation $D_2 \geq 0$ to $8 + 2 \operatorname{Re} \{C_1^2 C_2\} - 2|C_2|^2 - 4C_1^2 \geq 0$, or

$$(2.7) \quad 2C_2 = C_1^2 + x(4 - C_1^2), \quad \text{for some } x, |x| \leq 1;$$

and the condition $D_3 \geq 0$ is the same as

$$\begin{aligned} & |(4C_3 - 4C_1C_2 + C_1^3)(4 - C_1^2) + C_1(2C_2 - C_1^2)^2| \\ & \leq 2(4 - C_1^2)^2 - 2|2C_2 - C_1^2|^2, \end{aligned}$$

which, using (2.7), can be shown to be equivalent to

$$(2.8) \quad 4C_3 = C_1^3 + 2(4 - C_1^2)C_1x - C_1(4 - C_1^2)x^2 + 2(4 - C_1^2)(1 - |x|^2)z,$$

for a number z , $|z| \leq 1$.

Now, (2.7) and (2.8) enable us to rewrite the expression in (2.5) to be maximized as

$$(2.9) \quad \begin{aligned} \phi(C_1) = & -134C_1^4 - 336C_1^2(4 - C_1^2)x \\ & - (4 - C_1^2)(1764 - 201C_1^2)x + 480C_1(4 - C_1^2)(1 - |x|^2)z. \end{aligned}$$

Let $|x| = \rho$. Then

$$(2.10) \quad \begin{aligned} |\phi(C_1)| & \leq 134C_1^4 + 336C_1^2(4 - C_1^2)\rho + (4 - C_1^2)(1764 - 201C_1^2)\rho^2 \\ & \quad + 480C_1(4 - C_1^2)(1 - \rho^2) \\ & \leq \operatorname{Max} \left\{ [134C_1^4 + 480C_1(4 - C_1^2)] + 336C_1^2(4 - C_1^2)\rho \right. \\ & \quad \left. + (4 - C_1^2)(201)(2 - C_1) \left(C_1 + \frac{294}{67} \right) \rho^2 \right\} \\ & = 4(1764) - 1224C_1^2 - C_1^4 \\ & \leq 4(1764), \end{aligned}$$

and this is the bound for $|B_9|$ given in (2.1). The inequalities in (2.10) are rendered sharp by a function for which $C_1 = 0$, i.e., one of the form $P_1(z) = 1 + 2z^2 + 2z^4 + \dots$ and for which the corresponding member (1.10) of \mathcal{P} is $Q_1(z) = 1 + 2z^4 + \dots$. The odd member of \mathcal{X} related to $Q_1(z)$ is the function $F(z)$ mapping Δ onto a square and its inverse is of the form $\check{F}(w) = w + A_3w^3 + A_5w^5 + A_7w^7 + 17/360 w^9 + \dots$. This concludes our discussion of Theorem 1.

If $f(z)$ is an odd convex function, then it is starlike of order $1/2$ and has a Stieltjes integral representation of the form

$$(2.11) \quad f(z) = z \exp \left\{ -\frac{1}{2} \int_0^{2\pi} \log(1 - z^2 e^{-i\theta}) d\mu(\theta) \right\},$$

with $\mu(\theta)$ non-decreasing and chosen so that $\int_0^{2\pi} d\mu(\theta) = 1$, (see [4], for example). The coefficients of $\check{f}(w)$ have the representation

$$(2.12) \quad B_n = \frac{1}{2\pi i n} \int_{|z|=r} \frac{dz}{f(z)^n}.$$

Combining (2.11) and (2.12) gives

$$\begin{aligned} 2\pi n |B_n| &\leq r^n \int_{|z|=r} \left\{ \exp \left\{ \frac{n}{2} \int_0^{2\pi} \log (1 - z^2 e^{-i\theta}) d\mu(\theta) \right\} \right\} |dz| \\ &= r^{n+1} \int_0^{2\pi} \left\{ \exp \int_0^{2\pi} \log |1 - r^2 e^{2it} e^{-i\theta}|^{n/2} d\mu(\theta) \right\} dt \\ &\leq r^{n+1} \int_0^{2\pi} \int_0^{2\pi} |1 - r^2 e^{2it} e^{-i\theta}|^{n/2} d\mu(\theta) dt \\ &\leq \int_0^{2\pi} |1 - e^{2it}|^{n/2} dt \\ &= 2^{n/2-2} \int_0^\pi \sin^{n/2} t dt \\ &= 2\pi \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma^2\left(\frac{n}{4} + 1\right)}. \end{aligned}$$

Here we have used the integral generalization of the inequality between the arithmetic and geometric means (see [12, p. 110], for example) and standard tables ([13], see [8] for a similar argument). Re-indexing (2.13) gives (2.2).

3. Some Observations. (i) Are the coefficients of $\check{f}(w)$ for odd $f(z)$ in \mathcal{X} bounded? Functions of the form

$$f(z) = \int_0^z \frac{dz}{(1 + z^2)^\alpha (1 - z^2)^{1-\alpha}},$$

for sufficiently small positive α , map Δ onto rhombi with two vertices inside Δ ; the Cauchy-Hadamard formula applied to the coefficients of $\check{f}(w)$ guarantees that these coefficients cannot be bounded.

(ii) Functions of the form

$$g(z) = \int_0^z \frac{dt}{(1 - t^n)^{2/n}}, \quad n \geq 3,$$

map Δ onto a regular polygon with vertices outside Δ . Consequently, each such function is bi-univalent [12]; this is more generally the case for any function in \mathcal{X} mapping Δ onto a polygonal region whose sides are segments or circular arcs and all of whose vertices lie outside Δ .

(iii) Clunie [2] showed that if $M_n = \max \{|A_n|: f(z) \in \mathcal{X}\}$, then $M_n =$

$O(\log n \cdot n^{-3} \cdot 2^n)$, as $n \rightarrow \infty$, which improves the earlier estimate $M_n = O(n^{-3/2} \cdot 2^n)$, $n \rightarrow \infty$, supplied by Kirwan and Schober [7]. Our estimate (2.2) shows that, for odd members of \mathcal{X} , $M_n = O(\sqrt{2^n \cdot n^{-3}})$, as $n \rightarrow \infty$ and assuming n is odd. This is a better estimate than that of Clunie for the whole class and appears quite natural when compared to that of Kirwan and Schober. To justify our conclusion, we apply properties of the Γ function (see [6], for example) to (2.13) and obtain

$$\begin{aligned} |A_n| &\leq \frac{1}{n} \frac{\frac{n}{2} \Gamma\left(\frac{n}{2}\right)}{\left(\frac{n}{4} \Gamma\left(\frac{n}{4}\right)\right)^2} = \frac{8}{n^2} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma^2\left(\frac{n}{4}\right)} \\ &= \frac{4}{\sqrt{\pi}} (2^{n/2} n^{-3/2}) \left(\frac{\Gamma\left(\frac{n}{4} + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{4}\right)} \right) \end{aligned}$$

The last term is asymptotic to 1.

(iv) Earlier, Krzyż, Libera and Złotkiewicz, [8] showed that if $f(z)$ is starlike of order $1/2$, then

$$|A_n| \leq \frac{1}{n} \frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{2} + 1\right)^2}.$$

A computation like that above shows that this estimate is of higher order than the one given in (2.2), but is of the same order as that of Kirwan and Schober.

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF DELAWARE, NEWARK, DELAWARE 19716.

INSTYTUT MATEMATYKI UMCS, UL. NOWOTKI 10, 20–031 LUBLIN, POLAND.

