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TOMOGRAPHIC RECONSTRUCTION OF FUNCTIONS FROM THEIR SINGULARITIES

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The general problem of reconstructive tomography is to determine a density function $f(x)$, defined on $\mathbf{R}^n$, from its integrals on lower dimensional manifolds of some fixed dimension. Integrating f over hyperplanes has received considerable attention due to its relation to x-ray scanning in $\mathbf{R}^2$ and nuclear magnetic resonance scanning in $\mathbf{R}^3$ [**4**, **6**]. This leads to the consideration of the Radon transform of f, defined on $S^{n-1} \times \mathbf{R}^1$, by

$$(1) \qquad \hat{f}(\theta, p) = \int_{\mathbf{R}^n} f(x)\, \delta(p - \langle \theta, x \rangle) dx.$$

Here, δ denotes the Dirac delta mass and $\langle \cdot, \cdot \rangle$is the usual inner product. Thus, $\hat{f}$ is the integral of f over the hyperplane $\{x \,|\, \langle \theta, x \rangle = p\}$. It will be assumed throughout that the density function $f(x)$ is absolutely summable and compactly supported. This implies that, for each $\theta \in S^{n-1}$, $\hat{f}$ is defined for almost every p. Further, $\hat{f}$ is compactly supported on $S^{n-1} \times \mathbf{R}^1$.

Related to $\hat{f}$ is the so-called back projection $\check{f}$. This is defined on $\mathbf{R}^n$ by

$$(2) \qquad \check{f}(x) = \int_{S^{n-1}} \hat{f}(\theta, \langle \theta, x \rangle d\theta.$$

The function $\check{f}$ is directly related to f by the singular integral

$$(3) \qquad \check{f}(x) = \Omega_{n-1} \int_{\mathbf{R}^n} \frac{f(y)}{\|x - y\|}\, dy,$$

where Ω_n denotes the surface area of the unit sphere in $\mathbf{R}^n$. The value of Ω_n may be computed from the identity $\Omega_n = 2\pi^{n/2}/\Gamma(n/2)$, where Γ denotes the gamma function. It follows directly from (3) that $\check{f}$ is continuous whenever f is. By (1), the same implication also holds between $\hat{f}$ and f. Since identical results hold for the order of differentiability, it is quite often assumed in the literature that f is sufficiently smooth to justify inversion formulas such as those given in [**5**].

The point of view adopted in this paper is that physical densities have

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jumps along smooth curves. A simple model for such density functions is obtained by assuming

$$f(x) = \sum_{j=1}^{j=m} \alpha_j \chi_{E_j},$$

where $\alpha_j \in \mathbf{R}^1$ and χ_{E_j} denotes the characteristic function of the compact subset E_j of $\mathbf{R}^n$. By the linearity of (1), the Radon transform of such a density function is determined by each χ_{E_j}. As such, it is sufficient to consider the behavior of $\check{f}$ or $\hat{f}$, where f is a characteristic function. As will be shown, the location of the boundary may be easily determined from the smoothness of the Radon transform. It will be assumed that the boundary of each E_j, denoted $\partial(E_j)$, is a differentiable manifold of dimension $n - 1$. By taking linear combinations of the E_j's it is possible to generate certain objects with cusps or sharp corners on their boundary. In dealing with such objects the following subtlety should be noted. Due to cancellation effects, certain statements which would be valid for all $\theta \in S^{n-1}$ when f is a single characteristic function, will only apply for almost every θ if two or more sets are involved. Examples will be given later to illustrate this.

More generally, we consider density functions of the form

$$f(x) = \phi(x) + \sum_{j=1}^{j=m} \alpha_j \chi_{E_j}, \tag{4}$$

where ϕ is compactly supported and infinitely differentiable. Since the Radon transform of ϕ is infinitely differentiable, the discontinuities of such a density function are the same as for $\phi \equiv 0$. It will be shown that the order of differentiation required to detect the boundary is always less for $\hat{f}$ than $\check{f}$. Using $\hat{f}$, but without recourse to back projection, we show how to determine the singular support of f. More precisely, our method determines the supporting hyperplanes of the union of the E_j's. It remains to be seen what effective numerical algorithms can be developed to approximate the singular support based on finitely many hyperplanes. It seems likely that these techniques would be most appropriate for problems such as testing otherwise homogeneous materials for a small number of defects or impurities.

Our notation is as follows. The class of density functions whose partial derivatives up to order k satisfy a Lipschitz condition of order α ($0 < \alpha \leqq 1$) is denoted by $C^k_\alpha(\mathbf{R}^n)$. In the absence of any Lipschitz condition we write $C^k(\mathbf{R}^n)$ or simply $C(\mathbf{R}^n)$ when $k = 0$. A density function $f(x)$ is said to belong to one of these classes locally at $x_0 \in \mathbf{R}^n$ if the restriction of $f(x)$ to some neighborhood of x_0 belongs to the appropriate class. This notation will also apply to $\check{f}(x)$.

Given a density function $f(x)$, its Radon transform is said to belong to

$C^k_\alpha(\mathbf{R}^1)$ in the direction θ if $|\hat{f}^{(k)}(\theta, p) - \hat{f}^{(k)}(\theta, q)| \leqq M|p - q|^\alpha$, where $\hat{f}^{(k)}$ denotes the k-th partial derivative of $\hat{f}$ with respect to p. Similarly, we write $\hat{f} \in C^k(\mathbf{R}^1)$ in the direction θ if $\hat{f}^{(k)}$ is continuous in p. In referring to the local behavior of $\hat{f}$, the value of θ will also be held constant.

The mapping of f to $\check{f}$, defined by (3), is well understood in terms of the above classes. Indeed, using ideas developed in [**8**, p. 150], it is not difficult to establish the following fact. For $0 < \alpha < 1$, $f \in C^k_\alpha(\mathbf{R}^n)$ implies $\check{f} \in C^{k+n-1}_\alpha(\mathbf{R}^n)$ and conversely. Unfortunately, this does not hold for $C^k_1(\mathbf{R}^n)$ or $C^k(\mathbf{R}^n)$. Regarding the latter case, we can only assert that $\check{f} \in C^{k+n-1}(\mathbf{R}^n)$ implies $f \in C^k(\mathbf{R}^n)$ when n is odd. Of course, these negative results also apply to the local behavior of f and $\check{f}$. This complicates the problem of determining the discontinuities of a general density function.

Given a density function $f(x)$, define D_f to be the complement of the set of points $x_0 \in \mathbf{R}^n$ where $f(x) \in C(\mathbf{R}^n)$ locally. Next, define E^k_f to be the complement of the set of points $x_0 \in \mathbf{R}^n$, where $\check{f} \in C^k(\mathbf{R}^n)$ locally. It is evident that $E^k_f \subset E^{k+1}_f$ for all $k \geqq 0$. We seek the smallest k which is sufficient to determine D_f.

THEOREM 1. *Suppose that $f(x)$ is a density of the form* (4). *Then $D_f = E^{n-1}_f$, for n odd, and $D_f = E^n_f$, for n even.*

The proof of the theorem is an immediate consequence of the following two lemmas. It should be noted that the first of these applies to more general density functions.

LEMMA 1. (i) *$E^k_f (k \leqq n - 2)$ is empty if $f \in L^a(\mathbf{R}^n)$ for some $a > n(n - k - 1)^{-1}$.*

(ii) *$D_f \subset E^{n-1}_f$ for n odd.*

(iii) *$D_f \subset E^n_f$ for n even.*

PROOF. To establish (i), differentiate under the integral in (3) k times with respect to the coordinate x_i. This yields

$$(5) \qquad (\partial^k/\partial x_i^k)\check{f}(x) = \int_{\mathbf{R}^n} f(y)\, H_{i,k}(x - y)\, dy$$

where $H_{i,k}$ is a homogeneous function of degree $-k - 1$. For each coordinate, the corresponding $H_{i,k}$ is in $L^b(\mathbf{R}^n)$, locally for $b < n(k + 1)^{-1}$. Consequently, the convolution of $H_{i,k}$ with the compactly supported $f \in L^a(\mathbf{R}^n)$ must be continuous for a as stipulated. Since this also applies to each of the mixed partial derivatives of total order k, it follows that $\check{f} \in C^k(\mathbf{R}^n)$.

The proof of (ii) follows from the Radon inversion formula, for odd dimensions, given in [**3**, p. 20]. According to this formula,

(6) $$f(x) = \frac{(-1)^{(n-1)/2}}{(2\pi)^{n-1}} L^{(n-1)/2}[\check{f}(x)],$$

where $L^{(n-1)/2}$ denotes the Laplacian operator iterated $(n-1)/2$ times. It is evident from (6) that $\check{f} \in C^{n-1}(\mathbf{R}^n)$ locally at x_0 implies $f \in C(\mathbf{R}^n)$ locally at x_0. This is equivalent to the statement of (ii).

For n even, we require the following localization argument. Define $g(x) = f(x)\,\psi\,(x - x_0)$, where ψ is an infinitely differentiable cut off function. By this we mean ψ vanishes off an ε neighborhood of the origin and is equal to unity in some smaller neighborhood. Replacing f by g in (3) it remains only to show that g is continuous whenever $\check{g} \in C^n(\mathbf{R}^n)$ to obtain the analogous statement for the local behavior of f. The global continuity of g may be demonstrated by computing Fourier transforms in (3) or by simply invoking the previously mentioned result in [**8**, p. 150]. This establishes (iii) and completes the proof.

We remark that a variant of (ii) and (iii) could be obtained by considering the local behavior of f and $\check{f}$ in terms of $C^k_\alpha(\mathbf{R}^n)$. However, the conclusions of the lemma are best possible for $C^k(\mathbf{R}^n)$.

LEMMA 2. *If $f(x)$ is a density of the form* (4) *then $E^k_f \subset D_f$ for all $k \geqq 0$.*

PROOF. It is evident that $f \in C^\infty(R^n)$ locally at any point $x_0 \notin D_f$. Thus, D_f is the singular support of f. We apply the singular support lemma [**9**, p. 39] with $k(x, y) = \|x - y\|^{-1}$. Since $\check{f}(x)$ is a constant multiple of $\int_{\mathbf{R}^n} k(x, y)f(y)dy$, the singular support of $\check{f}$ is contained in D_f. This implies that $E^k_f \subset D_f$, for every k, and completes the proof.

For densities of the form (4), the derivatives of $\hat{f}$ could also be used to determine D_f. Indeed, for each fixed θ define $H_f(\theta)$ to be the complement of the set of hyperplanes $\langle\theta, x\rangle = p$ for which $\hat{f} \in C^\infty(\mathbf{R}^1)$ locally at p. It is not difficult to show that each hyperplane in $H_f(\theta)$ has a non-transversal intersection with the boundary of one of the E_j's. By varying θ, an approximation to the boundary of each E_j is obtained.

In terms of the required order of differentiation, it is usually easier to determine D_f from $\hat{f}$ than from $\check{f}$. Indeed, if $\hat{f} \in C^k(\mathbf{R}^1)$ uniformly in $\theta \in S^{n-1}$, then a differentiation under the integral in (2) yields

$$(\partial^k/\partial x_i^k)\check{f}(x) = \int_{S^{n-1}} \theta_i^k \hat{f}^{(k)}(\theta, \langle\theta, x\rangle)d\theta.$$

This implies that $\check{f} \in C^k(\mathbf{R}^n)$. The same implication also holds for $\hat{f} \in C^k_\alpha(\mathbf{R}^1)$ uniformly in θ. Thus, $\check{f}$ is at least as smooth as $\hat{f}$.

The difference in the smoothness of $\hat{f}$ and $\check{f}$ is extreme for certain characteristic functions. For example, suppose E is the unit cube rounded at the corners so that its boundary is smooth. Then the Radon trasnsform of $f = \chi_E$ is discontinuous for θ in the direction of each coordinate axis. However, $\hat{f}$ is continuous for every other direction.

In fact, if E is a convex set with a smooth boundary, then for almost every $\theta \in S^{n-1}$, $\hat{f}$ is differentiable up to some order depending on the dimension n. To show this, let $G(x)$ denote the Gauss mapping of each $x \in \partial(E)$ to the unit normal vector of $\partial(E)$ at x. Thus,

$$G:\partial(E) \to S^{n-1}$$

and the Jacobian of G, denoted $\kappa(x)$, is the curvature. The precise result is as follows.

THEOREM 2. *Suppose that* $f = \chi_E$, *where* $E \subset \mathbf{R}^n$ *is convex. Then* $\hat{f}(\theta, p) \in C^k_\alpha(\mathbf{R}^1)$ *for almost every* $\theta \in S^{n-1}$, *where*:

$$\text{(7)} \qquad \begin{aligned} &\text{(i) } k = (n-3)/2,\ \alpha = 1, \textit{ when } n \textit{ is odd};\\ &\text{(ii) } k = (n-2)/2,\ \alpha = 1/2, \textit{ when } n \textit{ is even}. \end{aligned}$$

PROOF. Since E is convex, each of the curvature components obtained from the Hessian matrix associated with each $x \in \partial(E)$ is non-negative. The vanishing of any of these curvature components at x implies that G is not invertible there. By Sard's theorem [**2**, p. 3], G is invertible for almost every $\theta \in S^{n-1}$. Equivalently, for almost every θ, the non-transversal intersection of $\langle\theta, x\rangle = p$ with $\partial(E)$ satisfies $\kappa(x) > 0$. By the Morse Lemma, the intersection of E with a hyperplane is asymptotic to an ellipsoid near any such x. Ignoring the relative differences in the curvature components, it suffices to establish (7) when E is a ball of radius ρ.

Let $B_\rho(x)$ denote the characteristic function of the ball of radius ρ centered at the origin. A straightforward computation yields

$$\hat{B}_\rho(\theta, p) = \begin{cases} \dfrac{\pi^{(n-1)/2}}{\Gamma(n/2 + 1/2)}(\rho^2 - p^2)^{(n-1)/2} & \text{for } |p| < \rho \\ 0 & \text{for } |p| > \rho. \end{cases}$$

The proof is completed by verifying that $\hat{B}_\rho \in C^k_\alpha(\mathbf{R}^1)$ with k and α as in (7).

The conclusion of Theorem 2 is sharp. In particular, if m is the greatest integer $\leqq n/2$, then $\hat{f}^{(m)}$ is discontinuous for every $\theta \in S^{n-1}$. Unfortunately, this does not carry over verbatim to combinations of convex sets. For example, in $\mathbf{R}^3$ the paraboloid $x_3 = x_1^2 + x_2^2$ and its mirror image reflected in the $x_1 - x_2$ plane constitute the boundaries of two convex sets A and B having the property that the Radon transform of $f = \chi_A - \chi_B$ is infinitely differentiable in p, for $\theta = (0, 0, 1)$. It should be noted that the curvature of $\partial(A)$ and $\partial(B)$ are both positive and that similar examples can be constructed whenever n is odd. When n is even the situation is somewhat better. It can be shown that $\hat{f}^{(n/2)}$ cannot be finite at points where the boundary curvature is positive. However, this does not hold for points where the curvature vanishes. To see what goes wrong, we modify the previous example to $\mathbf{R}^4$, with x_4 arbitrary. The curvature at the origin is zero, due to the boundary's cylindrical component, and $\hat{f}$ is infinitely differentiable in p, for $\theta = (0, 0, 1, 0)$.

In spite of the above mentioned pathologies, we can assert the following. For only finitely many θ can $\langle \theta, x \rangle = p$ be the supporting hyperplane of two or more E_j's. Thus, $\hat{f}^{(m)}$ is discontinuous, for almost every θ where m is the greatest integer $\leqq n/2$. This leads to an improvement in the earlier discussed technique to determine D_f from $\hat{f}$ for densities of the form (4). For this it is required only that D_f be expressible as a finite union of $\partial(E_j)$'s where each E_j is convex and has a smooth boundary. In this regard, we mention that the use of finitely many ellipses is well established as a method of generating $\mathbf{R}^2$ phantoms.

An analysis of the proof of Theorem 2 yields the following refinement. If, for a given θ, any one of the $n - 1$ curvature components is positive, then $\hat{\chi}_E \in C_{1/2}(\mathbf{R}^1)$ in that direction. While this may be generalized to the case where ℓ of the $n - 1$ curvature components are positive, the implication cannot be reversed. In particular, $\hat{\chi}_E \in C_{1/2}(\mathbf{R}^1)$ in the direction θ does not provide any information about the curvature components. For example, in $\mathbf{R}^3$, the set E whose boundary is asymptotic to $x_3 = x_1^4 + x_2^4$ at the origin is indistinguishable from a cylinder.

Matters are much simpler in $\mathbf{R}^2$ and it need only be assumed that E is simply connected. Then the boundary of E is a smooth curve depending on a parameter $t \in \mathbf{R}^1$. We denote, by $S(x_0, t)$, the local parametrization satisfying

$$S(x_0, 0) = \frac{d}{dt}[S(x_0, t)]_{t=0} = 0.$$

THEOREM 3. *Suppose that* $f = \chi_E$ *where* $E \subset \mathbf{R}^2$ *and the intersection of* $\partial(E)$ *with any line* $\langle \theta, x \rangle = p$ *is finite. Then* $\hat{f} \in C_{1/k}(\mathbf{R}^1)$ *for all* $\theta \in S^1$ *provided that the coefficient* $a_k(x_0)$ *in the asymptotic series*

$$S(x_0, t) = \sum_{i=2}^{\infty} a_i(x_0) t^i \tag{8}$$

is non-zero for all $x_0 \in \partial(E)$. *Further,* $\hat{f} \in C_{1/2}(\mathbf{R}^1)$, *for almost every* $\theta \in S^1$.

PROOF. The minimal possible smoothness of $\hat{f}$ is determined by what occurs locally at each point of intersection of a line with the boundary. Thus, there is no loss of generality in assuming that the non-transversal intersection of a given line with the boundary occurs at a single point x_0. Subject to the parametrization, this line is taken to be $x_2 = p$. When the leading exponent in (8) is odd, the limiting behavior of $S(x_0, t)$ depends on both the intervals $(-a, 0]$ and $[0, a)$. If the leading exponent is even, then only one of the intervals $(-a, 0]$ or $[0, a)$ is to be considered. The choice depends in an obvious way on the sign of the leading coefficient. In either case, the reasoning is the same for both intervals, so we confine our attention to $[0, a)$.

Having made these preliminary observations, suppose that (8) holds simultaneously at all x_0 for some k. Then $S(x_0, t)$ is locally invertible on some interval $[0, a)$. The unique value of t satisfying $S(x_0, t) = p$ is the contribution to $\hat{f}(p)$ in the half space $t \geqq 0$. This contribution to $\hat{f}(p)$ will satisfy a Lipschitz condition of order α, provided that $t \leqq M \mid S(x_0, t)|^\alpha$, or equivalently, when

$$\frac{t^{1/\alpha}}{|S(x_0, t)|} \leqq M^{1/\alpha}. \tag{9}$$

It is evident that this occurs precisely when $\alpha \leqq 1/k$.

The second assertion of the theorem follows immediately from the first and Sard's theorem. Indeed, the curvature $\kappa(x_0) = 2a_2(x_0)$ is non-zero for almost every $\theta \in S^1$. Thus, $\hat{f} \in C_{1/2}(\mathbf{R}^1)$ for almost every θ, and the proof is completed.

The modification of Theorem 3 to densities of the form (4) is evident. Of particular interest is the case where the boundary of each E_j is an analytic manifold. The intersection of $\partial(E_j)$ with any line must be finite. Indeed, an infinite intersection would imply the existence of a limit point. By a standard analytic identity argument, this leads to the contradiction that $\partial(E_j)$ coincides with a line. Now, for each E_j, define

$$G_j(k) = \{x \in \partial(E_j) \mid a_k(x) \neq 0\}$$

where $k \geqq 2$. Since $\partial(E_j)$ is compact and analytic, $G_j(2)$ contains all but finitely many points of $\partial(E_j)$. Each point in $\partial(E_j)\backslash G_j(2)$ must be in some $G_j(k)$ by the analyticity of the boundary. Choosing m to be the maximum value of k for these finitely many points yields $G_j(m) = \partial(E_j)$. Repeating this argument over finitely many j's, it follows that $\hat{f}$ is Lipschitz continuous.

Some remarks concerning the numerical implementation of Theorems 2 and 3 are in order. For reconstruction on $\mathbf{R}^2$ or $\mathbf{R}^3$, the discontinuities of $\partial\hat{f}/\partial p$ determine D_f, for densities of the form (4). If $\partial\hat{f}/\partial p$ is discontinuous at p_0 and $n = 2$, then, by Theorem 3,

$$\operatorname{Lim} h \to 0 \frac{|\hat{f}(\theta, p_0 + h) - \hat{f}(\theta, p_0)|^2}{8h}$$

is the reciprocal of the curvature at the boundary point x satisfying

$\langle\theta, x\rangle = p_0$, provided $\kappa(x) \neq 0$. For almost every x in the union of the boundaries of the E_j's, $\kappa(x)$ is finite and non-zero. If the given data is discrete, then Theorem 2 may be applied in the following way. For each θ, define $\Delta\hat{f}(\theta, p_j) = \hat{f}(\theta, p_{j+1}) - \hat{f}(\theta, p_j)$, where p_{j+1} and p_j are neighboring values of p. For $\mathbf{R}^2$ reconstruction, the boundary is indicated by the necessarily subjective process of looking for relatively large values of $\Delta\hat{f}$. For $\mathbf{R}^3$ reconstruction, it is changes in monotonicity between neighboring values of $\Delta\hat{f}$ that indicate the location of the boundary.

In the remainder of the paper we consider the problem of reconstructing density functions which are radial. While radial functions are seldom encountered in practice, they do provide some indication of what may be expected in general. Of course, if f is radial, then $\hat{f}(\theta, p) = \hat{f}(|p|)$.

It is possible that $\hat{f} \in C^{n-2}(\mathbf{R}^1)$ even though f has an unremovable discontinuity. An example is a compactly supported radial function which equals $\|x\|^{-1/2}$ in some deleted neighborhood of the origin and is infinitely differentiable on $\mathbf{R}^n\backslash\{0\}$. Thus, it is necessary to differentiate $\hat{f}$ $n-1$ times to determine that $0 \in D_f$.

This behavior is peculiar to the origin. It will be shown that, for radial functions, the set

$$D'_f = D_f \cap (\mathbf{R}^n\backslash\{0\})$$

is more easily detected. Let H^k_f denote the complement, relative to $\mathbf{R}^+$, of the set of points where $\hat{f} \in C^k(\mathbf{R}^1)$ locally.

THEOREM 4. *Suppose that $f(x)$ is a radial density function having countably many discontinuities and let m be the greatest integer $\leqq n/2$. If n is odd, then $H^m_f = D'_f$; if n is even, then $H^{m-1}_f \subset D'_f \subset H^m_f$.*

PROOF. Let a denote the radius of the ball in $\mathbf{R}^n$ which properly contains the support of f. Then the Radon transform of f is given by

$$\hat{f}(p) = \Omega_{n-1}\int_0^{\sqrt{a^2-p^2}} f(\sqrt{p^2+r^2})r^{n-2}\,dr$$

where $r = (\sum_{i=1}^{n-1} x_i^2)^{1/2}$. The change of variables $t^2 = p^2 + r^2$ yields

$$\hat{f}(p) = \Omega_{n-1}\int_p^a tf(t)\,(t^2-p^2)^{(n-3)/2}\,dt. \tag{10}$$

This may be differentiated $m-1$ times with respect to p. Specifically, if we apply the operator $p^{-1}d/dp$ $m-1$ times to (10) then the case for n odd reduces to $n = 3$. Performing another differentiation it now follows easily that $\hat{f}^{(m)}(p)$ is continuous, for any $p \neq 0$, if and only if $f(p)$ is. This proves the theorem for n odd.

Similarly, when n is even, it suffices to prove the theorem for $n = 2$. On $\mathbf{R}^2$, the Radon transform is given by

$$\hat{f}(p) = 2\int_p^a \frac{t f(t)}{(t^2 - p^2)^{1/2}}\, dt.$$

We make the change of variables $t^2 = u$, $p^2 = v$ and define $g(u) = f(t)$ and $\hat{g}(v) = \hat{f}(p)$. This yields

$$\hat{g}(v) = \int_v^{a^2} \frac{g(u)}{(u - v)^{1/2}}\, du \tag{11}$$

which is essentially the Abel transform of g. Thus, $\hat{g}$ is locally continuous at a point provided that g is, i.e., $H_f^0 \subset D_f'$. Since (11) is inverted by a convolution of $\hat{g}^{(1)}$, with $(-v)^{-1/2}$, the same argument yields $D_f' \subset H_f^1$. Extending this reasoning to $\mathbf{R}^n$ by applying the operator $p^{-1}\, d/dp$ $n/2 - 1$ times to (10) completes the proof of the theorem.

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