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CHAPTER 12 OF RAMANUJAN'S SECOND NOTEBOOK: CONTINUED FRACTIONS

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> We take up something—we know it is finite; but as soon as we begin to analyze it, it leads us beyond our reason, and we never find an end to all its qualities, its possibilities, its powers, its relations. It has become infinite.
>
> Vivekananda

Dedicated to the memory of R.A. Smith and E.G. Straus

1. Introduction. In assessing the content of Ramanujan's first letter, dated January 16, 1913, to him, Hardy [**34**, p. 9] remarked "... but (1.10) – (1.12) defeated me completely; I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them." These comments were directed at three continued fraction representations. Indeed, Ramanujan's contributions to the continued fraction expansions of analytic functions are one of his most spectacular achievements. The three formulas which challenged Hardy's acumen are not found in Chapter 12, but this chapter, which is almost entirely devoted to the study of continued fractions, contains many other beautiful and penetrating formulas. Unfortunately, Ramanujan left us no clues as to how he discovered these elegant continued fraction formulas. Especially enigmatic are the several representations for products and quotients of gamma functions. Three of the principal formulas involving gamma functions are Entries 34, 39, and 40. Entries 20 and 22, giving Gauss's and Euler's continued fractions, respectively, for a quotient of two hypergeometric functions, also play prominent roles. Several other formulas are dependent upon these five entries, and it may be helpful to schematically indicate these connections among entries.

The purpose of this paper is to prove each of the 113 theorems, corol-

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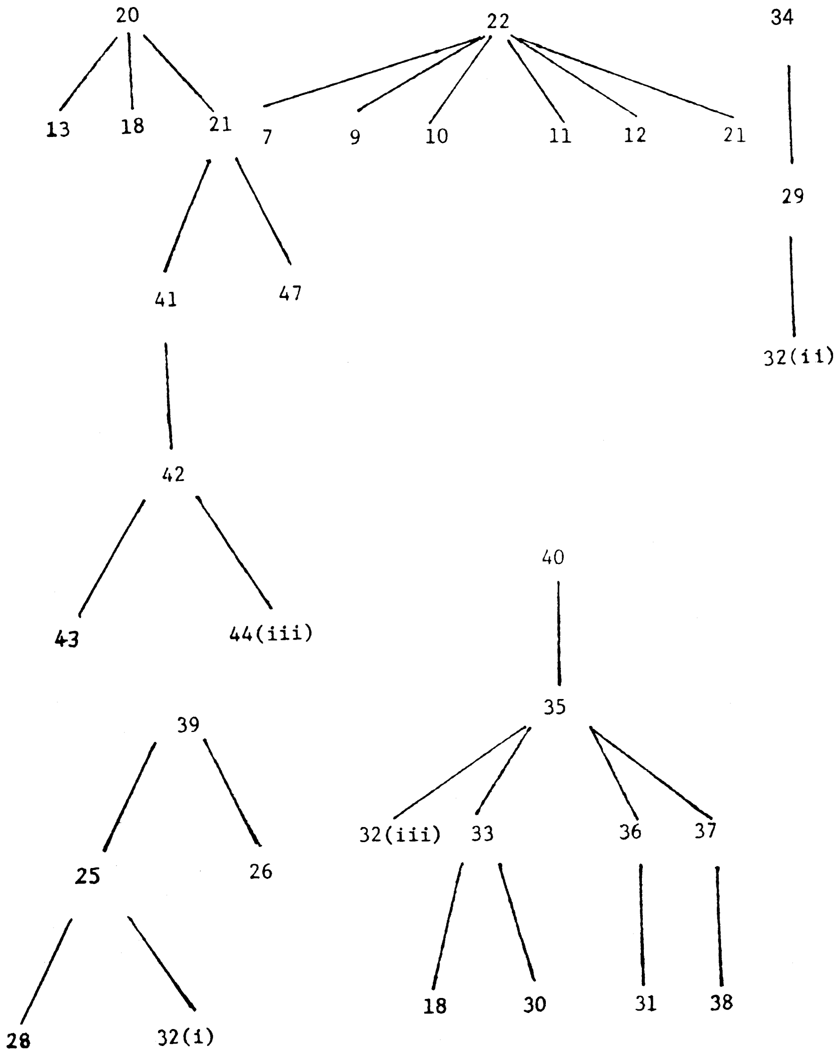

laries, and examples which are divided into 49 sections in Chapter 12. For those formulas which are found in the literature, we cite references wherein proofs may be found. Ramanujan never states conditions on the relevant parameters to insure that a formula holds. We have generally attached such hypotheses. Undoubtedly, in many instances, these conditions are more restrictive than necessary.

The first two named authors have been aided by notes left by the third named author. After devoting six years to the editing of Ramanujan's notebooks, B. M. Wilson prematurely passed away on March 18, 1935 at the age of 38. Two weeks earlier, he had entered a hospital for a hernia operation. Unfortunately, a blood infection incurred as a consequence of the surgery. In 1935, antibiotics were not yet available, and so Wilson was not able to recover from this infection. When he died, Wilson had held the chair of mathematics in Dundee for less than two years. The other leading candidate in 1933 for the professorship was E. T. Copson. Rather than reopen the competition for the chair, it was decided to appoint Copson as Wilson's successor. It may very well be that Wilson was working on Chapter 12 prior to his hospitalization, for his contributions to Chapter 12 abruptly end about the middle of the chapter.

We now offer a few comments about notation. As is customary, we put

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)},$$

where k denotes a nonnegative integer. The hypergeometric series ${}_pF_q$ is defined by

$${}_pF_q(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; x) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k \cdots (\alpha_p)_k\, x^k}{(\beta_1)_k(\beta_2)_k \cdots (\beta_q)_k\, k!},$$

where p and q are nonnegative integers with $p \leqq q + 1$ and $\alpha_1, \alpha_2, \ldots, \alpha_p, \beta_1, \beta_2, \ldots, \beta_q$ are complex numbers. If $p < q + 1$, ${}_pF_q$ converges for all complex numbers x, while if $p = q + 1$, ${}_pF_q$ converges for $|x| < 1$. However, in the latter case, ${}_pF_q$ can be analytically continued into the complex plane cut at $[1, \infty)$.

In the sequel, $\psi(z)$ always denotes $\Gamma'(z)/\Gamma(z)$. We shall employ the representation [**53**, p. 39]

$$\psi(x) = -\gamma + \sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+z}\right) \tag{0.1}$$

several times in this paper, usually without comment. Here γ denotes Euler's constant.

We shall adopt the notation

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots \tag{0.2}$$

for the continued fraction

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}.$$

The notation (0.2) appears to be the most convenient and widely used notation for continued fractions. We shall refer frequently to the well-known texts of Perron [**57**], Wall [**74**], Khovanskii [**38**], and Jones and Thron [**37**]. Because Perron's book contains several formulas that we shall employ and that are not found in the other texts, we shall make many references to this classic work.

2. Proofs.

ENTRY 1. *Let* $a_1, a_2, \ldots, a_r$ *and* $b_1, b_2, \ldots, b_r$ *be arbitrary complex numbers. Define* $N_{-1} = 0$, $N_0 = 1$, $D_{-1} = 0$, $D_0 = 1$,

$$N_{k-1} = b_k N_{k-2} + a_k N_{k-3}, \qquad k \geqq 2, \tag{1.1}$$

and $D_k = b_k D_{k-1} + a_k D_{k-2}$, $k \geqq 1$. *Then, for* $r \geqq 1$,

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_r}{b_r} = a_1 \frac{N_{r-1}}{D_r} = \sum_{k=1}^{r} \frac{(-1)^{k+1} a_1 \cdots a_k}{D_{k-1} D_k}. \tag{1.2}$$

PROOF. The first equality in (1.2) is a somewhat unusual formulation of a basic elementary formula in the theory of continued fractions [**74**, p. 15]. For future reference, we restate the first equality of (1.2) in a more familiar fashion. Let $A_{-1} = 1$, $A_0 = 0$, $B_{-1} = 0$, $B_0 = 1$,

$$A_k = b_k A_{k-1} + a_k A_{k-2}, \qquad k \geqq 1, \tag{1.3}$$

and

$$B_k = b_k B_{k-1} + a_k B_{k-2}, \qquad k \geqq 1. \tag{1.4}$$

Then, for $r \geqq 1$,

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_r}{b_r} = \frac{A_r}{B_r}. \tag{1.5}$$

Thus, $a_1 N_k = A_{k+1}$ and $D_k = B_k$, $k \geqq -1$. Note that if we define $N_{-2} = 1$, then (1.1) is valid for $k = 1$ as well. Recall that A_k and B_k are the kth numerator and denominator of the continued fraction (0.2).

The second equality in (1.2) is essentially another version of a well-known fact [**74**, p. 18] due to Euler [**22**].

COROLLARY. *If* $a_1, a_2, \ldots, a_r$ *are arbitrary complex numbers and* $r \geqq 3$, *then*

$$\sum_{k=1}^{r} a_k = \frac{a_1}{1} - \frac{a_2}{a_1 + a_2} - \frac{a_1 a_3}{a_2 + a_3} - \frac{a_2 a_4}{a_3 + a_4} - \cdots - \frac{a_{r-2} a_r}{a_{r-1} + a_r}.$$

This corollary is due to Euler [**22**], and a proof may be found in [**57**, p. 17].

ENTRY 2. *Let* $x, a_1, a_2, \ldots$ *denote nonzero complex numbers and define, for each nonnegative integer* n,

$$f_n(x) = \sum_{k=0}^{n} \frac{(-x)^k}{a_1 a_2 \cdots a_{k+1}}.$$

If

$$\lim_{n\to\infty} f_n(x) = \pm\infty, \tag{2.1}$$

then

$$x = x - a_1 + \frac{a_1 x}{x - a_2} + \frac{a_2 x}{x - a_3} + \cdots. \tag{2.2}$$

PROOF. For each nonnegative integer n [**14**, p. 516, eq. (14)],

$$\sum_{k=1}^{n+1} \frac{a_1 a_2 \cdots a_k x^k}{b_1 b_2 \cdots b_k} = \frac{a_1 x}{b_1} - \frac{b_1 a_2 x}{b_2 + a_2 x} - \frac{b_2 a_3 x}{b_3 + a_3 x} - \cdots - \frac{b_n a_{n+1} x}{b_{n+1} + a_{n+1} x}.$$

If we set $a_j = 1$ and replace b_j by $-a_j$, $j \geqq 1$, we find that $f_n(x) = 1/(a_1 - A)$, where

$$A = \frac{a_1 x}{x - a_2} + \frac{a_2 x}{x - a_3} + \cdots + \frac{a_n x}{x - a_{n+1}}.$$

Letting n tend to ∞ and using (2.1), we deduce that $a_1 - A = 0$, which is equivalent to (2.2).

Of course, we could impose several sets of conditions on $x, a_1, a_2, \ldots$ in order to insure that (2.1) holds. For example, if $\lim_{n\to\infty} |a_n| = \rho$, then, by the ratio test, (2.1) is valid if $|x| > \rho$.

It is easy to establish (2.2) formally, and this is probably how Ramanujan proceeded. Trivially,

$$a_k = \frac{a_k x}{x - a_{k+1} + a_{k+1}}, \qquad k \geqq 1. \tag{2.3}$$

If we successively employ (2.3) for $k = 1, 2, \ldots$, we find that

$$a_1 = \frac{a_1 x}{x - a_2 + a_2} = \frac{a_1 x}{x - a_2} + \frac{a_2 x}{x - a_3 + a_3}$$
$$= \cdots = \frac{a_1 x}{x - a_2} + \frac{a_2 x}{x - a_3} + \frac{a_3 x}{x - a_4} + \cdots,$$

which is equivalent to (2.2).

We shall interpret Entries 3 and 4 formally. There is a slight misprint in the formulation of Entry 3 [**64**, vol. 2, p. 143].

ENTRY 3. *If* $x, a_1, a_2, \ldots$ *are arbitrary complex numbers, then*

$$x = a_1 + \sqrt{\{x^2 + a_1(a_1 - 2a_2) - 2a_1\sqrt{\{x^2 + a_2(a_2 - 2a_3) - 2a_2\sqrt{\cdots}\}}\}}.$$

PROOF. It is easy to verify that

$$x - a_k = (x^2 + a_k(a_k - 2a_{k+1}) - 2a_k(z - a_{k+1}))^{1/2}, \qquad k \geqq 1. \tag{3.1}$$

Using (3.1) successively, we find that

$$\begin{aligned} x - a_1 &= (x^2 + a_1(a_1 - 2a_2) - 2a_1(x - a_2))^{1/2} \\ &= (x^2 + a_1(a_1 - 2a_2) - 2a_1(x^2 + a_2(a_2 - 2a_3) - 2a_2(x - a_3))^{1/2})^{1/2} \\ &= \cdots, \end{aligned}$$

and so the desired result follows.

ENTRY 4. *Let a, n, and x denote arbitrary complex numbers. Then*

$$f(x) \equiv x + n + a = \sqrt{\{ax + (n + a)^2 + x\sqrt{\{a(x + n) + (n + a)^2 + (x + n)\sqrt{\{a(x + 2n) + (n + a)^2 + (x + 2n)\sqrt{\cdots}\}}\}}\}}.$$

PROOF. By successively substituting, we find that

$$\begin{aligned} f(x) &= (ax + (n + a)^2 + xf(x + n))^{1/2} \\ &= (ax + (n + a)^2 + x(a(x + n) + (n + a)^2 + (x + n)f(x + 2n))^{1/2})^{1/2} \\ &= \cdots, \end{aligned}$$

and therefore we obtain the proposed formula.

EXAMPLES. We have

$$3 = \sqrt{\{1 + 2\sqrt{\{1 + 3\sqrt{\{1 + 4\sqrt{\{1 + \cdots\}}\}}\}}\}} \tag{i}$$

and

$$4 = \sqrt{\{6 + 2\sqrt{\{7 + 3\sqrt{\{8 + 4\sqrt{\{9 + \cdots\}}\}}\}}\}}. \tag{ii}$$

Examples (i) and (ii) were submitted by Ramanujan [**61**], [**63**, p. 323] as a problem in the Journal of the Indian Mathematical Society and solutions were subsequently given by him.

T. Vijayaraghavan [**63**, p. 348] has shown that

$$\sqrt{\{a_1 + \sqrt{\{a_2 + \sqrt{\{a_3 + \cdots + \sqrt{a_n}\}}\}}\}}, \qquad a_n \geqq 0,$$

tends to a limit as n tends to ∞ if and only if

$$\overline{\lim_{n\to\infty}} \frac{\operatorname{Log} a_n}{2^n} < \infty. \tag{4.1}$$

See also [**58**, pp. 37, 214]. Vijayaraghavan's theorem can be used to show that the infinite radicals in Examples (i) and (ii) are convergent [**63**, p. 348].

The literature on infinite radicals is rather scant, and so Herschfeld's paper [**35**] is to be particularly recommended. He points out that Ramanujan's proofs of (i) and (ii) are slightly incomplete, and he gives full rigorous solutions. This paper contains a good discussion on the convergence of infinite radicals.

We state Entry 5 (i) as Ramanujan records it. But, as we shall see, Entry 5 (i) is valid only for $\theta = 0$. We shall separate Entry 5 (ii) into two parts. The first part will be proved rigorously; the second will be regraded as a formal identity. However, we shall indicate some values of θ for which the second part of Entry 5 (ii) is rigorously true. We suggest to readers that they attempt to develop more thoroughly the theory of infinite radicals, so that perhaps concrete conditions may be imposed on the formal identities in §3–5 to insure their validity.

ENTRY 5(i). *We have*

$$2\cos\theta = (2 + 2\cos 2\theta)^{1/2} = (2 + (2 + 2\cos 4\theta)^{1/2})^{1/2}$$
$$= (2 + (2 + (2 + 2\cos 8\theta)^{1/2})^{1/2})^{1/2} = \cdots .$$

PROOF. Repeatedly apply the identity

$$2\cos(2^k\theta) = \pm(2 + 2\cos(2^{k+1}\theta))^{1/2}, \qquad k \geqq 0,$$

with the plus sign always chosen on the right side. However, unless $\theta = 0$, there clearly will be values of k when $\cos(2^k\theta) < 0$, and so we must choose the minus sign in such instances. If $\theta = 0$, Entry 5 (i) implies that $2 = (2 + (2 + (2 + \cdots)^{1/2})^{1/2})^{1/2}$, which is meaningful since (4.1) is easily seen to be satisfied. Furthermore, a direct proof may easily be given.

ENTRY 5 (ii) (first part). *Suppose that either* $|\theta| \leqq \pi/6$ *or* $5\pi/6 \leqq \theta \leqq 7\pi/6$. *Then*

$$2\cos\theta = \sqrt[3]{\{2\cos 3\theta + 3\sqrt[3]{\{2\cos 3\theta + 3\sqrt[3]{\{2\cos 3\theta + \cdots\}}\}}\}}.$$

PROOF. For $n \geqq 1$, let

$$R_n = (2\cos 3\theta + 3(2\cos 3\theta + 3(2\cos 3\theta + \cdots)^{1/3})^{1/3})^{1/3},$$

where n cube roots are taken. Observe that $R_n = (2\cos 3\theta + 3R_{n-1})^{1/3}$, for $n \geqq 2$.

First suppose that $|\theta| \leqq \pi/6$. Clearly, $R_{n-1} < R_n$ for each $n \geqq 2$. Thus,

$$R_n^3 = 2\cos 3\theta + 3R_{n-1} < 2\cos 3\theta + 3R_n. \tag{5.1}$$

The polynomial $x^3 - 3x - 2\cos 3\theta$ has three real roots, $2\cos\theta$ and $-\cos\theta \pm \sqrt{3}\,|\sin\theta|$. For $|\theta| \leqq \pi/6$, $-\cos\theta \pm \sqrt{3}\,|\sin\theta| \leqq 0$. Therefore $\{R_n\}$ is a nonnegative, increasing sequence bounded above by the root $2\cos\theta$. Thus, $\{R_n\}$ converges and, by (5.1), $\{R_n\}$ converges to a root of $x^3 - 3x - 2\cos 3\theta$. As we have just seen, this root must be $2\cos\theta$.

For $5\pi/6 \leqq \theta \leqq 7\pi/6$, consider $\alpha = \theta - \pi$. Thus, $|\alpha| \leqq \pi/6$. Using the foregoing analysis, we complete the proof.

We remark that if $\pi/2 < \theta < 5\pi/6$ or $7\pi/6 < \theta < 3\pi/2$, then $\{R_n\}$ converges to $-\cos\theta + \sqrt{3}\,|\sin\theta|$, while if $\pi/6 < \theta < \pi/2$ or $3\pi/2 < \theta < 11\pi/6$, $\{R_n\}$ converges to $-\cos\theta - \sqrt{3}\,|\sin\theta|$.

ENTRY 5(ii) (second part). *We have*

$$2\cos\theta = \sqrt[3]{\{6\cos\theta + \sqrt[3]{\{6\cos 3\theta + \sqrt[3]{\{6\cos 9\theta + \cdots\}}\}}\}}. \tag{5.2}$$

PROOF. Repeatedly employ the equality

$$2\cos(3^k\theta) = (6\cos(3^k\theta) + 2\cos(3^{k+1}\theta))^{1/3}$$

for $k = 0, 1, 2, \ldots$.

We now indicate some special cases when the second part of Entry 5(ii) may be established rigorously.

If $\theta = 0$, then (5.2) becomes

$$2 = (6 + (6 + (6 + \cdots)^{1/3})^{1/3})^{1/3}. \tag{5.3}$$

To prove (5.3), define

$$R_n = (6 + (6 + \cdots 6^{1/3} \cdots)^{1/3})^{1/3}, \qquad n \geqq 1,$$

where n cube roots are indicated. Observe that

$$R_n^3 = 6 + R_{n-1} < 6 + R_n, \qquad n \geqq 2. \tag{5.4}$$

Now $x = 2$ is the only real root of the equation $x^3 - x - 6 = 0$. It follows that $R_{n-1} < R_n < 2$, $n \geqq 2$. Thus, $\{R_n\}$ converges, and, by (5.4), the limit of $\{R_n\}$ equals 2.

If $\theta = \pi$, then (5.2) yields

$$\begin{aligned} -2 &= (-6 + (-6 + (-6 + \cdots)^{1/3})^{1/3})^{1/3} \\ &= -(6 + (6 + (6 + \cdots)^{1/3})^{1/3})^{1/3}, \end{aligned} \tag{5.5}$$

which is valid by (5.3).

If $\theta = \pi/3$, the right side of (5.2) becomes

$$(3 + (-6 + (-6 + \cdots)^{1/3})^{1/3})^{1/3} = (3 - 2)^{1/3} = 1,$$

by (5.5). Hence, (5.2) is valid for $\theta = \pi/3$. In fact, by induction, it is easy to show that (5.2) holds for $\theta = \pi/3^k$, $k \geqq 1$.

It may also be easily checked that (5.2) is valid if $\theta = \pi/2$ or $2\pi/3$, for example. If $\theta = \pi/4$, (5.2) holds, but the verification is more difficult.

ENTRY 6. *Let $a > 0$ but $a \neq 1$. Suppose that n is a nonnegative integer. In the field of formal power series, put*

$$f_n(v) = \sum_{j=1}^{\infty} a_j(n) v^j,$$

where $a_0(n) = 1$, $a_1(n) = -a^{-n}$, *and* $a_j(n)$, $j \geqq 2$, *is defined recursively by*

$$a_j(n) = \frac{1}{2(a^{j-1}-1)} \sum_{k=1}^{j-1} a_k(n) a_{j-k}(n). \tag{6.1}$$

Then for each nonnegative integer n,

$$\left(\frac{a(a-2)}{4} + \left(\frac{a(a-2)}{4} + \cdots + \left(\frac{a(a-2)}{4} + \frac{a}{2} f_0(v)\right)^{1/2} \cdots\right)^{1/2}\right)^{1/2} = \frac{a}{2} f_n(v), \tag{6.2}$$

where, on the left side, there are n iterated radicals. Furthermore,

$$\begin{aligned} f_n(v) = 1 - v/a^n &+ \frac{(v/a^n)^2}{2(a-1)} - \frac{(v/a^n)^3}{2(a-1)(a^2-1)} \\ &+ \frac{(v/a^n)^4(a+5)}{8(a-1)(a^2-1)(a^3-1)} - \frac{(v/a^n)^5(2a^2+3a+7)}{8(a-1)(a^2-1)(a^3-1)(a^4-1)} \\ &+ \cdots . \end{aligned} \tag{6.3}$$

PROOF. If $n = 0$, (6.2) is trivial. Thus, assume that $n > 0$. Proceeding by induction and squaring both sides of (6.2), we find that we must show that

$$\frac{a(a-2)}{4} + \frac{a}{2} f_{n-1}(v) = \frac{a^2}{4} f_n^2(v), \qquad n \geqq 1,$$

or, in other words,

$$\frac{a}{2} + \sum_{j=1}^{\infty} a_j(n-1) v^j = \frac{a}{2} f_n^2(v), \qquad n \geqq 1,$$

Now, by (6.1) and induction, $a_j(n-1) = a^j a_j(n)$, $j \geqq 0$, $n \geqq 1$, and so it suffices to show that

$$a^j a_j(n) = \frac{a}{2} \sum_{k=0}^{j} a_k(n) a_{j-k}(n), \qquad j \geqq 2.$$

But the latter equality is equivalent to (6.1), and so the proof of (6.2) is complete.

The expansion (6.3) is easily determined by employing (6.1).

Ramanujan's formulation of Entry 6 is slightly incorrect, for he claims that [**64**, vol. 2, p. 143]

$$a_j(n) = \frac{1}{2(a^{j-1}-1)} \sum_{k=0}^{j-1} a_k(n) a_{j-1-k}(n),$$

which should be compared with (6.1).

ENTRY 7. *If x is not a negative integer, then*

$$1 = \frac{x+1}{x} + \frac{x+2}{x+1} + \frac{x+3}{x+2} + \cdots. \tag{7.1}$$

PROOF. We first derive a consequence of Entry 22 that we shall employ several times in the sequel. In Entry 22, replace x by x/α and let α tend to ∞ to deduce that

$$\frac{{}_1F_1(\beta+1;\gamma+1;x)}{\gamma\, {}_1F_1(\beta;\gamma;x)} = \frac{1}{\gamma - x} + \frac{(\beta+1)x}{\gamma+1-x} + \frac{(\beta+2)x}{\gamma+2-x} + \cdots. \tag{7.2}$$

(An equivalent form of (7.2) was also found by Perron [**57**, p. 280, Satz 6.2].)

To prove (7.1), set $x = 1$ and $\beta = \gamma = x$ in (7.2). The result now easily follows.

It also should be remarked that Entry 7 follows from Entry 11 by setting $a = 1$ and $n = x + 1$.

Ramanujan [**64**, vol. 2, p. 143] has written x instead of 1 on the left side of (7.1).

COROLLARY. *We have*

$$1 = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \cdots.$$

PROOF. Set $x = 1$ in Entry 7.

ENTRY 8. *Let n denote a positive integer and suppose that $x \neq -ka$, $1 \leqq k \leqq n$. Then*

$$\begin{aligned}&\sum_{k=1}^{n} \frac{(-1)^{k+1}}{(x+a)(x+2a)\cdots(x+ka)} \\ &= \frac{1}{x+a} + \frac{x+a}{x+2a-1} + \frac{x+2a}{x+3a-1} + \cdots + \frac{x+(n-1)a}{x+na-1}.\end{aligned} \tag{8.1}$$

FIRST PROOF. Denote the right side of (8.1) by A_n/B_n in the notation of §1. Then by (1.3),

$$A_n = (x + na - 1)A_{n-1} + (x + (n-1)a)A_{n-2}, \qquad n \geqq 3,$$

or, upon iteration,

$$\begin{aligned} A_n - (x+na)A_{n-1} &= -\{A_{n-1} - (x + (n-1)a)A_{n-2}\} \\ &= \cdots = (-1)^n\{A_2 - (x+2a)A_1\} \\ &= (-1)^{n-1}, \qquad\qquad n \geqq 3, \end{aligned} \tag{8.2}$$

since $A_1 = 1$ and $A_2 = x + 2a - 1$.

Similarly, by (1.4),

$$\begin{aligned} B_n - (x + na)B_{n-1} &= -\{B_{n-1} - (x + (n-1)a)B_{n-2}\} \\ &= \cdots = (-1)^n\{B_2 - (x + 2a)B_1\} = 0, \qquad n \geqq 3, \end{aligned}$$

since $B_1 = x + a$ and $B_2 = (x + a)(x + 2a)$. Hence,

$$B_n = (x + a)(x + 2a) \cdots (x + na), \qquad n \geqq 1. \tag{8.3}$$

On the other hand, let the left side of (8.1) be denoted by the rational function P_n/Q_n. Clearly,

$$Q_n = (x + a)(x + 2a) \cdots (x + na), \qquad n \geqq 1. \tag{8.4}$$

Now, for $n \geqq 2$,

$$\frac{P_n}{Q_n} = \frac{P_{n-1}}{Q_{n-1}} + \frac{(-1)^{n+1}}{Q_n} = \frac{(x + na)P_{n-1} + (-1)^{n+1}}{Q_n},$$

i.e.,

$$P_n = (x + na)P_{n-1} + (-1)^{n+1}, \qquad n \geqq 2. \tag{8.5}$$

Hence, by (8.2) and (8.5), A_n and P_n satisfy the same recursion formula. Since $A_1 = P_1 = 1$ and $A_2 = P_2 = x + 2a - 1$, we conclude that $A_n = P_n$, $n \geqq 1$. Also, by (8.3) and (8.4), $B_n = Q_n$, $n \geqq 1$. Thus, the equality (8.1) has been established.

SECOND PROOF. We induct on n. For $n = 1$, (8.1) is trivially true.

Suppose that we denote the left side of (8.1) by $f_n(x)$. Proceeding by induction, we thus find that

$$\begin{aligned} (x + a)f_{n+1}(x) &= 1 - f_n(x + a) \\ &= 1 - \frac{1}{x + 2a +}\,\frac{x + 2a}{x + 3a - 1 +}\cdots\frac{x + na}{+\,x + (n+1)a - 1}. \end{aligned}$$

Letting

$$A = x + 3a - 1 + \frac{x + 3a}{x + 4a - 1 +}\cdots\frac{x + na}{+\,x + (n+1)\,a - 1},$$

we then deduce that

$$\begin{aligned} (x + a)\,f_{n+1}(x) &= 1 - \frac{1}{x + 2a + (x + 2a)/A} \\ &= \frac{(x + 2a - 1) + (x + 2a)/A}{(x + 2a - 1) + (x + 2a)/A + 1} \\ &= \frac{1}{1 + \dfrac{1}{(x + 2a - 1) + (x + 2a)/A}}. \end{aligned}$$

Upon dividing both sides of the equality above by $x + a$, we arrive at (8.1), but with n replaced by $n + 1$. This completes the induction.

COROLLARY. We have

$$\frac{1}{e-1} = \frac{1}{1+}\frac{2}{2+}\frac{3}{3+}\cdots.$$

PROOF. Let $x = 0$ and $a = 1$ in Entry 8 to obtain the equality

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k!} = \frac{1}{1+}\frac{1}{1+}\frac{2}{2+}\frac{3}{3+}\cdots+\frac{n-1}{n-1}.$$

Letting n tend to ∞ yields

$$1 - 1/e = \frac{1}{1+}\frac{1}{1+}\frac{2}{2+}\frac{3}{3+}\cdots.$$

The desired formula now readily follows by inverting the equality above. The previous Corollary is due to Euler [23].

ENTRY 9. *If* $x \neq -ka$, $1 \leqq k < \infty$, *then*

$$\frac{x+a+1}{x+1} = \frac{x+a}{x-1+}\frac{x+2a}{x+a-1+}\frac{x+3a}{x+2a-1+}\cdots. \tag{9.1}$$

PROOF. We first indicate a formal proof. Observe that for each positive integer n,

$$\frac{x+na+1}{x+(n-1)a+1} = \frac{x+na}{x+(n-1)a-1+\dfrac{x+(n+1)a+1}{x+na+1}}.$$

By applying this identity successively for $n = 1, 2, \ldots$, we formally derive (9.1).

We now give a rigorous proof based upon (7.2). Putting $x = 1/a$, $\beta = x/a$, and $\gamma = (x-a)/a$ in (7.2), we find that

$$\frac{1}{x-a}\,\frac{{}_1F_1\left(\dfrac{x+a}{a};\dfrac{x}{a};\dfrac{1}{a}\right)}{{}_1F_1\left(\dfrac{x}{a};\dfrac{x-a}{a};\dfrac{1}{a}\right)} \tag{9.2}$$
$$= \frac{1}{x-a-1+}\frac{x+a}{x-1+}\frac{x+2a}{x+a-1+}\frac{x+3a}{x+2a-1+}\cdots,$$

provided that $x \neq -ka$, $1 \leqq k < \infty$. But,

$$(9.3)\qquad \frac{{}_1F_1\left(\frac{x+a}{a};\frac{x}{a};\frac{1}{a}\right)}{{}_1F_1\left(\frac{x}{a};\frac{x-a}{a};\frac{1}{a}\right)}=\frac{x-a}{x}\,\frac{\frac{x}{a}e^{1/a}+\frac{1}{a}e^{1/a}}{\left(\frac{x}{a}-1\right)e^{1/a}+\frac{1}{a}e^{1/a}}$$

$$=\frac{(x-a)(x+1)}{x(x-a+1)}.$$

Substituting (9.3) into (9.2), taking the reciprocal of both sides, and simplifying, we arrive at (9.1).

EXAMPLES. We have

$$\text{(i)}\qquad \frac{4}{3}=\frac{3}{1}\,\frac{4}{+2}\,\frac{5}{+3}\,\frac{6}{+4}+\cdots$$

and

$$\text{(ii)}\qquad \frac{5}{3}=\frac{4}{1}\,\frac{6}{+3}\,\frac{8}{+5}\,\frac{10}{+7}+\cdots.$$

PROOF. Set $x = 2$ and $a = 1$ in Entry 9 to deduce (i); similarly, set $x = 2$ and $a = 2$ to obtain (ii).

ENTRY 10. *If n is a positive integer, then*

$$n=\frac{1}{1-n}\,\frac{2}{+2-n}\,\frac{3}{+3-n}+\cdots\frac{n}{+0}\,\frac{n+1}{+1}\,\frac{n+2}{+2}+\cdots.$$

FIRST PROOF. Putting $x = 1$, $\beta = 0$, and $\gamma = 1 - n$ in (7.2), we find that

$$0=\frac{(1-n)\,{}_1F_1(0;1-n;1)}{{}_1F_1(1;2-n;1)}=-n+\frac{1}{1-n}\,\frac{2}{+2-n}\,\frac{3}{+3-n}+\cdots,$$

which completes the proof.

SECOND PROOF. In (11.7), set $n = 1$ and replace a by n to deduce that

$$\frac{1}{1-n}\,\frac{2}{+2-n}+\cdots=1+\frac{n-1}{3-n}\,\frac{n-2}{+4-n}+\cdots.$$

We shall be finished if we can show that, for each positive integer n,

$$(10.1)\qquad \frac{n}{2-n}\,\frac{n-1}{+3-n}+\cdots=n.$$

We prove (10.1) by inducting on n. If $n = 1$, (10.1) is trivial. Assuming that (10.1) holds with n replaced by $n - 1$, $n > 1$, we see that

$$\frac{n}{2-n}\,\frac{n-1}{+3-n}+\cdots=\frac{n}{(2-n)+(n-1)}=n.$$

The interpretation of Entry 11 was made difficult because Ramanujan left most of his notation undefined. Furthermore, some of his notation is unnecessary and so will not be given.

ENTRY 11. *Suppose that a is a positive integer and that $-n$ is not a nonnegative integer. Define N_a and D_a by*

$$(11.1)\quad {}_1F_1(1-a; n+2-a; -1) = \frac{N_a}{(n+2-a)(n+3-a)\cdots n}$$

and

$$(11.2)\quad {}_1F_1(1-a; n+1-a; -1) = \frac{D_a}{(n+1-a)(n+2-a)\cdots(n-1)},$$

where if $a=1$, the denominators on the right sides of (11.1) *and* (11.2) *are understood to be equal to* 1. *Then*

$$(11.3)\qquad \frac{N_a}{D_a} = \frac{n}{n-a+}\,\frac{n+1}{n-a+1+}\,\frac{n+2}{n-a+2+}\cdots$$

and

$$(11.4)\quad \frac{N_{a+1}}{N_a} = n+2-a+\frac{a-1}{n+3-a+}\,\frac{a-2}{n+4-a+}\cdots.$$

PROOF. Since a is a positive integer, both ${}_1F_1(1-a; n+2-a; -1)$ and ${}_1F_1(1-a; n+1-a; -1)$ terminate, and so N_a and D_a are simply the numerators of the rational functions respectively obtained. In fact, N_a and D_a are polynomials in n of degree $a-1$.

Setting $\beta = n, \gamma = n+1-a$, and $x=1$ in (7.2), we find that

$$(11.5)\quad \begin{aligned}&\frac{n\,{}_1F_1(n+1; n+2-a; 1)}{(n+1-a)\,{}_1F_1(n; n+1-a; 1)}\\ &= \frac{n}{n-a+}\,\frac{n+1}{n-a+1+}\,\frac{n+2}{n-a+2+}\cdots.\end{aligned}$$

But by Kummer's theorem [**40**], (11.1), and (11.2),

$$(11.6)\quad \begin{aligned}&\frac{n\,{}_1F_1(n+1; n+2-a; 1)}{(n+1-a)\,{}_1F_1(n; n+1-a; 1)}\\ &= \frac{n\,{}_1F_1(1-a; n+2-a; -1)}{(n+1-a)\,{}_1F_1(1-a; n+1-a; -1)} = \frac{N_a}{D_a}.\end{aligned}$$

Thus, (11.3) follows from (11.5) and (11.6). (In fact, Kummer's theorem was rediscovered by Ramanujan [**64**, vol. 2, p. 125], [**9**, Entry 21].)

From (11.1),

$$\frac{N_{a+1}}{N_a} = \frac{(n+1-a)_1F_1(-a; n+1-a; -1)}{{}_1F_1(1-a; n+2-a; -1)}$$
$$= n+2-a+\frac{a-1}{n+3-a}+\frac{a-2}{n+4-a}+\cdots,$$

where we have applied (7.2) with $\beta = -a$, $\gamma = n+1-a$, and $x = -1$. This proves (11.4).

By generalizing the proof above, we can easily prove that

$$\text{(11.7)} \qquad \begin{aligned} &\frac{n}{n-a}+\frac{n+1}{n-a+1}+\frac{n+2}{n-a+2}+\cdots \\ &= 1+\frac{a-1}{n+2-a}+\frac{a-2}{n+3-a}+\cdots, \end{aligned}$$

provided that not both a and $-n$ are nonnegative integers.

COROLLARY 1. *If n is not a nonpositive integer, then*

$$\frac{n^2+n+1}{n^2-n+1} = \frac{n}{n-3}+\frac{n+1}{n-2}+\frac{n+2}{n-1}+\cdots.$$

PROOF. Let $a = 3$ in (11.3).

COROLLARY 2. *If n is not a nonpositive integer, then*

$$\frac{n^3+2n+1}{(n-1)^3+2(n-1)+1} = \frac{n}{n-4}+\frac{n+1}{n-3}+\frac{n+2}{n-2}+\cdots.$$

PROOF. Let $a = 4$ in (11.3).

ENTRY 12. *If $a \neq 0$ and $x \neq -ka$, $1 \leqq k < \infty$,*

$$1 = \frac{x+a}{a}+\frac{(x+a)^2-a^2}{a}+\frac{(x+2a)^2-a^2}{a}+\frac{(x+3a)^2-a^2}{a}+\cdots.$$

FIRST PROOF. In Entry 22, put $x = 1$, $\alpha = 0$, $\beta = (x-a)/a$, and $\gamma = (x+a)/a$. (To see that putting $x = 1$ in Entry 22 is justified, see an equivalent formulation of Entry 22 in [**57**, p. 299, eq. (21)].) After simplification, we find that

$$\frac{x-a}{x+a} = \frac{x-a}{a}+\frac{(x+a)^2-a^2}{a}+\frac{(x+2a)^2-a^2}{a}+\cdots.$$

Multiplying both sides by $(x+a)/(x-a)$, we complete the proof.

SECOND PROOF. In Entry 27, let $x = 1$. Then set $y = 1 + 2x/a$ and $n = -4$. We then find that

$$1+\frac{2x}{a}=1+\frac{4(x+a)^2/a^2-4}{2}+\frac{4(x+2a)^2/a^2-4}{2}+\frac{4(x+3a)^2/a^2-4}{2}+\cdots$$
$$=1+\frac{2}{a}\left\{\frac{(x+a)^2-a^2}{a}+\frac{(x+2a)^2-a^2}{a}+\frac{(x+3a)^2-a^2}{a}+\cdots\right\}$$
$$=1+\frac{2}{a}X,$$

say. Thus, $x = X$. Lastly,

$$1=\frac{a+x}{a+x}=\frac{a+x}{a+X},$$

which is the desired formula.

ENTRY 13. *If $a < b$, then*

$$a=\frac{ab}{a+b+d}-\frac{(a+d)(b+d)}{a+b+3d}-\frac{(a+2d)(b+2d)}{a+b+5d}-\cdots.$$

FIRST PROOF. Let $p_k = b + kd$, $k \geqq 0$. Then

$$p_n=a+b+(2n+1)d-\frac{(a+(n+1)d)(b+(n+1)d)}{p_{n+1}}, \qquad n \geqq 0.$$

Writing $p_n = x_n/x_{n+1}$, $n \geqq 0$, we may write the preceding formula in the form

$$x_n=(a+b+(2n+1)d)x_{n+1}-(a+(n+1)d)(b+(n+1)d)\,x_{n+2}.$$

Setting $a + nd = y_n/y_{n+1}$, $n \geqq 0$, we easily see that the same recurrence formula is satisfied by y_n.

Now if $x_0 = 1$,

$$x_{n+1}=\frac{x_{n+1}}{x_n}\frac{x_n}{x_{n-1}}\cdots\frac{x_1}{x_0}=\frac{1}{p_n}\frac{1}{p_{n-1}}\cdots\frac{1}{p_0}$$
$$=\frac{1}{(b+nd)(b+(n-1)d)\cdots b}.$$

Similarly, if $y_0 = 1$, $y_{n+1} = 1/((a + nd)(a + (n - 1)d) \cdots a)$. Thus, if $a < b$, then x_n/y_n tends to 0 as n tends to ∞.

We now apply a theorem in Perron's text [**57**, p. 97, Satz 2.46, C] to deduce that

$$\frac{x_0}{x_1}=b=a+b+d$$
$$-\frac{(a+d)(b+d)}{a+b+3d}-\frac{(a+2d)(b+2d)}{a+b+5d}-\frac{(a+3d)(b+3d)}{a+b+7d}-\cdots.$$

Now take the reciprocal of both sides above and then multiply both sides by ab to obtain the proposed continued fraction representation.

SECOND PROOF. In Entry 20, let $\alpha = b/(2d)$, $\beta = a/(2d)$, and $\gamma = (a + b + d)/(2d)$. Each of the two hypergeometric series in Entry 20 converges at $x = -1$ since $a < b$. Thus, we may let x tend to -1. After a slight amount of manipulation, we find that

$$\frac{ab}{a+b+d}\,\frac{{}_2F_1\left(\frac{a+d}{2d}, \frac{a+2d}{2d}; \frac{a+b+3d}{2d}; 1\right)}{{}_2F_1\left(\frac{a+d}{2d}, \frac{a}{2d}; \frac{a+b+d}{2d}; 1\right)} \tag{13.1}$$
$$= \frac{ab}{a+b+d-}\;\frac{(a+d)(b+d)}{a+b+3d}\;-\;\frac{(a+2d)(b+2d)}{a+b+5d}\;-\cdots.$$

If we now apply Gauss's theorem [**4**, p. 2] to each of the hypergeometric series above, we find that the left side of (13.1) becomes, for $a < b$,

$$\frac{ab}{a+b+d}\,\frac{\Gamma\left(\frac{a+b+3d}{2d}\right)\Gamma\left(\frac{b}{2d}\right)}{\Gamma\left(\frac{b+2d}{2d}\right)\Gamma\left(\frac{a+b+d}{2d}\right)} = a,$$

which completes the proof.

ENTRY 14. *If* $a_1, a_2, \ldots, a_{2n}$ *and* x *are arbitrary complex numbers, then*

$$\frac{a_1}{x+}\;\frac{a_2}{1+}\;\frac{a_3}{x+}\;\frac{a_4}{1+}\cdots+\frac{a_{2n}}{1} \tag{14.1}$$
$$= \frac{a_1}{x+a_2-}\;\frac{a_2a_3}{x+a_3+a_4-}\;\frac{a_4a_5}{x+a_5+a_6-}\cdots-\frac{a_{2n-2}a_{2n-1}}{x+a_{2n-1}+a_{2n}}.$$

PROOF. We shall induct on n. For $n = 1$, it is easy to verify that the proposed identity is valid. Now assume that (14.1) is true with n replaced by $n - 1$ for any fixed integer $n > 1$. Let

$$A = \frac{a_3}{x+}\;\frac{a_4}{1+}\;\frac{a_5}{x+}\;\frac{a_6}{1+}\cdots+\frac{a_{2n}}{1}$$

and

$$B = \frac{a_4a_5}{x+a_5+a_6-}\;\frac{a_6a_7}{x+a_7+a_8-}\cdots-\frac{a_{2n-2}a_{2n-1}}{x+a_{2n-1}+a_{2n}}.$$

Then, by induction,

$$\frac{a_1}{x+}\;\frac{a_2}{1+}\;\frac{a_3}{x+}\;\frac{a_4}{1+}\cdots+\frac{a_{2n}}{1} = \frac{a_1}{x+}\;\frac{a_2}{1+A}$$
$$= \frac{a_1}{x+}\;\frac{a_2}{1+\dfrac{a_3}{x+a_4-B}} = \frac{a_1}{x+a_2-\dfrac{a_2a_3}{x+a_3+a_4-B}},$$

which completes the proof.

If we let n tend to ∞ in (14.1), we obtain an identity given by Preece **[60]** in somewhat more general form. Rogers **[65]** also derived a similar result.

ENTRY 15. *If we assume that the continued fraction on the left side below is convergent, then*

$$\begin{aligned}\frac{a_1+h}{1}\,{}_{+}\,\frac{a_1}{x}\,{}_{+}\,\frac{a_2+h}{1}\,{}_{+}\,\frac{a_2}{x}\,{}_{+}\cdots \\ = h+\frac{a_1}{1}\,{}_{+}\,\frac{a_1+h}{x}\,{}_{+}\,\frac{a_2}{1}\,{}_{+}\,\frac{a_2+h}{x}\,{}_{+}\cdots.\end{aligned} \tag{15.1}$$

PROOF. Let

$$F_k = x+\frac{a_k+h}{1}\,{}_{+}\,\frac{a_k}{x}\,{}_{+}\,\frac{a_{k+1}+h}{1}\,{}_{+}\,\frac{a_{k+1}}{x}\,{}_{+}\cdots, \qquad k \geqq 2.$$

Denoting the left side of (15.1) by F, we find that

$$\begin{aligned}F &= \frac{a_1+h}{1+a_1/F_2} = \frac{h(F_2+a_1)+a_1(F_2-h)}{F_2+a_1} \\ &= h+\frac{a_1(F_2-h)}{F_2+a_1} = h+\frac{a_1(F_2-h)}{(F_2-h)+(a_1+h)} \\ &= h+\cfrac{a_1}{1+\cfrac{a_1+h}{F_2-h}}.\end{aligned} \tag{15.2}$$

Next, for $k \geqq 2$,

$$\begin{aligned}F_k-h &= x-h+\frac{a_k+h}{1+a_k/F_{k+1}} = x+\frac{a_k\left(1-\dfrac{h}{F_{k+1}}\right)}{1+\dfrac{a_k}{F_{k+1}}} \\ &= x+\frac{a_k\left(1-\dfrac{h}{F_{k+1}}\right)}{\left(1-\dfrac{h}{F_{k+1}}\right)+\left(\dfrac{a_k+h}{F_{k+1}}\right)} = x+\cfrac{a_k}{1+\cfrac{a_k+h}{F_{k+1}-h}}.\end{aligned} \tag{15.3}$$

If we now use (15.3) successively in (15.2) beginning with $k=2$, we complete the proof.

ENTRY 16. *If neither m nor n is a negative integer, then*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(m+k)(n+k)}$$
$$=\frac{1}{(m+1)(n+1)}\,{}_{+}\,\frac{(m+1)^2(n+1)^2}{m+n+3}\,{}_{+}\,\frac{(m+2)^2(n+2)^2}{m+n+5}\,{}_{+}\,\frac{(m+3)^2(n+3)^2}{m+n+7}\,{}_{+}\cdots.$$

PROOF. We shall employ the Corollary in §1. Letting $a_k = (-1)^{k+1}/(m+k)(n+k)$, and letting r tend to ∞, we find that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(m+k)(n+k)}$$

$$= \frac{(m+1)^{-1}(n+1)^{-1}}{1} + \frac{(m+2)^{-1}(n+2)^{-1}}{(m+1)^{-1}(n+1)^{-1} - (m+2)^{-1}(n+2)^{-1}}$$

$$- \frac{(m+1)^{-1}(n+1)^{-1}(m+3)^{-1}(n+3)^{-1}}{-(m+2)^{-1}(n+2)^{-1}+(m+3)^{-1}(n+3)^{-1}} - \cdots$$

$$= \frac{1}{(m+1)(n+1)} + \frac{(m+1)^2(n+1)^2}{(m+2)(n+2) - (m+1)(n+1)}$$

$$+ \frac{(m+2)^2(n+2)^2}{(m+3)(n+3) - (m+2)^{-1}(n+2)} + \cdots,$$

from which the proposed identity readily follows.

ENTRY 17. *Write*

$$\frac{1}{1} + \frac{a_1 x}{1} + \frac{a_2 x}{1} + \frac{a_3 x}{1} + \cdots = \sum_{k=0}^{\infty} A_k(-x)^k, \tag{17.1}$$

where $A_0 = 1$. *Let*

$$P_n = a_1 a_2 \cdots a_{n-1}(a_1 + a_2 + \cdots + a_n),\ n \geqq 1.$$

Then

$$P_1 = A_1,$$
$$P_2 = A_2,$$
$$P_3 = A_3 - a_1 A_2,$$
$$P_4 = A_4 - (a_1 + a_2)A_3,$$
$$P_5 = A_5 - (a_1 + a_2 + a_3)A_4 + a_1 a_3 A_3,$$
$$P_6 = A_6 - (a_1 + a_2 + a_3 + a_4)A_5 + (a_1 a_3 + a_2 a_4 + a_1 a_4)A_4.$$

In general, for $n \geqq 1$,

$$P_n = \sum_{0 \le k < n/2} (-1)^k \varphi_k(n) A_{n-k}, \tag{17.2}$$

where $\varphi_0(n) \equiv 1$ *and* $\varphi_r(n)$, $r \geqq 1$, *is defined recursively by*

$$\varphi_r(n+1) - \varphi_r(n) = a_{n-1}\varphi_{r-1}(n-1). \tag{17.3}$$

FIRST PROOF. Let $C_n = C_n(x)$ and $B_n = B_n(x)$ denote the numerator and denominator, respectively, of the nth convergent of the continued fraction (17.1). Then, from (1.3) and (1.4), (17.4) $$\begin{cases} C_1 = C_2 = 1,\ C_n = C_{n-1} + a_{n-1}x\, C_{n-2}, \\ B_1 = 1,\ B_2 = 1 + a_1 x,\ B_n = B_{n-1} + a_{n-1}x\, B_{n-2}, \qquad n \geqq 3. \end{cases}$$

By induction, it is easily seen that C_{2n-1}, C_{2n}, and B_{2n-1} are polynomials in x of degree $n - 1$, while B_{2n} is of degree $2n$, where $n \geqq 1$. Thus, for $n \geqq 1$, set

(17.5) $$B_n(x) = \sum_{k=0}^{[n/2]} \beta_k(n+1)x^k.$$

We make the convention that $\beta_k(n+1) = 0$ if $k > [n/2]$. From (17.4), it is obvious that $\beta_0(n+1) = 1$ for each $n \geqq 1$. Using (17.5) in the recursion formula for B_n given in (17.4) and equating coefficients of x^r, we readily deduce that

(17.6) $$\beta_r(n+1) - \beta_r(n) = a_{n-1}\beta_{r-1}(n-1), \qquad r \geqq 1.$$

Thus, by (17.3) and (17.6), we see that $\varphi_r(n)$ and $\beta_r(n)$ satisfy the same recursion formula. Since, furthermore, $\varphi_0(n) \equiv 1 \equiv \beta_0(n)$, we conclude that $\varphi_r(n) = \beta_r(n)$, $r \geqq 0$. Also note that $\varphi_r(n) \equiv 0$ if $r \geqq [n/2]$.

Put

(17.7) $$E_n = \frac{a_n}{1} \, + \, \frac{a_{n+1}x}{1} \, + \, \frac{a_{n+2}x}{1} \, + \cdots,$$

where $n \geqq 0$ and $a_0 = 1$. We shall show, by induction, that (see also [**65**, p. 72, eq. (1)])

(17.8) $$E_0B_n - C_n = (-1)^n E_0E_1 \cdots E_n x^n, \qquad n \geqq 1.$$

Since, by (17.7), $E_0 = 1/(1 + xE_1)$, (17.8) is easy to establish for $n = 1$. Assume now that (17.8) is valid for each nonnegative integer up to and including n. Then, by (17.4),

$$\begin{aligned} E_0B_{n+1} - C_{n+1} &= E_0B_n - C_n + a_nx(E_0B_{n-1} - C_{n-1}) \\ &= (-1)^n E_0E_1 \cdots E_nx^n + (-1)^{n-1} a_nxE_0E_1 \cdots E_{n-1}x^{n-1} \\ &= (-1)^n E_0E_1 \cdots E_{n-1}x^n(E_n - a_n) \\ &= (-1)^n E_0E_1 \cdots E_{n-1}x^n\left(\frac{a_n}{1 + xE_{n+1}} - a_n\right) \\ &= (-1)^{n+1} E_0E_1 \cdots E_nE_{n+1}x^{n+1}, \end{aligned}$$

and so the induction is complete.

Write

(17.9) $$E_0E_1 \cdots E_n = \sum_{k=0}^{\infty} e_k(n)x^k.$$

Setting $x = 0$, we find that

$$e_0(n) = a_1 a_2 \cdots a_n. \tag{17.10}$$

Next rewrite (17.9) in the form

$$\frac{1}{1+xE_1}\,\frac{a_1}{1+xE_2}\cdots\frac{a_n}{1+xE_{n+1}} - a_1a_2\cdots a_n = \sum_{k=1}^{\infty} e_k(n)x^k.$$

Dividing both sides by x and then letting x tend to 0, we deduce that

$$e_1(n) = -a_1a_2\cdots a_n(a_1 + a_2 + \cdots + a_{n+1}) = -P_{n+1}, \tag{17.11}$$

for each nonnegative integer n.

In (17.8) replace n by $n-1$. Then, by (17.1), (17.5), (17.9), (17.10), and (17.11),

$$\begin{aligned}
&\sum_{j=0}^{\infty} A_j(-x)^j \sum_{k=0}^{[(n-1)/2]} \varphi_k(n)x^k - C_{n-1}(x) \\
&= E_0B_{n-1}(x) - C_{n-1}(x) \\
&= (-1)^{n-1}a_1a_2\cdots a_{n-1}x^{n-1} + (-1)^nP_nx^n + \cdots .
\end{aligned}$$

Equating coefficients of x^n, $n \geqq 2$, yields

$$(-1)^nP_n = \sum_{k=0}^{[(n-1)/2]} (-1)^{n-k}A_{n-k}\varphi_k(n),$$

which is precisely (17.2). Since the case $n = 1$ of (17.2) is readily verified, the proof is complete.

Essentially the same proof that we have given above was independently and almost simultaneously discovered by Goulden and Jackson **[31]**. They **[31]** have also found a beautiful combinatorial proof of (17.3) by enumerating certain paths.

Before proceeding further, we shall find an exact formula for $\varphi_k(n)$, defined by (17.3).

First, it is not difficult to show that

$$\varphi_1(n) = \sum_{j=1}^{n-2} a_j$$

and

$$\varphi_2(n) = \sum_{\substack{1\leqq i\leqq j-2\\ 3\leqq j\leqq n-2}} a_ia_j.$$

We shall show by induction on k that

$$\varphi_k(n) = \sum_{\substack{1\leqq j_1\leqq j_2-2\\ 1\leqq j_2\leqq j_3-2\\ \vdots\\ 1\leqq j_k\leqq n-2}} a_{j_1}a_{j_2}\cdots a_{j_k}. \tag{17.12}$$

We have already indicated that (17.12) is true for $k = 1, 2$. Proceeding by induction and employing (17.3), we find that

$$\varphi_k(n) - \varphi_k(n-1) = a_{n-2} \sum_{\substack{1 \leqq j_1 \leqq j_2 - 2 \\ \vdots \\ 1 \leqq j_{r-1} \leqq n-4}} a_{j_1} a_{j_2} \cdots a_{j_{r-1}},$$

$$\varphi_k(n-1) - \varphi_k(n-2) = a_{n-3} \sum_{\substack{1 \leqq j_1 \leqq j_2 - 2 \\ \vdots \\ 1 \leqq j_{r-1} \leqq n-5}} a_{j_1} a_{j_2} \cdots a_{j_{r-1}},$$

$$\vdots$$

Adding together all of the equalities above, we deduce (17.12). This completes the proof of the desired exact formula for $\varphi_k(n)$.

Rogers **[65]** has expressed $\varphi_k(n)$ by a determinant.

We are extremely grateful to G. E. Andrews for providing us with the following elegant, second proof of Entry 17. In fact, this proof was found prior to the proofs of Goulden and Jackson **[31]** and the authors. The first part of Andrews' argument was anticipated by De Morgan **[18]**.

SECOND PROOF OF ENTRY 17. We first obtain a recursion formula for the coefficients A_k, $k \geqq 0$. In order to do this, we introduce auxiliary coefficients $\bar{A}_k$, $k \geqq 0$, which we now define. Of course, each coefficient A_k can be written in terms of $a_1, a_2, \ldots$. We define $\bar{A}_k$ by the same expression for A_k except that the subscript of each a_j appearing in A_k is increased by 1. For example, since $A_2 = a_1^2 + a_1 a_2$, we define $\bar{A}_2 = a_2^2 + a_2 a_3$.

Now, by (17.1),

$$\begin{aligned}
\sum_{k=0}^{\infty} (-1)^k A_k x^k &= \cfrac{1}{1 + a_1 x\left(\cfrac{1}{1} + \cfrac{a_2 x}{1} + \cfrac{a_3 x}{1} + \cdots\right)} \\
&= \frac{1}{1 + a_1 x \sum_{k=0}^{\infty} (-1)^k \bar{A}_k x^k} \\
&= \frac{1}{\sum_{k=0}^{\infty} (-1)^{k-1} a_1 \bar{A}_{k-1} x^k},
\end{aligned}$$

where $\bar{A}_{-1} = -1/a_1$. Multiply both sides of the extremal equation above by the denominator on the right side and equate coefficients of x^n on both sides to deduce that, for $n \geqq 1$, $\sum_{k=-1}^{n-1} a_1 \bar{A}_k A_{n-k-1} = 0$, or

$$A_n = \sum_{k=0}^{n-1} a_1 \bar{A}_k A_{n-k-1}, \qquad n \geqq 1, \tag{17.13}$$

which is the recurrence formula that we sought.

We now show that

(17.14) A_n is a homogeneous polynomial of degree n in the noncommutative variables $a_1, a_2, \ldots, a_n$, where the subscripts $j_1 j_2, \ldots, j_n$ of the monomials comprising A_n are precisely those sequences of positive integers starting at 1 for which $j_{k+1} - j_k \leqq 1, j_k \geqq 1$.

In order to make clearer the assertion above, we record the following examples:

$$A_1 = a_1,$$
$$A_2 = a_1a_2 + a_1a_1,$$
$$A_3 = a_1a_2a_2 + a_1a_2a_3 + a_1a_2a_1 + a_1a_1a_2 + a_1a_1a_1.$$

We now prove the assertion (17.14) by inducting on n. By using (17.13), we easily verify that (17.14) is true for $n = 1, 2, 3$, as indicated above. Assume that (17.14) is true up to but not including a specific integer n. Let A_n^* denote the polynomial described by (17.14). We shall show that A_n^* is equal to the right side of (17.13). Thus, $A_n^* = A_n$, which completes the induction. Let us divide the monomials comprising A_n^* into n classes. The kth class, $0 \leqq k \leqq n - 1$, consists of all monomials in A_n^* wherein the second appearance of a_1 is the $(k + 2)$nd term in the monomial. (Recall that a_1 begins each monomial.) Thus, the entries of the kth class are produced in the following manner. Start with a_1, adjoin a string of k a_j's, $j \geqq 2$, that starts with a_2 and follows the appropriate subscript rules, and lastly adjoin a string of $n - k - 1$ a_j's that starts with a_1 and follows the prescribed subscript rules. But the entries for the string of k terms are generated by $\bar{A}_k$ and the entries for the remaining $n - k - 1$ terms are generated by A_{n-k-1}, by induction. Hence, the monomials in the kth class are generated by $a_1\bar{A}_kA_{n-k-1}$. Summing on $k, 0 \leqq k \leqq n - 1$, we find that

$$A_n^* = \sum_{k=0}^{n-1} a_1\bar{A}_kA_{n-k-1},$$

which, by (17.13), completes the induction.

We now have a combinatorial interpretation (17.14) for A_n. After finding combinatorial interpretations for P_n and $\varphi_k(n)$, we shall use a sieving process to establish (17.2).

Let us say that a word of the type generated by A_n, i.e., $a_{j_1}a_{j_2} \ldots a_{j_n}$, where $j_1 = 1$ and $j_{k+1} - j_k \leqq 1$, with $j_k \geqq 1$, has an "internal drop" if $j_{k+1} - j_k \neq 1$ for some k, $1 \leqq k < n - 1$. Then we see that P_n is the polynomial in $a_1, a_2, \ldots, a_n$ composed of all words without internal drops.

From (17.12), observe that $\varphi_k(n)$ is a homogeneous polynomial of degree k in the noncommuting variables $a_1, a_2, \ldots, a_{n-2}$ wherein the subscripts of each monomial $a_{j_1}a_{j_2} \ldots a_{j_k}$ satisfy the inequalities $j_{i+1} - j_i \geqq 2$, $1 \leqq i \leqq k - 1$.

We now begin the sieving procedure. We first examine A_n. Recall that an internal drop occurs when $j_{i+1} - j_i \neq 1$ and $1 \leqq i < n - 1$. Let us call a_{j_k} the "top of the last internal drop" if k is maximal for internal drops, i.e., if $j_{i+1} - j_i \neq 1$, $1 \leqq i < n - 1$, then $j_{k+1} - j_k \neq 1$, $1 \leqq k < n - 1$, and $i \leqq k$. The top of the last internal drop must be one of the letters $a_1, a_2, \ldots, a_{n-2}$, since neither a_{n-1} nor a_n can be far enough to the left in a word to be at the top of an internal drop.

In order to eliminate all words from A_n with internal drops, we take the words from A_{n-1} and insert a_j, $1 \leqq j \leqq n - 2$, in the last position where it forms the top of an internal drop. Thus,

$$A_n - (a_1 + a_2 + \cdots + a_{n-2})A_{n-1} \tag{17.15}$$

does not possess any internal drops. (Note that we have written (17.15) commutatively; the correct noncommutative expression would have a_j, $1 \leqq j \leqq n - 2$, inserted as described above.) Unfortunately, there are words in (17.15) that were not originally in A_n. These words arose when the insertion of an a_j produced a subscript increase greater than or equal to 2 from the a_i immediately to the left of the inserted a_j. Of course, we must eliminate these undesirable words. We do this by taking the words of A_{n-2} and inserting pairs $a_i a_j$ with $j - i \geqq 2$ so that a_j is at the top of the last internal drop. Hence,

$$A_n - \varphi_1(n)A_{n-1} + \varphi_2(n)A_{n-2} \tag{17.16}$$

does not possess internal drops. (Again, note that (17.16) is a commutative representation of what is really a noncommutative polynomial in $a_1, a_2, \ldots, a_n$.) Unfortunately, we have now introduced some new words which were not originally under consideration. These new words have triples $a_i a_j a_k$ with $j - i \geqq 2$ and $k - j \geqq 2$, and with a_k at the top of the last internal drop.

We continue the process described above by induction. At each stage we must introduce a term

$$(-1)^k \varphi_k(n) A_{n-k}$$

to compensate for unwanted terms introduced at the previous stage. Fortunately, $\varphi_k(n) = 0$ for $k \geqq n/2$, which is evident from (17.12). Thus, the sieving process terminates, and we reach the desired formula (17.2).

Among others, Muir **[49]** and Rogers **[65]** have studied the problem of deriving a continued fraction expansion from the coefficients of a power series. Both De Morgan **[18]** and Rogers **[65]** have commented on the fact that it is extremely more difficult to determine the power series coefficients A_k, $0 \leqq k < \infty$, from a continued fraction of the form (17.1).

Ramanujan's Entry 17 is a fascinating contribution to this more recondite converse problem.

By a theorem of Euler [22], [37, p. 37] (see also (1.2)),

$$\frac{1}{1}+\frac{a_1x}{1}+\frac{a_2x}{1}+\cdots=\sum_{k=0}^{\infty}\frac{(-1)^k a_1a_2\cdots a_k}{B_kB_{k+1}}x^k,$$

where $B_k = B_k(x)$ is given by (1.4) and (17.5). Thus, $(-1)^n A_n$ is equal to the coefficient of x^n in

$$\sum_{k=0}^{n}\frac{(-1)^k a_1\cdots a_k}{B_k(x)B_{k+1}(x)}x^k.$$

Obtaining a general formula for A_n in this manner seems hopeless.

However, a very complicated formula for A_n can be established combinatorially by counting planted plane trees with respect to their heights in two different ways. For a nice exposition of this proof, see the book of Goulden and Jackson [30]. See also a paper of Flajolet [28].

COROLLARY (i). *Write*

$$\frac{1}{1+b_1x}+\frac{a_1x}{1+b_2x}+\frac{a_2x}{1+b_3x}+\cdots=\sum_{k=0}^{\infty}A_k(-x)^k, \tag{17.17}$$

where $A_0 = 1$. *Define*

$$P_n = a_1a_2\cdots a_{n-1}(a_1+b_1+a_2+b_2+\cdots+a_n+b_n), \qquad n \geqq 1.$$

Then, for $n \geqq 1$,

$$P_n=\sum_{k=0}^{n-1}(-1)^k\varphi_k(n)A_{n-k},$$

where $\varphi_0(n) \equiv 1$ *and* $\varphi_r(n)$, $r \geqq 1$, *is defined recursively by*

$$\varphi_r(n+1)-\varphi_r(n)=b_n\varphi_{r-1}(n)+a_{n-1}\varphi_{r-1}(n-1).$$

As with Entry 17, Goulden and Jackson [31] independently and simultaneously discovered the proof that the authors found and record below. Goulden and Jackson [31] have also derived a combinatorial proof. Since the proof below is very similar to the first proof of Entry 17, we give only a brief sketch.

PROOF. Let $C_n = C_n(x)$ and $B_n = B_n(x)$ denote the numerator and denominator, respectively, of the nth convergent of the continued fraction (17.17). Then

$$\begin{cases} C_1=1,\ C_2=1+b_2x,\ C_n=(1+b_nx)C_{n-1}+a_{n-1}x\,C_{n-2},\\ B_1=1+b_1x,\ B_2=1+(a_1+b_1+b_2)x+b_1b_2x^2,\\ B_n=(1+b_nx)B_{n-1}+a_{n-1}xB_{n-2}, \end{cases} \tag{17.18}$$

where $n \geqq 3$. Observe that $C_n(x)$ has degree $n-1$ and $B_n(x)$ has degree n, $n \geqq 1$. Thus, put

$$B_n(x) = \sum_{k=0}^{n} \beta_k(n+1)x^k, \qquad n \geqq 1. \tag{17.19}$$

By substituting (17.19) into the recursion formula for $B_n(x)$ in (17.18) and equating coefficients of x^r, we deduce that $\beta_r(n) = \varphi_r(n)$, $r \geqq 0$, $n \geqq 2$.

The remainder of the proof is exactly parallel to that of the first proof of Entry 17.

COROLLARY (ii). *Let $B_n(x)$ be defined as at the beginning of the proof of* Entry 17. *Then, for $n \geqq 1$,*

$$B_n(x) = \sum_{k=0}^{[n/2]} \varphi_k(n+1)x^k.$$

PROOF. Corollary (ii) was established in the course of proving Entry 17. In particular, recall that $\beta_k(n) = \varphi_k(n)$ and consult (17.5).

EXAMPLE. We have

$$ {}_2F_1^2(1/2, 1/2; 1; x) = \frac{1}{1}\,-\,\frac{x}{2}\,-\,\frac{3x}{8}\,-\,\frac{5x}{2}\,-\,\frac{17x}{40}\,-\,\frac{23x}{2}\,-\,\frac{1395x}{3128}\,-\cdots. \tag{17.22}$$

PROOF. Ramanujan evidently intends this example to be an illustration for Entry 17. In the notation of Entry 17, $a_1 = -1/2$, $a_2 = -3/16$, $a_3 = -5/16$, $a_4 = -17/80$, $a_5 = -23/80$, and $a_6 = -1395/6256$. Squaring ${}_2F_1(1/2, 1/2; 1; x)$, we find, after some laborious computing, that $A_1 = -1/2$, $A_2 = 11/32$, $A_3 = -17/64$, $A_4 = 1787/2^{13}$, $A_5 = -3047/2^{14}$, and $A_6 = 42631/2^{18}$. Lastly, $P_1 = -1/2$, $P_2 = 11/32$, $P_3 = -3/32$, $P_4 = 291/2^{13}$, $P_5 = -153/2^{14}$, and $P_6 = 32337/5\cdot 2^{21}$. All of these calculations are in agreement with Entry 17, and so (17.22) is, indeed, correct.

ENTRY 18. *Let x be any complex number outside the real interval* $[-1, 1]$ *and let n be any real number. Then*

$$\frac{(x+1)^n-(x-1)^n}{(x+1)^n+(x-1)^n} = \frac{n}{x}\,+\,\frac{n^2-1^2}{3x}\,+\,\frac{n^2-2^2}{5x}\,+\,\frac{n^2-3^2}{7x}\,+\cdots. \tag{18.1}$$

FIRST PROOF. If we replace x by $1/x$ in [**57**, p. 153, eq. (9)], we obtain a continued fraction representation easily found to be equivalent to (18.1).

Perron's derivation of Entry 18 arises from Entry 20.

SECOND PROOF. Let

$$g(m, n, x) = \frac{\Gamma(\tfrac{1}{2}(mx+m-n+1))\Gamma(\tfrac{1}{2}(mx-m+n+1))}{\Gamma(\tfrac{1}{2}(mx+m+n+1))\Gamma(\tfrac{1}{2}(mx-m-n+1))}.$$

Replacing x by mx in Entry 33, we find that

$$(18.2)\qquad \begin{aligned}\frac{1-g(m,n,x)}{1+g(m,n,x)}&=\frac{mn}{mx+}\,\frac{(m^2-1^2)(n^2-1^2)}{3mx}\,\frac{}{+}\,\frac{(m^2-2^2)(n^2-2^2)}{5mx}\,\frac{}{+}\cdots\\ &=\frac{n}{x+}\,\frac{(1-1/m^2)(n^2-1^2)}{3x}\,\frac{}{+}\,\frac{(1-2^2/m^2)(n^2-2^2)}{5x}\,\frac{}{+}\cdots.\end{aligned}$$

Now let m tend to ∞ in (18.2). By using an asymptotic formula for the quotient of Γ-functions [**47**, p. 33] or Stirling's formula, we find that $\lim_{m\to\infty} g(m, n, x) = (x+1)^{-n}(x-1)^n$. Formula (18.1) now easily follows.

Entry 18 is due to Euler [**27**] and easily implies a continued fraction expansion for $(x+1)^n/(x-1)^n$ due to Laguerre [**42**], [**57**, p. 153, eq. (10)].

If V_n denotes the left side of (18.1), then Ramanujan remarks that $V_n + 1/V_n = 2/V_{2n}$, a fact that is easily verified.

COROLLARY 1. *Let x be any complex number outside the cuts $(-i\infty, -i]$ and $[i, i\infty)$. Then*

$$\tan^{-1} x = \frac{x}{1+}\,\frac{x^2}{3+}\,\frac{(2x)^2}{5}\,\frac{}{+}\,\frac{(3x)^2}{7}\,\frac{}{+}\cdots.$$

For a proof see [**57**, p. 155]. Early proofs of Corollary 1 were given by Lambert [**44**], Lagrange [**41**], and Euler [**27**].

COROLLARY 2. *Let x be any complex number outside the cuts $(-\infty, -1]$ and $[1, \infty)$. Then*

$$\operatorname{Log}\frac{1+x}{1-x} = \frac{2x}{1-}\,\frac{x^2}{3-}\,\frac{(2x)^2}{5}\,\frac{}{-}\,\frac{(3x)^2}{9}\,\frac{}{-}\cdots.$$

For a proof see [**57**, p. 154]. Corollary 2 is due to Euler [**27**]. For an application of Corollary 2 to product-weighted lead codes, see a paper of Jackson [**36**].

COROLLARY 3. *For any complex number x,*

$$\tan x = \frac{x}{1-}\,\frac{x^2}{3-}\,\frac{x^2}{5-}\,\frac{x^2}{7-}\cdots.$$

Corollary 3 was initially discovered by Lambert [**43**], [**45**]. A proof may be found in Perron's book [**57**, p. 157].

COROLLARY 4. *For any complex number x,*

$$\frac{e^x-1}{e^x+1} = \frac{x}{2+}\,\frac{x^2}{6+}\,\frac{x^2}{10+}\,\frac{x^2}{14+}\cdots.$$

Corollary 4 is due to Euler [**25**] and a proof may be found in Perron's text [**57**, p. 157].

ENTRY 19. *If n and x are arbitrary complex numbers, then*

$$\frac{x\,{}_0F_1(n+1;x)}{n\,{}_0F_1(n;x)} = \frac{\sqrt{x}\,J_n(2i\sqrt{x})}{iJ_{n-1}(2i\sqrt{x})} = \frac{x}{n} \, \frac{x}{+\,n+1} \, \frac{x}{+\,n+2} + \cdots,$$

where J_ν denotes the ordinary Bessel function of order ν.

FIRST PROOF. By a theorem of Euler [**25**], [**57**, p. 281, Satz 6.3],

$$c + \frac{a}{c+d} \, \frac{a}{+\,c+2d} \, \frac{a}{+\,c+3d} + \cdots = c\,\frac{{}_0F_1(c/d;\, a/d^2)}{{}_0F_1(c/d+1;\, a/d^2)}, \tag{19.1}$$

where $d \neq 0$. Let $c = n$, $a = x$, and $d = 1$ to find that

$$n + \frac{x}{n+1} \, \frac{x}{+\,n+2} \, \frac{x}{+\,n+3} + \cdots = n\,\frac{{}_0F_1(n;\, x)}{{}_0F_1(n+1;\, x)}.$$

Taking the reciprocal of both sides above and then multiplying both sides by x, we deduce the desired result.

SECOND Proof. This proof is similar to the proof above, but employs a "finite" version of (19.1), namely Entry 24. Simply let r tend to ∞ in Entry 24. After multiplying both sides by x/n, we complete the proof.

See also [**74**, p. 349].

ENTRY 20. *If x is any complex number outside the interval $(-\infty, -1]$, then*

$$\frac{\alpha\beta x}{\gamma}\,\frac{{}_2F_1(\gamma-\alpha, \beta+1; \gamma+1; -x)}{{}_2F_1(\gamma-\alpha, \beta; \gamma; -x)} = \frac{\alpha\beta x}{\gamma} \, \frac{(\alpha-\gamma)(\beta-\gamma)x}{+\,\gamma+1} \, \frac{(\alpha+1)(\beta+1)x}{+\,\gamma+2}$$
$$+ \frac{(\alpha-\gamma-1)(\beta-\gamma-1)x}{\gamma+3} \, \frac{(\alpha+2)(\beta+2)x}{+\,\gamma+4} + \cdots.$$

This result is very famous and is known as Gauss's continued fraction [**29**]. A proof may be found in any of the standard texts [**37**], [**38**], [**57**], [**74**]. It might be mentioned that Gauss's continued fraction may be found in Carr's book [**14**, p. 97], which was the most influential book in Ramanujan's development. Recent work on Gauss's continued fraction may be found in [**7**].

ENTRY 21. *We have*

$$\frac{\beta x}{\gamma}\,{}_2F_1(\beta+1, 1; \gamma+1; -x)$$

$$= \frac{\beta x}{\gamma} \, \frac{\gamma(\beta+1)x}{+\,\gamma+1} \, \frac{1(\gamma-\beta)x}{+\,\gamma+2} \, \frac{(\gamma+1)(\beta+2)x}{+\,\gamma+3} \tag{21.1}$$
$$+ \frac{2(\gamma+1-\beta)x}{\gamma+4} + \cdots, \qquad \textit{if } x \notin (-\infty, -1],$$

$$(21.2)\quad = \frac{\beta x}{\gamma} + \frac{(\beta+1)x}{1} + \frac{1(1+x)}{\gamma} + \frac{(\beta+2)x}{1} + \frac{2(1+x)}{\gamma} + \cdots,$$

$$\text{if } \operatorname{Re}(x) > -\frac{1}{2},$$

$$(21.3)\quad = \frac{\beta x}{\gamma+x(\beta+1)} - \frac{1(\beta+1)x(x+1)}{\gamma+1+x(\beta+3)} - \frac{2(\beta+2)x(x+1)}{\gamma+2+x(\beta+5)} + \cdots,$$

$$\text{if } \operatorname{Re}(x) > -\frac{1}{2}.$$

PROOF. The proof that we give of (21.1) is indicated by Ramanujan in the first notebook [**64**, vol. 1, p. 217]. Let

$$G = \frac{(\beta+1)x}{(\gamma+1)} {}_2F_1(\beta+2, 1; \gamma+2; -x).$$

Then

$$(21.4)\quad \frac{\beta x}{\gamma} {}_2F_1(\beta+1, 1; \gamma+1; -x) = \frac{\beta x}{\gamma}(1-G) = \frac{\beta x}{\gamma} \frac{1}{1+\dfrac{G}{1-G}}.$$

Now in Entry 20, replace β by $\beta+1$ and γ by $\gamma+1$ and then set $\alpha = \gamma$. This yields

$$\frac{G}{1-G} = \frac{(\beta+1)x}{\gamma+1} \frac{{}_2F_1(\beta+2, 1; \gamma+2; -x)}{{}_2F_1(\beta+1, 1; \gamma+1; -x)}$$

$$(21.5)\quad = \frac{(\beta+1)x}{\gamma+1} + \frac{1(\gamma-\beta)x}{\gamma+2} + \frac{(\gamma+1)(\beta+2)x}{\gamma+3} + \frac{2(\gamma-\beta+1)x}{\gamma+4} + \frac{(\gamma+2)(\beta+3)x}{\gamma+5} + \cdots.$$

If we substitute (21.5) into (21.4), we complete the proof of (21.1).

We next prove (21.3). If $\operatorname{Re}(x) < 1/2$, $\alpha+1$ and $\gamma-\alpha$ are not both nonpositive integers, and $\beta+1$ and $\gamma-\beta$ are not both nonpositive integers, then by a result of Nörlund [**52**], [**57**, p. 286, eq. (10)],

$$\frac{{}_2F_1(\alpha+1, \beta+1; \gamma+1; x)}{\gamma\, {}_2F_1(\alpha, \beta; \gamma; x)}$$

$$(21.6)\quad = \frac{1}{\gamma-(1+\alpha+\beta)x} + \frac{(\alpha+1)(\beta+1)(x-x^2)}{\gamma+1-(3+\alpha+\beta)x} + \frac{(\alpha+2)(\beta+2)(x-x^2)}{\gamma+2-(5+\alpha+\beta)x} + \cdots.$$

Setting $\alpha = 0$, replacing x by $-x$, and lastly multiplying both sides by βx, we complete the proof of (21.3).

Lastly, we prove that the continued fractions (21.2) and (21.3) are equal. To do this, simply apply Entry 14 and then let n tend to ∞.

COROLLARY 1. *For every complex number x, we have*

$$\frac{x}{n}\,{}_1F_1(1; n+1; x)$$

$$= \frac{x}{n} - \frac{nx}{n+1} + \frac{x}{n+2} - \frac{(n+1)x}{n+3} + \frac{2x}{n+4} - \cdots$$

$$= \frac{x}{n-x} + \frac{x}{n+1-x} + \frac{2x}{n+2-x} + \frac{3x}{n+3-x} + \cdots.$$

PROOF. To prove the first equality, replace γ by n and x by $-x/\beta$ in (21.1). Letting β tend to ∞, we easily deduce the desired result.

To prove the second equality, employ (21.3) and proceed in precisely the same manner as above.

COROLLARY 2. *If x is any complex number, then*

$${}_1F_1(1; x+1; x) = 1 + \frac{2x}{2} + \frac{3x}{3} + \frac{4x}{4} + \frac{5x}{5} + \cdots.$$

PROOF. Let $n = x$ in the second equality of Corollary 1.

ENTRY 22. *If* $|x| < 1$, *then*

$$\frac{\beta x}{\gamma}\,\frac{{}_2F_1(-\alpha, \beta+1; \gamma+1; -x)}{{}_2F_1(-\alpha, \beta; \gamma; -x)}$$

$$= \frac{\beta x}{\gamma - (\alpha+\beta+1)x} + \frac{(\beta+1)(\alpha+\gamma+1)x}{\gamma+1-(\alpha+\beta+2)x} + \frac{(\beta+2)(\alpha+\gamma+2)x}{\gamma+2-(\alpha+\beta+3)x} + \cdots.$$

PROOF. Since [47, p. 67, eq. (4)]

$${}_2F_1(a, b; c; x) = (1-x)^{c-a-b}\,{}_2F_1\left(c-a, b; c; \frac{x}{x-1}\right),$$

we may write (21.6) in the form

$$\frac{x}{\gamma(1-x)}\,\frac{{}_2F_1\left(\gamma-\alpha, \beta+1; \gamma+1; \frac{1}{x-1}\right)}{{}_2F_1\left(\gamma-\alpha, \beta; \gamma; \frac{1}{x-1}\right)}$$

$$= \frac{x}{\gamma-(1+\alpha+\beta)x} + \frac{(\alpha+1)(\beta+1)(x-x^2)}{\gamma+1-(3+\alpha+\beta)x} + \frac{(\alpha+2)(\beta+2)(x-x^2)}{\gamma+1-(5+\alpha+\beta)x} + \cdots,$$

provided that $\mathrm{Re}(x) < 1/2$. Letting $u = x/(1-x)$, we find, after simplification, that

$$\frac{u\,{}_2F_1(\gamma-\alpha,\beta+1;\gamma+1;-u)}{\gamma\,{}_2F_1(\gamma-\alpha,\beta;\gamma;-u)}$$
$$=\frac{u}{\gamma(u+1)-(1+\alpha+\beta)u}+\frac{(\alpha+1)(\beta+1)u}{(\gamma+1)(u+1)-(3+\alpha+\beta)u}$$
$$+\frac{(\alpha+2)(\beta+2)u}{(\gamma+2)(u+1)-(5+\alpha+\beta)u}+\cdots,$$

provided that $|u|<1$. Replacing α by $\alpha+\gamma$ in the foregoing equality, we readily complete the proof.

Perron [**57**, p. 306] attributes Entry 22 to Andoyer [**2**], and, as we have seen, Entry 22 is equivalent to Nörlund's result (21.6). However, in the appendix to this paper, R. Askey points out that Entry 22 is really due to Euler.

ENTRY 23. *Write, for each nonnegative integer n,*

$$\frac{a_n}{b_nx}+\frac{a_{n+1}}{b_{n+1}x}+\frac{a_{n+2}}{b_{n+2}x}+\cdots=c_n\sum_{k=0}^{\infty}A_n(k)(-x)^k, \tag{23.1}$$

where $A_n(0)=1$. *Then* $c_nc_{n+1}=a_n$,

$$A_n(1)+A_{n+1}(1)=\frac{b_n}{c_{n+1}}=\frac{b_nc_n}{a_n}, \tag{23.2}$$

$$\begin{aligned}A_n(2)+A_{n+1}(2)&=A_n^2(1),\\ A_n(3)+A_{n+1}(3)&=A_n(1)\{A_n(2)-A_{n+1}(2)\},\\ A_n(4)+A_{n+1}(4)&=A_n(1)\{A_n(3)-A_{n+1}(3)\}-A_n(2)A_{n+1}(2),\end{aligned} \tag{23.3}$$

and, in general, for $k\geqq 3$,

$$\begin{aligned}&A_n(k)+A_{n+1}(k)\\&=A_n(1)\{A_n(k-1)-A_{n+1}(k-1)\}-\sum_{j=2}^{k-2}A_n(j)A_{n+1}(k-j).\end{aligned} \tag{23.4}$$

PROOF. From (23.1),

$$\frac{a_n}{b_nx+c_{n+1}\sum_{k=0}^{\infty}A_{n+1}(k)(-x)^k}=c_n\sum_{k=0}^{\infty}A_n(k)(-x)^k,$$

or

$$\begin{aligned}a_n&=c_n\Big(b_nx+c_{n+1}\sum_{k=0}^{\infty}A_{n+1}(k)(-x)^k\Big)\sum_{k=0}^{\infty}A_n(k)(-x)^k\\&=-c_nb_n\sum_{k=1}^{\infty}A_n(k-1)(-x)^k+c_nc_{n+1}\sum_{k=0}^{\infty}\sum_{j=0}^{k}A_n(j)A_{n+1}(k-j)(-x)^k.\end{aligned}$$

Now equate coefficients of x^k, $k \geqq 0$, on both sides. For $k = 0$, we find that $c_n c_{n+1} = a_n$, and, for $k \geqq 1$, we deduce that

$$(23.5) \quad A_n(k) + A_{n+1}(k) = \frac{b_n}{c_{n+1}} A_n(k-1) - \sum_{j=1}^{k-1} A_n(j) A_{n+1}(k-j).$$

Letting $k = 1$, we immediately deduce (23.2). Using (23.2) in (23.5), we find that, for $k \geqq 2$,

$$A_n(k) + A_{n+1}(k) = \{A_n(1) + A_{n+1}(1)\} A_n(k-1) - \sum_{j=1}^{k-1} A_n(j) A_{n+1}(k-j).$$

Upon simplifying the equality above, we deduce both (23.3) and (23.4).

EXAMPLE. We have

$$\lim_{x\to\infty} \left\{ \sqrt{\frac{2x}{\pi}} - \frac{x}{1+} \frac{2x}{2+} \frac{3x}{3+} \frac{4x}{4+} \cdots \right\} = \frac{2}{3\pi}.$$

PROOF. From Entry 47, for $x > 0$,

$$1 + \frac{e^x \Gamma(x+1)}{x^x} - \int_0^\infty e^{-t}\left(1 + \frac{t}{x}\right)^x dt = 1 + \frac{2x}{2+} \frac{3x}{3+} \frac{4x}{4+} \cdots.$$

Taking the reciprocal of both sides, we find that

$$L(x) \equiv \frac{x}{1 + \dfrac{e^x \Gamma(x+1)}{x^x} - \displaystyle\int_0^\infty e^{-t}\left(1 + \frac{t}{x}\right)^x dt} = \frac{x}{1+} \frac{2x}{2+} \frac{3x}{3+} \frac{4x}{4+} \cdots.$$

It therefore remains to show that

$$(23.6) \qquad L(x) = \sqrt{\frac{2x}{\pi}} - \frac{2}{3\pi} + o(1),$$

as x tends to ∞.

Write

$$L(x) = \frac{x}{\frac{1}{2} e^x x^{-x} \Gamma(x+1) + \theta_x}.$$

When x is a positive integer, Ramanujan **[62]**, **[63**, p. 324] derived an asymptotic expansion for θ_x as x tends to ∞. Watson **[75]** later established the expansion for general $x > 0$. See also the corollary to Entry 48 below and Entry 6 of Chapter 13 [**64**, p. 156], **[11]**. Using this asymptotic series (48.4) and Stirling's formula, we find that

$$L(x) = \frac{x}{\sqrt{\frac{\pi x}{2}} + 0(x^{-1/2}) + \frac{1}{3} + O(1/x)}$$

$$= \sqrt{\frac{2x}{\pi}} - \frac{2}{3\pi} + O(x^{-1/2}),$$

as x tends to ∞. Thus, (23.6) is established, and the proof is completed.

ENTRY 24. *Let n and x be any complex numbers, and let r be any positive integer. Let*

$$f(n, r, x) = \sum_{k=0}^{\infty} \frac{(-r+k-1)_k x^k}{(n)_k(-n-r)_k k!}.$$

Then

$$\text{(24.1)} \quad \frac{n}{n} + \frac{x}{n+1} + \frac{x}{n+2} + \frac{x}{n+3} + \cdots + \frac{x}{n+r} = \frac{f(n+1, r-1, x)}{f(n, r, x)}.$$

PROOF. We shall induct on r. For $r = 1$,

$$\frac{n}{n} + \frac{x}{n+1} = \frac{n}{n + x/(n+1)} = \frac{1}{1 + \dfrac{x}{n(n+1)}},$$

and so (24.1) is established for $r = 1$.

Now assume that (24.1) is true when r is replaced by $r - 1$ for some fixed integer r, $r \geqq 2$. Then applying the induction hypothesis with n replaced by $n + 1$, we find that

$$\text{(24.2)} \quad \begin{aligned} \frac{n}{n} + \frac{x}{n+1} + \frac{x}{n+2} + \cdots + \frac{x}{n+r} &= \frac{n}{n + \dfrac{x}{n+1} \dfrac{f(n+2, r-2, x)}{f(n+1, r-1, x)}} \\ &= \frac{nf(n+1, r-1, x)}{nf(n+1, r-1, x) + \dfrac{x}{n+1} f(n+2, r-2, x)}. \end{aligned}$$

We are thus led to examine, for $k \geqq 1$,

$$\frac{n(-r+k)_k}{(n+1)_k(-n-r)_k k!} + \frac{(-r+k)_{k-1}}{(n+1)_k(-n-r)_{k-1}(k-1)!}$$

$$= \frac{(-r+k)_{k-1}}{(n+1)_k(-n-r)_{k-1}(k-1)!} \left\{ \frac{n(-r+2k-1)}{(-n-r+k-1)k} + 1 \right\}$$

$$= \frac{(-r+k)_{k-1}}{(n+1)_k(-n-r)_{k-1}(k-1)!} \frac{(n+k)(-r+k-1)}{(-n-r+k-1)k}$$

$$= \frac{(-r+k-1)_k}{(n+1)_{k-1}(-n-r)_k k!} = \frac{n(-r+k-1)_k}{(n)_k(-n-r)_k k!}.$$

Hence,

$$nf(n+1, r-1, x) + \frac{x}{n+1} f(n+2, r-2, x) = nf(n, r, x). \tag{24.3}$$

Substituting (24.3) into (24.2), we complete the induction.

Entry 24 is a rather remarkable result, for it gives a continued fraction expansion for the quotient of hypergeometric polynomials,

$$\frac{{}_2F_3\left(\frac{1-r}{2}, -\frac{r}{2}; -r, n+1, -r-n; x\right)}{{}_2F_3\left(\frac{-1-r}{2}, -\frac{r}{2}; -r-1, n, -r-n; x\right)}.$$

ENTRY 25. *Suppose that either* $x > 0$ *and* $n^2 < 1$ *or that* $|x| > 1$ *and* n^2 *is real. Then*

$$\frac{\Gamma(\frac{1}{4}(x+n+1))\Gamma(\frac{1}{4}(x-n+1))}{\Gamma(\frac{1}{4}(x+n+3))\Gamma(\frac{1}{4}(x-n+3))} = \frac{4}{x} - \frac{n^2-1^2}{2x} - \frac{n^2-3^2}{2x} - \frac{n^2-5^2}{2x} - \cdots.$$

Entry 25 is originally due to Euler [**26**, §67]. Stieltjes [**68**], [**70**, pp. 329–394] derived Entry 25 from Entry 22. Still another proof may be found in Perron's book [**57**, p. 35].

PROOF. We offer another proof which is based upon Entry 39 and so requires that $|x| > 1$. First, rewrite Entry 39 in the form

$$\begin{aligned}&\frac{8}{P} + \frac{1}{2}(x^2 + \ell^2 - n^2 - 1)\\ &= x^2 - 1 + \frac{1^2-n^2}{1} + \frac{1^2-\ell^2}{x^2-1} + \frac{3^2-n^2}{1} + \frac{3^2-\ell^2}{x^2-1} + \cdots,\end{aligned}$$

or

$$\begin{aligned}&\frac{1}{8/P + \frac{1}{2}(x^2 + \ell^2 - n^2 - 1)}\\ &= \frac{1}{x^2-1} + \frac{1^2-n^2}{1} + \frac{1^2-\ell^2}{x^2-1} + \frac{3^2-n^2}{1} + \frac{3^2-\ell^2}{x^2-1} + \cdots\\ &= \frac{1}{x^2-n^2} - \frac{(1^2-n^2)(1^2-\ell^2)}{x^2-\ell^2-n^2+9} - \frac{(3^2-n^2)(3^2-\ell^2)}{x^2-\ell^2-n^2+33} - \cdots,\end{aligned}$$

by Entry 14. Now take the reciprocal of both sides above and then solve for P, which again involves taking reciprocals. Hence,

$$\begin{aligned}P = \frac{8}{\frac{1}{2}(x^2-\ell^2-n^2+1)} &- \frac{(1^2-n^2)(1^2-\ell^2)}{x^2-\ell^2-n^2+9}\\ &- \frac{(3^2-n^2)(3^2-\ell^2)}{x^2-\ell^2-n^2+33} - \cdots.\end{aligned}$$

Replacing x by $x + \ell$, we find that

$$\begin{aligned}
&\frac{\Gamma(\frac{1}{4}(x+n+1))\Gamma(\frac{1}{4}(x-n+1))}{\Gamma(\frac{1}{4}(x+n+3))\Gamma(\frac{1}{4}(x-n+3))} \\
&\times \ell \frac{\Gamma(\frac{1}{4}(x+2\ell+n+1))\Gamma(\frac{1}{4}(x+2\ell-n+1))}{\Gamma(\frac{1}{4}(x+2\ell+n+3))\Gamma(\frac{1}{4}(x+2\ell-n+3))} \\
&= \frac{8\ell}{\frac{1}{2}(x^2+2x\ell-n^2+1)} - \frac{(1^2-n^2)(1^2-\ell^2)}{x^2+2x\ell-n^2+9} \\
&\qquad - \frac{(3^2-n^2)(3^2-\ell^2)}{x^2+2x\ell-n^2+33} - \cdots \\
&= \frac{8}{\frac{1}{2}(2x+(x^2-n^2+1)/\ell)} - \frac{(1^2-n^2)(1/\ell^2-1)}{2x+(x^2-n^2+9/\ell} \\
&\qquad - \frac{(3^2-n^2)(3^2/\ell^2-1)}{2x+(x^2-n^2+33)/\ell} - \cdots.
\end{aligned}$$

Now let ℓ tend to $-\infty$. By using the reflection formula for the Γ-function and Stirling's formula, we deduce that

$$\lim_{\ell\to-\infty} \ell \frac{\Gamma(\frac{1}{4}(x+2\ell+n+1))\Gamma(\frac{1}{4}(x+2\ell-n+1))}{\Gamma(\frac{1}{4}(x+2\ell+n+3))\Gamma(\frac{1}{4}(x+2\ell-n+3))} = 2,$$

and so Entry 25 readily follows.

Corollary 1. *If* $x > 0$, *then*

$$\frac{\Gamma^2(\frac{1}{4}(x+1))}{\Gamma^2(\frac{1}{4}(x+3))} = \frac{4}{x} + \frac{1^2}{2x} + \frac{3^2}{2x} + \frac{5^2}{2x} + \cdots.$$

Proof. Set $n = 0$ in Entry 25.

Corollary 1 was first proved by Bauer [**6**] in 1872 and was communicated by Ramanujan [**63**, p. xxvii] in his first letter to Hardy.

If we put $x = 1$ in Corollary 1, we obtain Lord Brouncker's continued fraction for π,

$$\pi = \frac{4}{1} + \frac{1^2}{2} + \frac{3^2}{2} + \frac{5^2}{2} + \cdots.$$

For a very interesting historical account of Brouncker's continued fraction, see Dutka's paper [**19**].

Corollary 2. *If* $x > 0$, *then*

$$\frac{\Gamma\left(\frac{1}{8}(x+3)\right)\Gamma\left(\frac{1}{8}(x+1)\right)}{\Gamma\left(\frac{1}{8}(x+7)\right)\Gamma\left(\frac{1}{8}(x+5)\right)} = \frac{8}{x} + \frac{1\cdot 3}{2x} + \frac{5\cdot 7}{2x} + \frac{9\cdot 11}{2x} + \cdots.$$

Proof. Replace x by $x/2$ and n by $1/2$ in Entry 25.

ENTRY 26. *If* $|x| > 1$ *and* n^2 *is real, then*

$$\frac{\Gamma^2(\frac{1}{4}(x+n+1))\Gamma^2(\frac{1}{4}(x-n+1))}{\Gamma^2(\frac{1}{4}(x+n+3))\Gamma^2(\frac{1}{4}(x-n+3))}$$
$$= \frac{8}{\frac{1}{2}(x^2+n^2-1)+} \frac{1^2-n^2}{1} + \frac{1^2}{x^2-1+} \frac{3^2-n^2}{1} + \frac{3^2}{x^2-1+} \cdots$$
$$= \frac{8}{\frac{1}{2}(x^2-n^2-1)} + \frac{1}{1} + \frac{1^2-n^2}{x^2-1} + \frac{3^2}{1} + \frac{3^2-n^2}{x^2-1} + \cdots.$$

PROOF. To obtain the first equality, set $\ell = 0$ in Entry 39. The second equality follows from Entry 39 by letting $n = 0$ and replacing ℓ by n.

Alternatively, the two continued fractions can be shown to be equal by an application of Entry 15. Let $h = -n^2$ and $a_k = (2k-1)^2$, $k \geqq 1$, and also replace x by $x^2 - 1$ in Entry 15. The desired equality easily follows.

COROLLARY. *If* $|x| > 1$, *then*

$$\frac{\Gamma^4(\frac{1}{4}(x+1))}{\Gamma^4(\frac{1}{4}(x+3))} = \frac{8}{\frac{1}{2}(x^2-1)} + \frac{1}{1} + \frac{1^2}{x^2-1} + \frac{3^2}{1} + \frac{3^2}{x^2-1} + \cdots.$$

PROOF. Set $n = 0$ in Entry 26.

The next theorem is found in Ramanujan's [**63**, p. xxix] second letter to Hardy. The first proof in print was provided by Preece [**59**]. Entry 27 can also be found in Perron's book [**57**, p. 37, eq. (31)].

ENTRY 27. *Suppose that* $x, y > 0$. *Then*

$$x + \frac{(1+y)^2+n}{2x} + \frac{(3+y)^2+n}{2x} + \frac{(5+y)^2+n}{2x} + \cdots$$
$$= y + \frac{(1+x)^2+n}{2y} + \frac{(3+x)^2+n}{2y} + \frac{(5+x)^2+n}{2y} + \cdots.$$

ENTRY 28. *Let* $x > 0$. *Then*

$$\lim_{n\to\infty} \frac{x + \dfrac{n^2+1^2}{2x} + \dfrac{n^2+3^2}{2x} + \cdots}{n + \dfrac{x^2-1^2}{2n} + \dfrac{x^2-3^2}{2n} + \cdots} = 1. \tag{28.1}$$

PROOF. Apply Entry 25 with n replaced by in to find that, for $x > 0$,

$$\frac{\Gamma(\frac{1}{4}(x+in+1))\Gamma(\frac{1}{4}(x-in+1))}{\Gamma(\frac{1}{4}(x+in+3))\Gamma(\frac{1}{4}(x-in+3))}$$
$$= \frac{4}{x} + \frac{n^2+1^2}{2x} + \frac{n^2+3^2}{2x} + \cdots,$$

or

$$4\frac{\Gamma(\frac14(x+in+3))\Gamma(\frac14(x-in+3))}{\Gamma(\frac14(x+in+1))\Gamma(\frac14(x-in+1))}$$
$$= x + \frac{n^2+1^2}{2x} + \frac{n^2+3^2}{2x} + \cdots.$$

Secondly, apply Entry 25 with x and n interchanged to obtain, for $n > 0$,

$$4\frac{\Gamma(\frac14(n+x+3))\Gamma(\frac14(n-x+3))}{\Gamma(\frac14(n+x+1))\Gamma(\frac14(n-x+1))}$$
$$= n - \frac{x^2-1^2}{2n} - \frac{x^2-3^2}{2n} - \cdots.$$

Now,

$$\lim_{n\to\infty}\frac{\Gamma(\frac14(x+in+3))\Gamma(\frac14(x-in+3))\Gamma(\frac14(n+x+1))\Gamma(\frac14(n-x+1))}{\Gamma(\frac14(x+in+1))\Gamma(\frac14(x-in+1))\Gamma(\frac14(n+x+3))\Gamma(\frac14(n-x+3))} = 1,$$

where we have applied Stirling's formula for the quotient of 2 Γ-functions [**47**, p. 33]. Thus, we have shown that

$$\lim_{n\to\infty}\frac{x + \dfrac{n^2+1^2}{2x} + \dfrac{n^2+3^2}{2x} + \cdots}{n - \dfrac{x^2-1^2}{2n} - \dfrac{x^2-3^2}{2n} - \cdots} = 1. \tag{28.2}$$

However,

$$\lim_{n\to\infty}\frac{n - \dfrac{x^2-1^2}{2n} - \dfrac{x^2-3^2}{2x} - \cdots}{n + \dfrac{x^2-1^2}{2n} + \dfrac{x^2-3^2}{2n} + \cdots} = 1, \tag{28.3}$$

because the numerator and denominator above are both of the form $n + O(1/n)$ as n tends to ∞. Combining (28.2) and (28.3), we deduce (28.1).

In his first notebook [**64**, vol. 1, p. 160], Ramanujan states a more precise version of Entry 28,

$$\frac{x + \dfrac{n^2+1^2}{2x} + \dfrac{n^2+3^2}{2x} + \cdots}{n + \dfrac{x^2-1^2}{2n} + \dfrac{x^2-3^2}{2n} + \cdots} = \frac{1-e^{-\pi n}}{1 - 2e^{-\pi n/2}\sin(\pi x/2) + e^{-\pi n}}.$$

Ramanujan probably intends the right side to be an approximation to the left side for n large. However, the right side is $1 + O(e^{-\pi n/2})$ as n tends to ∞. A close analysis of our proof of Entry 28 shows that the left side of (28.1) is of the form $1 + O(1/n)$ as n tends to ∞ and that the expression which is equal to $O(1/n)$ cannot be improved. Thus, Ramanujan's claim does not appear to have a valid interpretation.

ENTRY 29. *If* $x > 0$ *and* $n^2 < 1$, *then*

$$\sum_{k=1}^{\infty}\left\{\frac{(-1)^{k+1}}{x+n+2k-1}+\frac{(-1)^{k+1}}{x-n+2k-1}\right\}$$
$$=\frac{1}{x+}\,\frac{1^2-n^2}{x}\,\frac{}{+}\,\frac{2^2}{x+}\,\frac{3^2-n^2}{x}\,\frac{}{+}\,\frac{4^2}{x+}\,\frac{5^2-n^2}{x}\,\frac{}{+}\cdots.$$

FIRST PROOF. Our first proof merely consists of a reformulation of a result found in Perron's book [**57**, p. 33, eq. (12)],

$$\text{(29.1)}\qquad \begin{aligned}&\frac{4}{x+}\,\frac{1^2-n^2}{x}\,\frac{}{+}\,\frac{2^2}{x+}\,\frac{3^2-n^2}{x}\,\frac{}{+}\,\frac{4^2}{x+}\,\frac{5^2-n^2}{x}\,\frac{}{+}\cdots\\ &=\psi\left(\frac{x+n+3}{4}\right)+\psi\left(\frac{x-n+3}{4}\right)-\psi\left(\frac{x+n+1}{4}\right)-\psi\left(\frac{x-n+1}{4}\right),\end{aligned}$$

where $x > 0$ and $n^2 < 1$. Now employ (0.1) and simplify to complete the proof.

In fact, Entry 29 was first proved in print in 1953 by Perron [**56**] who derived it from Entry 34 below.

SECOND PROOF. Since

$$\frac{1}{1+t^2}=\sum_{k=0}^{\infty}(-1)^k t^{2k},\qquad |t|<1,$$

we find that, for $\operatorname{Re}(x) > -1$,

$$\text{(29.2)}\qquad H(x)\equiv\int_0^1\frac{t^x}{1+t^2}\,dt=\sum_{k=0}^{\infty}\frac{(-1)^k}{x+2k+1}.$$

Then for $\operatorname{Re}(x \pm n) > -1$,

$$\text{(29.3)}\qquad \begin{aligned}H(x+n)+H(x-n)&=\int_0^1\frac{t^x(t^n+t^{-n})}{1+t^2}\,dt\\ &=\int_0^{\infty}e^{-ux}\frac{\cosh(nu)}{\cosh u}\,du,\end{aligned}$$

where we have made the change of variable $t = e^{-u}$. But for $x > 0$ and $n^2 < 1$, Rogers [**66**] has shown that

$$\text{(29.4)}\qquad \int_0^{\infty}e^{-xu}\frac{\cosh(nu)}{\cosh u}\,du=\frac{1}{x+}\,\frac{1^2-n^2}{x}\,\frac{}{+}\,\frac{2^2}{x+}\,\frac{3^2-n^2}{x}\,\frac{}{+}\,\frac{4^2}{x+}\cdots.$$

Employing (29.2) and (29.4) in (29.3), we arrive at the desired formula.

COROLLARY. *If* $x > 0$, *then*

$$2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{x+2k-1}=\frac{1}{x+}\,\frac{1^2}{x+}\,\frac{2^2}{x+}\,\frac{3^2}{x+}\,\frac{4^2}{x+}\cdots.$$

PROOF. Set $n = 0$ in Entry 29.

ENTRY 30. *Suppose that* $x > 0$ *and* n^2 *is real. Then*

$$\sum_{k=0}^{\infty}\left\{\frac{1}{x-n+2k+1}-\frac{1}{x+n+2k+1}\right\} \tag{30.1}$$
$$=\frac{n}{x+}\,\frac{1^2(1^2-n^2)}{3x}\,\underset{+}{}\,\frac{2^2(2^2-n^2)}{5x}\,\underset{+}{}\,\frac{3^2(3^2-n^2)}{7x}\,\underset{+}{}\cdots.$$

FIRST PROOF. Letting

$$R = \Gamma(\tfrac{1}{2}(x+m+n+1))\,\Gamma(\tfrac{1}{2}(x-m-n+1))$$

and

$$T = \Gamma(\tfrac{1}{2}(x+m-n+1))\,\Gamma(\tfrac{1}{2}(x-m+n+1)),$$

we first write Entry 33 in the form

$$\frac{R-T}{R+T}=\frac{mn}{x}\,\underset{+}{}\,\frac{(m^2-1^2)(n^2-1^2)}{3x}\,\underset{+}{}\,\frac{(m^2-2^2)(n^2-2^2)}{5x}\,\underset{+}{}\cdots.$$

Thus,

$$\lim_{m\to\infty}\frac{1}{m}\frac{R-T}{R+T}=\frac{n}{x+}\,\frac{1^2(1^2-n^2)}{3x}\,\underset{+}{}\,\frac{2^2(2^2-n^2)}{5x}\,\underset{+}{}\cdots. \tag{30.2}$$

On the other hand, a direct calculation with the use of L'Hospital's rule shows that

$$\lim_{m\to\infty}\frac{1}{m}\frac{R-T}{R+T}=\frac{1}{2}\psi\left(\frac{x+n+1}{2}\right)-\frac{1}{2}\psi\left(\frac{x-n+1}{2}\right) \tag{30.3}$$
$$=\sum_{k=0}^{\infty}\left\{\frac{1}{x-n+2k+1}-\frac{1}{x+n+2k+1}\right\}.$$

Combining (30.2) and (30.3), we finish the proof.

SECOND PROOF. This proof requires that $n^2 < 1$. Proceeding in somewhat the same way as in the second proof of Entry 29, we find that, for $\operatorname{Re}(x \pm n) > -1$,

$$\int_0^1\frac{t^{x-n}-t^{x+n}}{1-t^2}\,dt=\sum_{k=0}^{\infty}\left\{\frac{1}{x-n+2k+1}-\frac{1}{x+n+2k+1}\right\}. \tag{30.4}$$

On the other hand, letting $t = e^{-u}$ and using a theorem of Stieltjes [67], [**70**, pp. 378–391], which was also proved by Rogers [**66**], we find that, for $x > 0$ and $n^2 < 1$,

$$\int_0^1\frac{t^{x-n}-t^{x+n}}{1-t^2}\,dt=\int_0^{\infty}e^{-xu}\frac{\sinh(nu)}{\sinh u}\,du \tag{30.5}$$
$$=\frac{n}{x+}\,\frac{1^2(1^2-n^2)}{3x}\,\underset{+}{}\,\frac{2^2(2^2-n^2)}{5x}\,\underset{+}{}\cdots.$$

Combining (30.4) and (30.5), we complete the second proof.

COROLLARY. *If* $x > 0$, *then*

$$2\sum_{k=0}^{\infty}\frac{1}{(x+2k+1)^2}=\frac{1}{x+}\frac{1^4}{3x+}\frac{2^4}{5x+}\frac{3^4}{7x+}\cdots.$$

PROOF. Divide both sides of (30.1) by n and let n tend to 0.

If we set $x = 1$ in the Corollary above, we deduce that

$$\frac{1}{2}\zeta(2)=\frac{\pi^2}{12}=\frac{1}{1+}\frac{1^4}{3+}\frac{2^4}{5+}\frac{3^4}{7+}\cdots.$$

For a simple proof of this expansion, see a note by Madhava [**48**].

ENTRY 31. *Suppose that* $|x| > 1$ *and* n^2 *is real, or suppose that* $x > 0$ *and* $n^2 < 1$. *Then*

$$\begin{aligned}(31.1)\qquad &\sum_{k=0}^{\infty}\left\{\frac{(-1)^k}{x-n+2k+1}-\frac{(-1)^k}{x+n+2k+1}\right\}\\ &=\frac{n}{x^2-1+}\frac{2^2-n^2}{1}\frac{2^2}{+x^2-1+}\frac{4^2-n^2}{1}\frac{4^2}{+x^2-1+}\cdots.\end{aligned}$$

FIRST PROOF. From Entry 36, if $|x| > 1$ and n^2 is real,

$$(31.2)\qquad \lim_{\ell\to 0}\frac{1}{\ell}\frac{1-P}{1+P}=\frac{n}{x^2-1+}\frac{2^2-n^2}{1}\frac{2^2}{+x^2-1+}\frac{4^2-n^2}{1}\frac{4^2}{+x^2-1+}\cdots.$$

On the other hand, a direct calculation with the use of L'Hospital's rule gives

$$\begin{aligned}&\lim_{\ell\to 0}\frac{1}{\ell}\frac{1-P}{1+P}\\ &=\frac{1}{4}\left\{\psi\left(\frac{x+n+1}{4}\right)+\psi\left(\frac{x-n+3}{4}\right)-\psi\left(\frac{x-n+1}{4}\right)-\psi\left(\frac{x+n+3}{4}\right)\right\}\\ (31.3)\qquad &=\sum_{k=0}^{\infty}\left\{-\frac{1}{x+n+4k-3}-\frac{1}{x-n+4k-1}+\frac{1}{x-n+4k-3}\right.\\ &\qquad\left.+\frac{1}{x+n+4k-1}\right\}.\end{aligned}$$

Equalities (31.2) and (31.3) taken together yield (31.1).

SECOND PROOF. As in the second proofs of Entries 29 and 30, we easily find that, for $\mathrm{Re}(x \pm n) > -1$,

$$\sum_{k=0}^{\infty}\left\{\frac{(-1)^k}{x-n+2k+1}-\frac{(-1)^k}{x+n+2k+1}\right\}=\int_0^{\infty}e^{-xu}\frac{\sinh(nu)}{\cosh u}\,du.$$

But Stieltjes [**69**], [**70**, pp. 402–566] and later Rogers [**66**] have shown that, for $x > 0$ and $n^2 < 1$,

$$\int_0^{\infty}e^{-xu}\frac{\sinh(nu)}{\cosh u}\,du=\frac{n}{x^2-1+}\frac{2^2-n^2}{1}\frac{2^2}{+x^2-1+}\frac{4^2-n^2}{1}\frac{4^2}{+x^2-1+}\cdots.$$

The foregoing two equalities imply (31.1).

COROLLARY. *If $x > 0$, then*

$$2\sum_{k=0}^{\infty}\frac{(-1)^k}{(x+2k+1)^2} = \frac{1}{x^2-1}\,+\,\frac{2^2}{1}\,+\,\frac{2^2}{x^2-1}\,+\,\frac{4^2}{1}\,+\,\frac{4^2}{x^2-1}\,+\cdots.$$

PROOF. Divide both sides of (31.1) by n and then let n tend to 0.

If we put $x = 2$ in the foregoing Corollary, we obtain the following elegant continued fraction for Catalan's constant G.

$$2G \equiv 2\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2} = 2 - \frac{1}{3}\,+\,\frac{2^2}{1}\,+\,\frac{2^2}{3}\,+\,\frac{4^2}{1}\,+\,\frac{4^2}{3}\,+\cdots.$$

Of course, similar continued fraction expansions for G can be obtained by setting $x = 2n$, where n is any positive integer, in the Corollary above. This same infinite set of continued fractions for G was independently found by H. Cohen (personal communication) who obtained them from a different formula.

ENTRY 32(i). *If $x > 0$, then*

$$1 + 2x\sum_{k=1}^{\infty}\frac{(-1)^k}{x+2k} = \frac{1}{x}\,+\,\frac{1\cdot 2}{x}\,+\,\frac{2\cdot 3}{x}\,+\,\frac{3\cdot 4}{x}\,+\cdots. \tag{32.1}$$

PROOF. Let

$$P = P(x, n) = \frac{\Gamma\left(\frac{1}{4}(x+n+3)\right)\Gamma\left(\frac{1}{4}(x-n+3)\right)}{\Gamma\left(\frac{1}{4}(x+n+1)\right)\Gamma\left(\frac{1}{4}(x-n+1)\right)}.$$

Then by Entry 25, for $x > 0$ and $n \neq 1$,

$$4P = x + \frac{1^2-n^2}{2x}\,+\,\frac{3^2-n^2}{2x}\,+\,\frac{5^2-n^2}{2x}\,+\cdots,$$

or

$$\frac{4P-x}{1-n} = \frac{1+n}{2x}\,+\,\frac{3^2-n^2}{2x}\,+\,\frac{5^2-n^2}{2x}\,+\cdots. \tag{32.2}$$

Note that $P(x, 1) = x/4$. We now let n tend to 1 in (32.2) and apply L'Hospital's rule on the left side. We then find that

$$\frac{x}{4}\left\{2\,\psi\left(\frac{x+2}{4}\right) - \psi\left(\frac{x}{4}+1\right) - \psi\left(\frac{x}{4}\right)\right\}$$
$$= \frac{2}{2x}\,+\,\frac{2\cdot 4}{2x}\,+\,\frac{4\cdot 6}{2x}\,+\,\frac{6\cdot 8}{2x}\,+\cdots.$$

Simplifying each side above, we arrive at (32.1).

ENTRY 32(ii). *If* $x > 0$, *then*

$$1 + 2x^2 \sum_{k=1}^{\infty} \frac{(-1)^k}{(x+k)^2} = \frac{1}{x} + \frac{1^2}{x} + \frac{1 \cdot 2}{x} + \frac{2^2}{x} + \frac{2 \cdot 3}{x} + \frac{3^2}{x} + \cdots.$$

PROOF. Let

$$P = P(x,n) = \psi\left(\frac{x+n+3}{4}\right) + \psi\left(\frac{x-n+3}{4}\right) - \psi\left(\frac{x+n+1}{4}\right) - \psi\left(\frac{x-n+1}{4}\right).$$

Then from (29.1), we find that, for $x > 0$ and $n^2 < 1$,

$$\frac{4/P - x}{1-n} = \frac{1+n}{x} + \frac{2^2}{x} + \frac{3^2 - n^2}{x} + \frac{4^2}{x} + \frac{5^2 - n^2}{x} + \cdots. \tag{32.3}$$

Observe that $P(x, 1) = 4/x$. Letting n tend to 1 in (32.3) and employing L'Hospital's rule, we find that

$$\frac{4\frac{\partial}{\partial n}P(x,n)|_{n=1}}{P^2(x,1)} = x^2 \sum_{k=1}^{\infty} \left\{\frac{1}{(x+4k)^2} - \frac{2}{(x+4k-2)^2} + \frac{1}{(x+4k-4)^2}\right\}$$
$$= \frac{2}{x} + \frac{2^2}{x} + \frac{2 \cdot 4}{x} + \frac{4^2}{x} + \frac{4 \cdot 6}{x} + \cdots.$$

Replacing x by $2x$, we deduce that

$$1 + 2x^2 \sum_{k=1}^{\infty} \frac{(-1)^k}{(x+k)^2} = \frac{2}{2x} + \frac{2^2}{2x} + \frac{2 \cdot 4}{2x} + \frac{4^2}{2x} + \frac{4 \cdot 6}{2x} + \cdots.$$

Simplifying the right side, we complete the proof.

If we set $x = 1$ in Entry 32(ii), we deduce that

$$\zeta(2) = 1 + \frac{1}{1} + \frac{1^2}{1} + \frac{1 \cdot 2}{1} + \frac{2^2}{1} + \frac{2 \cdot 3}{1} + \frac{3^2}{1} + \cdots.$$

Putting $x = 1/2$ in Entry 32(ii) yields another continued fraction for G,

$$2G = 1 + \frac{1}{1/2} + \frac{1^2}{1/2} + \frac{1 \cdot 2}{1/2} + \frac{2^2}{1/2} + \frac{2 \cdot 3}{1/2} + \frac{3^2}{1/2} + \cdots.$$

ENTRY 32(iii). *If* $x > 0$, *then*

$$\zeta(3, x+1) \equiv \sum_{k=1}^{\infty} \frac{1}{(x+k)^3}$$

$$(32.4) \quad = \frac{1}{2x(x+1)} + \frac{1^3}{1} + \frac{1^3}{6x(x+1)} + \frac{2^3}{1} + \frac{2^3}{10x(x+1)} + \cdots$$
$$= \frac{1}{2x^2+2x+1} - \frac{1^6}{3(2x^2+2x+3)} - \frac{2^6}{5(2x^2+2x+7)} - \frac{3^6}{7(2x^2+2x+13)} - \cdots.$$

Proof. In Entry 35, replace x by $2x + 1$. Then $y = 4x(x + 1) + 2m - m^2$, and we need to require that $x > 0$. Also let $\ell = n = m$. Noting that $t = 0$ and using the second continued fraction of Entry 35, we find that

$$\frac{1-P}{1+P} = \frac{1 - \dfrac{\Gamma\left(\frac{1}{2}(2x+2+3m)\right)\Gamma^3\left(\frac{1}{2}(2x+2-m)\right)}{\Gamma\left(\frac{1}{2}(2x+2-3m)\right)\Gamma^3\left(\frac{1}{2}(2x+2+m)\right)}}{1 + \dfrac{\Gamma\left(\frac{1}{2}(2x+2+3m)\right)\Gamma^3\left(\frac{1}{2}(2x+2-m)\right)}{\Gamma\left(\frac{1}{2}(2x+2-3m)\right)\Gamma^3\left(\frac{1}{2}(2x+2+m)\right)}} \tag{32.5}$$

$$= \frac{2m^3}{y - 2m^3 +} \, \frac{2(1-m)(1^2-m^2)}{1} + \frac{2(1+m)(1^2-m^2)}{3y} + \frac{2(2-m)(2^2-m^2)}{1} + \frac{2(2+m)(2^2-m^2)}{5y} + \cdots.$$

Now divide both sides of (32.5) by m^3 and let m tend to 0. On the right side, we arrive at

$$\frac{2}{4x(x+1)} + \frac{2\cdot 1^3}{1} + \frac{2\cdot 1^3}{12x(x+1)} + \frac{2\cdot 2^3}{1} + \frac{2\cdot 2^3}{20x(x+1)} + \cdots.$$

Simplifying above, we obtain the former continued fraction of (32.4).

Next, write the aforementioned continued fraction in the equivalent form

$$\frac{1}{2x(x+1)} + \frac{1^3}{1} + \frac{1^3/3}{2x(x+1)} + \frac{2^3/3}{1} + \frac{2^3/5}{2x(x+1)} + \frac{3^3/5}{1} + \cdots.$$

Applying Entry 14 to this continued fraction, we deduce the equality between the continued fractions of (32.4).

For brevity, set $z = x + 1$. For $z > 1$, it remains to examine, by (32.5),

$$\begin{aligned}
&\lim_{m\to 0} \frac{1}{m^3}\frac{1-P}{1+P} \\
&= \lim_{m\to 0}\frac{1}{2m^3\Gamma^4(z)}\left(\left\{\Gamma(z) - \frac{3m}{2}\Gamma'(z) + \frac{3^2m^2}{2^3}\Gamma''(z) - \frac{3^2m^3}{2^4}\Gamma'''(z) + \cdots\right\}\right. \\
&\quad \times\left\{\Gamma(z) + \frac{m}{2}\Gamma'(z) + \frac{m^2}{2^3}\Gamma''(z) + \frac{m^3}{2^4\cdot 3}\Gamma'''(z) + \cdots\right\}^3 \\
&\quad - \left\{\Gamma(z) + \frac{3m}{2}\Gamma'(z) + \frac{3^2m^2}{2^3}\Gamma''(z) + \frac{3^2m^3}{2^4}\Gamma'''(z) + \cdots\right\} \\
&\quad \left.\times\left\{\Gamma(z) - \frac{m}{2}\Gamma'(z) + \frac{m^2}{2^3}\Gamma''(z) - \frac{m^3}{2^4\cdot 3}\Gamma'''(z) + \cdots\right\}^3\right) \\
&= \frac{1}{\Gamma^4(z)}\left(\left\{-\frac{3^2}{2^4} + \frac{1}{2^4}\right\}\Gamma'''(z)\Gamma^3(z) + \left\{\frac{3^3}{2^4} - \frac{3^2}{2^4} + \frac{6}{2^4}\right\}\Gamma''(z)\Gamma'(z)\Gamma^2(z)\right. \\
&\quad \left. + \left\{\frac{1}{2^3} - \frac{3^2}{2^3}\right\}\Gamma'(z)^3\Gamma(z)\right)
\end{aligned}$$

$$= -\frac{\Gamma''''(z)}{2\Gamma(z)} + \frac{3\Gamma'''(z)\Gamma'(z)}{2\Gamma^2(z)} - \frac{\Gamma'(z)^3}{\Gamma^3(z)}$$

$$= -\frac{1}{2}\frac{d^2}{dz^2}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = -\frac{1}{2}\psi''(z) = \sum_{k=0}^{\infty}\frac{1}{(z+k)^3}.$$

The proof is now complete.

Ramanujan's second continued fraction in Entry 32(iii) is slightly in error [**64**, vol. 2, p. 149].

We might compare Entry 32(iii) with another continued fraction for $\zeta(3, x)$,

$$4x^3\zeta(3, x) = 2x + 2 + \frac{1}{x} + \frac{p_1}{x} + \frac{q_1}{x} + \frac{p_2}{x} + \frac{q_2}{x} + \cdots,$$

where, for $k \geqq 1$, $p_k = (k^2(k+1))/(4k+2)$ and $q_k = (k(k+1)^2)/(4k+2)$. The last result was discovered by Stieltjes [**67**], [**70**, pp. 378–391].

Setting $x = 1$ in Entry 32(iii), we deduce the following beautiful continued fraction for $\zeta(3)$.

$$\zeta(3) = 1 + \frac{1}{2\cdot 2} + \frac{1^3}{1} + \frac{1^3}{6\cdot 2} + \frac{2^3}{1} + \frac{2^3}{10\cdot 2} + \cdots.$$

This continued fraction also follows from work of Apéry [**3**].

Entry 33. *Let $x > 0$ and suppose that m^2 and n^2 are real. Then*

$$\frac{\Gamma(\tfrac12(x+m+n+1))\Gamma(\tfrac12(x-m-n+1)) - \Gamma(\tfrac12(x+m-n+1))\Gamma(\tfrac12(x-m+n+1))}{\Gamma(\tfrac12(x+m+n+1))\Gamma(\tfrac12(x-m-n+1)) + \Gamma(\tfrac12(x+m-n+1))\Gamma(\tfrac12(x-m+n+1))}$$

$$= \frac{mn}{x} + \frac{(m^2-1^2)(n^2-1^2)}{3x} + \frac{(m^2-2^2)(n^2-2^2)}{5x} + \frac{(m^2-3^2)(n^2-3^2)}{7x} + \cdots.$$

Proof. Set

$$R(m) = \frac{\Gamma\left(\frac{1}{2}(x+\ell+n+1)+m\right)\Gamma\left(\frac{1}{2}(x-\ell-n+1)+m\right)}{\Gamma\left(\frac{1}{2}(x-\ell+n+1)+m\right)\Gamma\left(\frac{1}{2}(x+\ell-n+1)+m\right)}$$

and

$$T = \frac{\Gamma\left(\frac{1}{2}(x+\ell-n+1)\right)\Gamma\left(\frac{1}{2}(x-\ell+n+1)\right)}{\Gamma\left(\frac{1}{2}(x+\ell+n+1)\right)\Gamma\left(\frac{1}{2}(x-\ell-n+1)\right)}.$$

Suppose that m is a positive integer in Entry 35. Replacing x by $x + m$ in Entry 35, we find that

$$\begin{aligned}
&\frac{1 - R(m)T}{1 + R(m)T} \\
&\quad = \frac{2\ell mn}{x^2 + 2mx - \ell^2 - n^2 + 1} + \frac{4(\ell^2 - 1^2)(m^2 - 1^2)(n^2 - 1^2)}{3(x^2 + 2mx - \ell^2 - n^2 + 5)} \\
&\qquad + \frac{4(\ell^2 - 2^2)(m^2 - 2^2)(n^2 - 2^2)}{5(x^2 + 2mx - \ell^2 - n^2 + 13)} + \cdots \\
&\quad = \frac{\ell n}{x + (x^2 - \ell^2 - n^2 + 1)/2m} + \frac{(\ell^2 - 1^2)(n^2 - 1^2)(1 - 1/m^2)}{3(x + (x^2 - \ell^2 - n^2 + 5)/2m)} \\
&\qquad + \frac{(\ell^2 - 2^2)(n^2 - 2^2)(1 - 2^2/m^2)}{5(x + (x^2 - \ell^2 - n^2 + 13)/2m)} + \cdots.
\end{aligned} \tag{33.1}$$

Now let m tend to ∞ in (33.1). By Stirling's formula, $R(m)$ tends to 1 as m tends to ∞. Hence,

$$\frac{1 - T}{T + 1} = \frac{\ell n}{x} + \frac{(\ell^2 - 1^2)(n^2 - 1^2)}{3x} + \frac{(\ell^2 - 2^2)(n^2 - 2^2)}{5x} + \cdots.$$

The convergence of this continued fraction is guaranteed by a theorem of Pringsheim [**57**, p. 47, Satz 2.11]. Replacing ℓ by m above, we complete the proof.

In fact, Entry 33 was first proved in print by Nörlund [**52**].

The continued fraction in Entry 33 is a special case of a more general continued fraction for a quotient of two integrals involving hypergeometric functions that was discovered by Stieltjes [**67**], [**70**, p. 389, eq. (29)].

ENTRY 34. *Let $x > 0$ and suppose that ℓ^2 and n^2 are real. Define*

$$P = \frac{\Gamma\left(\frac{1}{4}(x + \ell + n + 1)\right)\Gamma\left(\frac{1}{4}(x + \ell - n + 1)\right)}{\Gamma\left(\frac{1}{4}(x - \ell + n + 1)\right)\Gamma\left(\frac{1}{4}(x - \ell - n + 1)\right)} \cdot \frac{\Gamma\left(\frac{1}{4}(x - \ell + n + 3)\right)\Gamma\left(\frac{1}{4}(x - \ell - n + 3)\right)}{\Gamma\left(\frac{1}{4}(x + \ell + n + 3)\right)\Gamma\left(\frac{1}{4}(x + \ell - n + 3)\right)}.$$

Then

$$\frac{1 - P}{1 + P} = \frac{\ell}{x} + \frac{1^2 - n^2}{x} + \frac{2^2 - \ell^2}{x} + \frac{3^2 - n^2}{x} + \frac{4^2 - \ell^2}{x} + \cdots.$$

Entry 34 was stated by Ramanujan [**63**, p. 350] in his first letter to Hardy. The first published proof was provided by Preece [**60**]. Another proof has been devised by Perron [**55**], [**57**, p. 34, eq. (15)].

COROLLARY. *Suppose that* $x > 0$, $-\alpha^2 < 4\gamma^2$, *and* $-\beta^2 < \gamma^2$. *Put*

$$F(\alpha, \beta) = \tan^{-1}\left\{\frac{\alpha}{x+}\ \frac{\beta^2+\gamma^2}{x+}\ \frac{\alpha^2+(2\gamma^2)}{x+}\ \frac{\beta^2+(3\gamma)^2}{x}\ + \cdots\right\}.$$

Then

$$F(\alpha, \beta) + F(\beta, \alpha) = 2\,F\left\{\frac{1}{2}(\alpha+\beta), \frac{1}{2}(\alpha+\beta)\right\}.$$

This corollary was communicated by Ramanujan [**63**, p. 353] in his second letter to Hardy. Again, the first published proof was given by Preece [**60**], and, indeed, this result is a corollary of Entry 34.

ENTRY 35. *Suppose that either* ℓ, m, *or* n *is a positive integer, or that* $y > 0$ *and* m, ℓ^2, *and* n^2 *are real, where* $y = x^2 - (1-m)^2$. *Define* $t = (n^2 - \ell^2)(1-2m)$ *and*

$$P = \frac{\Gamma\left(\frac{1}{2}(x+\ell+m+n+1)\right)\Gamma\left(\frac{1}{2}(x+\ell-m-n+1)\right)}{\Gamma\left(\frac{1}{2}(x-\ell-m-n+1)\right)\Gamma\left(\frac{1}{2}(x-\ell+m+n+1)\right)} \cdot \frac{\Gamma\left(\frac{1}{2}(x-\ell+m-n+1)\right)\Gamma\left(\frac{1}{2}(x-\ell-m+n+1)\right)}{\Gamma\left(\frac{1}{2}(x+\ell-m+n+1)\right)\Gamma\left(\frac{1}{2}(x+\ell+m-n+1)\right)}.$$

Then

$$\begin{aligned}\frac{1-P}{1+P} &= \frac{2\ell mn}{x^2-\ell^2-m^2-n^2+1+}\ \frac{4(\ell^2-1^2)(m^2-1^2)(n^2-1^2)}{3(x^2-\ell^2-m^2-n^2+5)} \\ &\qquad + \frac{4(\ell^2-2^2)(m^2-2^2)(n^2-2^2)}{5(x^2-\ell^2-m^2-n^2+13)} + \cdots \qquad (35.1)\\ &= \frac{2\ell mn}{y+t-2\ell^2 m+}\ \frac{2(1-m)(1^2-n^2)}{1+}\ \frac{2(1+m)(1^2-\ell^2)}{3y+t} \\ &\qquad + \frac{2(2-m)(2^2-n^2)}{1+}\ \frac{2(2+m)(2^2-\ell^2)}{5y+t} + \cdots.\end{aligned}$$

PROOF. The first equality was shown by Watson [**77**] to be a corollary of Entry 40. If either ℓ, m, or n is a positive integer, Watson's limiting process is trivially justified. If neither ℓ, m, nor n is a positive integer, then the alternative set of proposed conditions is sufficient to insure the convergence of the second displayed continued fraction and to justify the limiting process, as we shall see below.

To prove the second equality, we employ the following generalization of Entry 14.

$$\frac{a_1}{x_1}+\frac{a_2}{1}+\frac{a_3}{x_3}+\frac{a_4}{1}+\cdots+\frac{a_{2k-1}}{x_{2k-1}}+\frac{a_{2k}}{1}$$
$$=\frac{a_1}{x_1+a_2}-\frac{a_2a_3}{x_3+a_3+a_4}-\frac{a_4a_5}{x_5+a_5+a_6}-\cdots$$
$$-\frac{a_{2k-2}a_{2k-1}}{x_{2k-1}+a_{2k-1}+a_{2k}},$$

which can be proved in precisely the same manner as Entry 14. Thus, with $a_1 = 2\ell mn$, $a_2 = 2(1-m)(1-n^2), \ldots$ and $x_1 = y+t-2\ell^2m$, $x_3 = 3y+t^2, \ldots$, we find that

$$\frac{2\ell mn}{y+t-2\ell^2m+}\ \frac{2(1-m)(1-n^2)}{1}+\frac{2(1+m)(1-\ell^2)}{3y+t}+\frac{2(2-m)(2^2-n^2)}{1}$$
$$+\frac{2(2+m)(2^2-\ell^2)}{5y+t}+\cdots+\frac{2(k-m)(k^2-n^2)}{1}$$
$$(35.2)\qquad =\frac{2\ell mn}{x^2-\ell^2-m^2-n^2+1}-\frac{4(1-m^2)(1-\ell^2)(1-n^2)}{3(x^2-\ell^2-m^2-n^2+5)}$$
$$-\frac{4(2^2-m^2)(2^2-\ell^2)(2^2-n^2)}{5(x^2-\ell^2-m^2-n^2+13)}-\cdots$$
$$-\frac{4((k-1)^2-m^2)((k-1)^2-\ell^2)((k-1)^2-n^2)}{(2k-1)(x^2-\ell^2-m^2-n^2+2k^2-2k+1)},$$

where we have used the easily proven identity

$$(2j+1)y+t+2(j+m)(j^2-\ell^2)+2(j+1-m)((j+1)^2-n^2)$$
$$=(2j+1)(x^2-\ell^2-m^2-n^2+2j^2+2j+1).$$

Now let k tend to ∞ in (35.2) to formally establish the second equality in (35.1). The second alternative set of hypotheses of Entry 35 insures that the jth numerator and jth denominator of the second continued fraction in (35.1) are positive for j sufficiently large. By a theorem in Perron's text [**57**, p. 47, Satz 2.11], this continued fraction converges. Hence, by (35.2), the first continued fraction in (35.1) converges as well. Thus, Watson's restriction that ℓ, m, or n is a positive integer is not necessary.

ENTRY 36. *Suppose either that n or ℓ is an even positive integer or that $|x| > 1$ and ℓ^2 and n^2 are real. Let*

$$P=\frac{\Gamma\left(\frac{1}{4}(x+\ell+n+3)\right)\Gamma\left(\frac{1}{4}(x-\ell-n+3)\right)}{\Gamma\left(\frac{1}{4}(x+\ell+n+1)\right)\Gamma\left(\frac{1}{4}(x-\ell-n+1)\right)}$$
$$\cdot\frac{\Gamma\left(\frac{1}{4}(x+\ell-n+1)\right)\Gamma\left(\frac{1}{4}(x-\ell+n+1)\right)}{\Gamma\left(\frac{1}{4}(x+\ell-n+3)\right)\Gamma\left(\frac{1}{4}(x-\ell+n+3)\right)}.$$

Then

$$\frac{1-P}{1+P}=\frac{\ell n}{x^2-1-\ell^2+}\;\frac{2^2-n^2}{1}\;\frac{2^2-\ell^2}{+x^2-1+}\;\frac{4^2-n^2}{1}\;\frac{4^2-\ell^2}{+x^2-1+}\cdots.$$

PROOF. In the second equality of Entry 35, let $m = 1/2$ and replace x, n, and ℓ by $x/2$, $n/2$, and $\ell/2$, respectively. After simplification, the proposed identity follows.

ENTRY 37. *Suppose that* $|x| > 1$ *and that* ℓ^2 *and* n^2 *are real. Then*

$$\begin{aligned}\frac{1}{2}\Bigg\{\psi\left(\frac{x+\ell-n+1}{2}\right)+\psi\left(\frac{x-\ell+n+1}{2}\right)&\\ -\psi\left(\frac{x+\ell+n+1}{2}\right)-\psi\left(\frac{x-\ell-n+1}{2}\right)\Bigg\}& \qquad (37.1)\\ =\frac{2\ell n}{x^2-1+n^2-\ell^2+}\;\frac{2(1^2-n^2)}{1}\;\frac{2(1^2-\ell^2)}{+3(x^2-1)+n^2-\ell^2}&\\ +\frac{4(2^2-n^2)}{1}\;\frac{4(2^2-\ell^2)}{+5(x^2-1)+n^2-\ell^2+}\cdots.&\end{aligned}$$

PROOF. Taking the second equality in (35.1), divide both sides by m and then let m tend to 0. Applying L'Hospital's rule on the left side, we readily deduce the desired formula with no difficulty.

ENTRY 38. *Assume that* $|x| > 1$ *and that* n^2 *is real. Then*

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{1}{(x-n+2k+1)^2}-\sum_{k=0}^{\infty}\frac{1}{(x+n+2k+1)^2}&\\ =\frac{n}{x^2-1+n^2+}\;\frac{2(1^2-n^2)}{1}\;\frac{2\cdot 1^2}{+3(x^2-1)+n^2}&\qquad (38.1)\\ +\frac{4(2^2-n^2)}{1}\;\frac{4\cdot 2^2}{+5(x^2-1)+n^2+}\cdots&\\ =\frac{n}{x^2-n^2+1-}\;\frac{4(1^2-n^2)1^4}{3(x^2-n^2+5)-}\;\frac{4(2^2-n^2)2^4}{5(x^2-n^2+13)-}\cdots.&\end{aligned}$$

PROOF. To prove the first equality in (38.1), divide both sides of (37.1) by 2ℓ and let ℓ tend to 0. Applying L'Hospital's rule on the left side, we easily achieve the desired equality.

The second equality in (38.1) is also easily established. First, divide both sides of the first equality in (35.1) by m and then let m tend to 0. Of course, this gives a second continued fraction for the left side of (37.1). Now divide both sides by ℓ and let ℓ tend to 0.

ENTRY 39. *Suppose that* $|x| > 1$ *and that* ℓ^2 *and* n^2 *are real. Then*

$$P \equiv \frac{\Gamma\left(\frac{1}{4}(x+\ell+n+1)\right)\Gamma\left(\frac{1}{4}(x-\ell+n+1)\right)}{\Gamma\left(\frac{1}{4}(x+\ell+n+3)\right)\Gamma\left(\frac{1}{4}(x-\ell+n+3)\right)}$$

(39.1)
$$\cdot\frac{\Gamma\left(\frac{1}{4}(x+\ell-n+1)\right)\Gamma\left(\frac{1}{4}(x-\ell-n+1)\right)}{\Gamma\left(\frac{1}{4}(x+\ell-n+3)\right)\Gamma\left(\frac{1}{4}(x-\ell-n+3)\right)}$$

$$= \frac{8}{(x^2-\ell^2+n^2-1)/2}\,\frac{}{+}\,\frac{1^2-n^2}{1}\,\frac{}{+}\,\frac{1^2-\ell^2}{x^2-1}\,\frac{}{+}\,\frac{3^2-n^2}{1}\,\frac{}{+}\,\frac{3^2-\ell^2}{x^2-1}\,\frac{}{+}\cdots.$$

PROOF. First, we remark that a theorem of Pringsheim [57, p. 47] insures that the continued fraction on the right side of (39.1) converges for the designated ranges of the parameters ℓ, n, and x.

To prove Entry 39, we employ the following theorem found in [57, p. 27, Satz 1.13]. Suppose that all of the elements are positive in both continued fractions below. (Actually, the partial numerators and denominators only need to be positive if their indices are sufficiently large.) Assume also that each continued fraction converges. Then

(39.2)
$$b_0 + \frac{a_1}{b_1}\,\frac{}{+}\,\frac{a_2}{b_2}\,\frac{}{+}\,\frac{a_3}{b_3}\,\frac{}{+}\cdots$$
$$= b_0 + r_0 + \frac{\varphi_1}{b_1+r_1}\,\frac{}{+}\,\frac{a_1\varphi_2/\varphi_1}{b_2+r_2-r_0\varphi_2/\varphi_1}\,\frac{}{+}\,\frac{a_2\varphi_3/\varphi_2}{b_3+r_3-r_1\varphi_3/\varphi_2}\,\frac{}{+}\cdots,$$

where

(39.3)
$$\varphi_k = a_k - r_{k-1}(b_k + r_k), \qquad k \geqq 1.$$

(The parameters r_k, $k \geqq 0$, have no restrictions other than those imposed above.)

Let

(39.4)
$$F(x) = F(x, \ell, n) = \frac{x^2-\ell^2+n^2-1}{2} + \frac{1^2-n^2}{1}\,\frac{}{+}\,\frac{1^2-\ell^2}{x^2-1}\,\frac{}{+}\,\frac{3^2-n^2}{1}\,\frac{}{+}\,\frac{3^2-\ell^2}{x^2-1}\,\frac{}{+}\cdots.$$

In the notation above, $a_{2k} = (2k-1)^2 - \ell^2$, $a_{2k-1} = (2k-1)^2 - n^2$, $b_{2k} = x^2 - 1$, and $b_{2k-1} = 1$, where $k \geqq 1$. Write

(39.5)
$$r_{2k} = d_1 k + c_1 \quad \text{and} \quad r_{2k-1} = d_2 k + c_2, \qquad k \geqq 1.$$

Our first goal is to determine c_1, c_2, d_1, and d_2 so that φ_k is constant for $k \geqq 1$.

From (39.3), (39.5), and the aformentioned formula for a_{2k}, it follows that

$$d_1 d_2 = 4. \tag{39.6}$$

Thus, from (39.3), we find that

$$\begin{aligned} \varphi_{2k} &= (2k-1)^2 - \ell^2 - (d_2 k + c_2)(x^2 - 1 + d_1 k + c_1) \\ &= -\{4 + d_2(x^2 - 1 + c_1) + c_2 d_1\}k + 1 - \ell^2 - c_2(x^2 - 1 + c_1) \end{aligned} \tag{39.7}$$

and

$$\begin{aligned} \varphi_{2k-1} &= (2k-1)^2 - n^2 - \{d_1(k-1) + c_1\}\{1 + d_2 k + c_2\} \\ &= -\{4 + d_1(1 + c_2) - d_2(d_1 - c_1)\}k + 1 - n^2 + (d_1 - c_1)(1 + c_2), \end{aligned} \tag{39.8}$$

where $k \geqq 1$. By our prescriptions, we require that

$$d_2(x^2 - 1 + c_1) + c_2 d_1 = -4 = d_1(1 + c_2) - d_2(d_1 - c_1). \tag{39.9}$$

Using (39.6) and simplifying the extremal equality above, we find that $d_1^2 - 4d_1 + 4(1 - x^2) = 0$. We shall choose the positive root $d_1 = 2x + 2$. Thus, by (39.6), $d_2 = 2/(x + 1)$.

Since we wish φ_k to be constant, by (39.7) and (39.8), we need to stipulate that $1 - \ell^2 - c_2(x^2 - 1 + c_1) = 1 - n^2 + (d_1 - c_1)(1 + c_2)$. Simplifying, we find that

$$c_1 - c_2(x + 1)^2 = \ell^2 - n^2 + 2(x + 1). \tag{39.10}$$

On the other hand, from (39.9),

$$c_1 + c_2(x + 1)^2 = -(x + 1)^2. \tag{39.11}$$

Adding (39.10) and (39.11), we deduce that $c_1 = \frac{1}{2}(\ell^2 - n^2 - x^2 + 1)$, and so

$$c_2 = -1 - \frac{\ell^2 - n^2 - x^2 + 1}{2(x + 1)^2}.$$

Hence, we have determined the parameters c_1, c_2, d_1, and d_2 so that φ_k is constant, viz., from (39.5),

$$r_{2k} = 2(x + 1)k + \frac{1}{2}(\ell^2 - n^2 - x^2 + 1), \qquad k \geqq 1, \tag{39.12}$$

and

$$r_{2k-1} = \frac{2}{x + 1}k - 1 - \frac{\ell^2 - n^2 - x^2 + 1}{2(x + 1)^2}. \tag{39.13}$$

Let us set $\varphi_k = \alpha$. By (39.8) and our determinations above,

$$\begin{aligned}
\alpha &= 1 - n^2 + (d_1 - c_1)(1 + c_2)\\
&= \frac{4(1-n^2)(x+1)^2 - 4(x+1)(\ell^2-n^2-x^2+1) + (\ell^2-n^2-x^2+1)^2}{4(x+1)^2}\\
&= \frac{-4n^2(x+1)^2 + \{\ell^2 - n^2 - x^2 + 1 - 2(x+1)\}^2}{4(x+1)^2}\\
&= \frac{\{\ell^2-n^2-x^2+1-2(1+n)(x+1)\}\{\ell^2-n^2-x^2+1-2(1-n)(x+1)\}}{4(x+1)^2}.
\end{aligned}$$

The numerator above is a polynomial in x of degree 4. It is easily checked that the four roots of this polynomial are $x + 1 = \pm\ell \pm n$, where all four possible combinations of signs are taken. Hence,

$$\alpha = \frac{(x+1+\ell+n)(x+1-\ell-n)(x+1+\ell-n)(x+1-\ell+n)}{4(x+1)^2}. \tag{39.14}$$

Recalling the definition (39.4), applying (39.2), and employing (39.12) and (39.13), we have shown that

$$\begin{aligned}
F(x) &= \frac{\alpha}{\dfrac{2}{x+1} - \dfrac{\ell^2-n^2-x^2+1}{2(x+1)^2}} + \frac{1^2-n^2}{x^2-1+2x+2}\\
&\quad + \frac{1^2-\ell^2}{1+2/(x+1)} + \frac{3^2-n^2}{x^2-1+2x+2} + \frac{3^2-\ell^2}{1+2/(x+1)} + \cdots\\
&= \frac{\alpha(x+1)^2}{\{(x+2)^2-\ell^2+n^2-1\}/2} + \frac{1^2-n^2}{1} + \frac{1^2-\ell^2}{(x+1)(x+3)}\\
&\quad + \frac{3^2-n^2}{1} + \frac{3^2-\ell^2}{(x+1)(x+3)} + \cdots\\
&= \frac{\alpha(x+1)^2}{F(x+2)}.
\end{aligned} \tag{39.15}$$

For brevity, set, for any function f,

$$\begin{aligned}
&\textstyle\prod_{\pm} f(x + k \pm \ell \pm n)\\
&\quad = f(x+k+\ell+n)f(x+k-\ell-n)f(x+k+\ell-n)f(x+k-\ell+n).
\end{aligned}$$

Hence, from (39.14) and (39.15),

$$F(x)F(x+2) = \frac{1}{4}\prod_{\pm}(x + 1 \pm \ell \pm n),$$

and so

$$\frac{F(x)F(x+2)}{F(x+2)F(x+4)} = \prod_{\pm}\left(\frac{x+1\pm\ell\pm n}{x+3\pm\ell\pm n}\right).$$

By iteration of this formula, we find that, for each positive integer m,

$$\frac{F(x)}{F(x+4m)}=\prod_{k=0}^{m-1}\prod_{\pm}\left(\frac{x+4k+1\pm\ell\pm n}{x+4k+3\pm\ell\pm n}\right)$$

$$=\frac{1}{m^2}\prod_{\pm}\prod_{k=0}^{m-1}\frac{\left(\frac{1}{4}(x+1\pm\ell\pm n)+k\right)m!m^{(x-1\pm\ell\pm n)/4}}{\left(\frac{1}{4}(x+3\pm\ell\pm n)+k\right)m!m^{(x-3\pm\ell\pm n)/4}}.$$

Hence,

$$\text{(39.16)}\qquad \lim_{m\to\infty}\frac{F(x)m^2}{F(x+4m)}=\prod_{\pm}\frac{\Gamma\left(\frac{1}{4}(x+3\pm\ell\pm n)\right)}{\Gamma\left(\frac{1}{4}(x+1\pm\ell\pm n)\right)}=\frac{1}{P}.$$

From the definition of $F(x)$ in (39.4), we easily see that

$$\text{(39.17)}\qquad \lim_{m\to\infty}\frac{m^2}{F(x+4m)}=\frac{1}{8}.$$

Combining (39.16) and (39.17), we deduce (39.1).

H. Cohen has communicated to us another proof of Entry 39. His proof is based upon Apéry's method for accelerating the convergence of a continued fraction. For a complete description of this method, see Cohen's seminar notes [**16**]. Accounts are also given in [**3**] and [**5**].

We might note an interesting consequence of Entry 39. From Malmstén's integral representation for Log $\Gamma(z)$ [**78**, p. 249], we find that

$$\operatorname{Log} P=\int_0^\infty\left(\frac{\sum_\pm e^{-(x\pm\ell\pm n+1)t/4}-\sum_\pm e^{-(x\pm\ell\pm n+3)t/4}}{1-e^{-t}}-2e^{-t}\right)\frac{dt}{t},$$

where $\sum_\pm$ indicates a sum of four terms with each possible combination of signs taken. Simplifying, we find that

$$\operatorname{Log} P=2\int_0^\infty\left(\frac{e^{-tx/4}\cosh(\ell t/4)\cosh(nt/4)}{\cosh(t/4)}-e^{-t}\right)\frac{dt}{t}$$

$$\text{(39.18)}\qquad =2\int_0^\infty e^{-tx}\left(\frac{\cosh(\ell t)\cosh(nt)}{\cosh t}-1\right)\frac{dt}{t}-2\int_0^\infty\frac{e^{-4t}-e^{-xt}}{t}\,dt$$

$$=2\int_0^\infty e^{-tx}\left(\frac{\cosh(\ell t)\cosh(nt)}{\cosh t}-1\right)\frac{dt}{t}+2\operatorname{Log}(4/x),$$

by Frullani's theorem [**20**, pp. 337–342]. Exponentiating (39.18) and combining the result with (39.1), we deduce that

$$\text{(39.19)}\qquad \exp\left(2\int_0^\infty e^{-tx}\left(\frac{\cosh(\ell t)\cosh(nt)}{\cosh t}-1\right)\frac{dt}{t}\right)$$

$$=\frac{x^2/2}{(x^2-\ell^2+n^2-1)/2+}\,\frac{1^2-n^2}{1}\,\frac{1^2-\ell^2}{+x^2-1+}\,\frac{3^2-n^2}{1}\,\frac{3^2-\ell^2}{+x^2-1+}\cdots,$$

where $0 \leqq |\ell|, |n| < 1$ and $x > 1$.

The expansion (39.19) appears to be new. It generalizes a result of Rogers [**66**] and is similar to results of both Rogers [**66**] and Stieltjes [**67**], [**70**, pp. 368–391].

ENTRY 40. *Let* $P = \prod \Gamma(\frac{1}{2}(\alpha \pm \beta \pm \gamma \pm \delta \pm \varepsilon + 1))$, *where the product contains* 8 *gamma functions and where the argument of each gamma function contains an even number of minus signs. Let* $Q = \prod \Gamma(\frac{1}{2}(\alpha \pm \beta \pm \gamma \pm \delta \pm \varepsilon + 1))$, *where the product contains* 8 *gamma functions and where the argument of each gamma function contains an odd number of minus signs. Suppose that at least one of the parameters* $\beta, \gamma, \delta, \varepsilon$ *is equal to a nonzero integer. Then*

$$(40.1) \qquad \frac{P-Q}{P+Q} =$$

$$\cfrac{8\alpha\beta\gamma\delta\varepsilon}{1\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\varepsilon^4+1)-(\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2-1)^2-2^2\} + \cfrac{64(\alpha^2-1^2)(\beta^2-1^2)(\gamma^2-1^2)(\delta^2-1^2)(\varepsilon^2-1^2)}{3\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\varepsilon^4+1)-(\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2-5)^2-6^2\} + \cfrac{64(\alpha^2-2^2)(\beta^2-2^2)(\gamma^2-2^2)(\delta^2-2^2)(\varepsilon^2-2^2)}{5\{2(\alpha^4+\beta^4+\gamma^4+\delta^4+\varepsilon^4+1)-(\alpha^2+\beta^2+\gamma^2+\delta^2+\varepsilon^2-13)^2-14^2\}+\cdots}}}.$$

Entry 40 is certainly one of Ramanujan's crowning achievements in the theory of continued fractions. Watson [77] has given the only published proof of Entry 40. Of course, under Watson's hypotheses, the continued fraction terminates. It would be very desirable to have another proof which does not require that at least one of the parameters $\beta, \gamma, \delta, \varepsilon$ be a nonzero integer.

In an address before the London Mathematical Society in 1931, Watson [**76**] discussed Entry 40 but incorrectly wrote 9 and 10 instead of 13 and 14, respectively, in the last recorded denominator above. In a footnote of [**77**], Watson remarked that "Through an error in copying which occurred when I previously published an enunciation of the theorem. . . ." However, Watson did copy the result faithfully; Ramanujan had made the same error in [**64**, vol. 2, p. 152]. Throughout the notebooks, Ramanujan normally did not completely state identities involving sequences, but he did usually give enough terms to determine the sequence. In particular, if a sequence is linear, Ramanujan often gave only two terms, while if a sequence is quadratic, he would give three. In the first notebook, he only stated two terms of the sequences $2n^2 + 2n + 1$ and $2n^2 + 2n + 2$; i.e., 1, 5 and 2, 6, respectively, that occur on the right side of (40.1). This was probably carelessness on his part for he most likely knew the quadratic patterns of the sequences. When he wrote his second notebook, a revised enlargement of the first, he decided to add one more term. However, he evidently did not rederive his identity and erroneously assumed that the two sequences are linear. Ironically, Watson's statement of Entry 40 in [**77**] also contains a misprint. Watson [**77**] also obtained a q-analogue of Entry 40.

ENTRY 41. *If* $\mathrm{Re}(x) > 0$ *or if* $\mathrm{Re}(x) < -2$, *then*

$$\begin{aligned} &{}_2F_1(-\beta, 1; \gamma+1; -x) \\ &= \frac{\Gamma(\beta+1)\Gamma(\gamma+1)(1+x)^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)x^{\gamma}} - \frac{\gamma}{(\beta+1)x+1-\gamma} \\ &\quad - \frac{1(1-\gamma)(x+1)}{(\beta+2)x+3-\gamma} - \frac{2(2-\gamma)(x+1)}{(\beta+3)x+5-\gamma} - \cdots. \end{aligned}$$

PROOF. From [**21**, p. 108, formula (2)],

$$\begin{aligned} &{}_2F_1(-\beta, 1; \gamma+1; -x) \\ &= \frac{\Gamma(-\beta-1)\Gamma(\gamma+1)}{\Gamma(-\beta)\Gamma(\gamma)x}\, {}_2F_1(1, 1-\gamma; \beta+2; -1/x) \\ &\quad + \frac{\Gamma(\beta+1)\Gamma(\gamma+1)x^{\beta}}{\Gamma(\beta+\gamma+1)}\, {}_2F_1(-\beta, -\beta-\gamma; -\beta; -1/x) \qquad (41.1) \\ &= -\frac{\gamma}{(\beta+1)x}\, {}_2F_1(1, 1-\gamma; \beta+2; -1/x) \\ &\quad + \frac{\Gamma(\beta+1)\Gamma(\gamma+1)(x+1)^{\beta+\gamma}}{\Gamma(\beta+\gamma+1)x^{\gamma}}. \end{aligned}$$

Now apply (21.3) with β, γ, and x replaced by $-\gamma$, $\beta+1$, and $1/x$, respectively, to deduce that

$$\begin{aligned} &-\frac{\gamma}{(\beta+1)x}\, {}_2F_1(1, 1-\gamma; \beta+2; -1/x) \\ &= -\frac{\gamma/x}{\beta+1+(1-\gamma)/x} - \frac{1(1-\gamma)(1+1/x)/x}{\beta+2+(3-\gamma)/x} \qquad (41.2) \\ &\quad - \frac{2(2-\gamma)(1+1/x)/x}{\beta+3+(5-\gamma)/x} - \cdots. \end{aligned}$$

Combining (41.1) and (41.2), we obtain an equivalent form of the proposed formula.

ENTRY 42. *If* $x > 0$, *then*

$$
\begin{aligned}
&{}_1F_1(1; n+1; x)\\
(42.1)\quad &= \frac{e^x\Gamma(n+1)}{x^n} - \frac{n}{x+}\,\frac{1-n}{1+}\,\frac{1}{x+}\,\frac{2-n}{1+}\,\frac{2}{x+}\,\frac{3-n}{1+}\,\frac{3}{x+}\cdots\\
&= \frac{e^x\Gamma(n+1)}{x^n} - \frac{n}{x+1-n-}\,\frac{1(1-n)}{x+3-n-}\,\frac{2(2-n)}{x+5-n-}\,\frac{3(3-n)}{x+7-n-}\cdots.
\end{aligned}
$$

PROOF. In Entry 41, replace x by x/β and γ by n to find that

$$
\begin{aligned}
&{}_2F_1(-\beta, 1; n+1; -x/\beta)\\
&= \frac{\Gamma(\beta+1)\Gamma(n+1)(1+x/\beta)^{\beta+n}}{\Gamma(\beta+n+1)(x/\beta)^n} - \frac{\gamma}{(\beta+1)x/\beta+1-n}\\
&\quad - \frac{1(1-n)(1+x/\beta)}{(\beta+2)x/\beta+3-n-}\,\frac{2(2-n)(1+x/\beta)}{(\beta+3)x/\beta+5-n-}\cdots.
\end{aligned}
$$

Letting β tend to ∞, we deduce the second equality in (42.1) at once.

To obtain the first equality in (42.1), apply Entry 14.

Entry 42 was first discovered by Legendre **[46]**. See also Nielsen's book [**50**, p. 217] for a proof.

COROLLARY. *If $x > 0$, then*

$$
\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)} = \frac{\Gamma(n)}{x^n} - \frac{e^{-x}}{x+}\,\frac{1-n}{1+}\,\frac{1}{x+}\,\frac{2-x}{1+}\,\frac{2}{x+}\cdots.
$$

PROOF. Multiplying both sides of (42.1) by e^{-x}/n and comparing the resulting equality with that above, we see that we must show that

$$
e^{-x}\sum_{k=0}^{\infty} \frac{x^k}{(n)_{k+1}} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!(n+k)}. \tag{42.2}
$$

However, (42.2) is just a special case of a theorem due to Kummer **[40]** and rediscovered by Ramanujan in Chapter 10. In fact, in [**9**, Entry 21], replace x by $-x$ and n by $n+1$, and lastly set $m = n$ to deduce (42.2).

ENTRY 43. *If $x > 0$, then*

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{x^k}{1\cdot 3\cdots(2k+1)}\\
(43.1)\quad &= \sqrt{\frac{\pi}{2x}}\,e^{x/2} - \frac{1}{x+}\,\frac{1}{1+}\,\frac{2}{x+}\,\frac{3}{1+}\,\frac{4}{x+}\,\frac{5}{1+}\cdots\\
&= \sqrt{\frac{\pi}{2x}}\,e^{x/2} - \frac{1}{x+1-}\,\frac{1\cdot 2}{x+5-}\,\frac{3\cdot 4}{x+9-}\,\frac{5\cdot 6}{x+13-}\cdots.
\end{aligned}
$$

PROOF. Putting $n = 1/2$ in Entry 42, we find that

$$\sum_{k=0}^{\infty} \frac{(2x)^k}{1 \cdot 3 \cdots (2k+1)}$$
$$= \sqrt{\frac{\pi}{4x}}e^x - \frac{1/2}{x} + \frac{1/2}{1} + \frac{1}{x} + \frac{3/2}{1} + \frac{2}{x} + \frac{5/2}{x} + \cdots.$$

Replacing x by $x/2$, we obtain an equivalent form of the first continued fraction of (43.1).

The second continued fraction in (43.1) follows in the same way from the second continued fraction of (42.1). Alternatively, apply Entry 14 to the first continued fraction in (43.1).

COROLLARY 1. *For* $x > 0$,

$$F(x) \equiv \int_0^x e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2} - \frac{e^{-x^2}}{2x} + \frac{1}{x} + \frac{2}{2x} + \frac{3}{x} + \frac{4}{2x} + \cdots.$$

PROOF. By (42.2), for $x, n > 0$,

$$\int_0^x e^{-t}t^{n-1}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k x^{n+k}}{k!(n+k)} = e^{-x}\sum_{k=0}^{\infty} \frac{x^{n+k}}{(n)_{k+1}}. \tag{43.2}$$

Let $n = 1/2$ and replace t by t^2 and x by x^2. Applying Entry 43, we then find that

$$F(x) = xe^{-x^2}\sum_{k=0}^{\infty} \frac{x^{2k}}{(3/2)_k} = xe^{-x^2}\sum_{k=0}^{\infty} \frac{(2x^2)^k}{1 \cdot 3 \cdots (2k+1)}$$
$$= xe^{-x^2}\left\{\sqrt{\frac{\pi}{4x^2}}e^{x^2} - \frac{1}{2x^2} + \frac{1}{1} + \frac{2}{2x^2} + \frac{3}{1} + \frac{4}{2x^2} + \frac{5}{1} + \cdots\right\},$$

which is equivalent to the proposed formula.

COROLLARY 2. *As* x *tends to* ∞,

$$\int_0^x \frac{F(t)}{t}\,dt = \frac{\sqrt{\pi}}{2}\left(\frac{\gamma}{2} + \operatorname{Log}(2x)\right) + o(1), \tag{43.3}$$

where F *is defined in Corollary* 1 *and* γ *denotes Euler's constant.*

PROOF. Integrating by parts, we find that

$$\begin{aligned} \int_0^x \frac{F(t)}{t}\,dt &= F(x)\operatorname{Log} x - \int_0^x e^{-t^2}\operatorname{Log} t\,dt \\ &= \left(\int_0^\infty e^{-t^2}\,dt - \int_x^\infty e^{-t^2}\,dt\right)\operatorname{Log} x \\ &\quad - \left(\int_0^\infty e^{-t^2}\operatorname{Log} t\,dt - \int_x^\infty e^{-t^2}\operatorname{Log} t\,dt\right) \\ &= \frac{\sqrt{\pi}}{2}\operatorname{Log} x - \int_0^\infty e^{-t^2}\operatorname{Log} t\,dt + o(1), \end{aligned} \tag{43.4}$$

as x tends to ∞.

From the definition of $\Gamma(x)$,

$$\Gamma'(x) = 4\int_0^\infty e^{-t^2} t^{2x-1} \operatorname{Log} t dt.$$

In particular[**47**, p. 13],

$$\Gamma'(1/2) = 4\int_0^\infty e^{-t^2} \operatorname{Log} t dt = -\sqrt{\pi}(\gamma + 2 \operatorname{Log} 2). \tag{43.5}$$

(In fact, Ramanujan [**64**, vol. 2, p. 92], [**8**, §6, cor. 3] established (43.5) in Chapter 8.) Employing (43.5) in (43.4), we deduce (43.3) at once.

ENTRY 44. *For $x > 0$, define*

$$\varphi(x) = \int_0^\infty e^{-t}/(x+t)\, dt.$$

Then, for $x > 0$,

$$\int_0^x \frac{1-e^{-t}}{t} dt = \sum_{k=1}^\infty \frac{(-1)^{k-1}x^k}{k!k} = \gamma + \operatorname{Log} x + e^{-x}\varphi(x), \tag{44.1}$$

where γ denotes Euler's constant.

PROOF. At the outset, we remark that essentially the same calculations are made in slightly more detail in our edited version of Chapter 4[**12**, p. 103].

The first equality in (44.1) is readily established by writing the integrand as a Maclaurin series and inverting the order of summation and integration.

Next, making a simple change of variable in the definition of φ and using a well-known integral representation for γ [**53**, p. 40], we find that

$$e^{-x}\varphi(x) + \gamma + \operatorname{Log} x = \int_x^\infty \frac{e^{-t}}{t} dt + \int_0^1 \frac{1-e^{-t}}{t} dt - \int_1^\infty \frac{e^{-t}}{t} dt + \int_1^x \frac{dt}{t}.$$

Upon simplification, we complete the proof of the second equality in (44.1).

ENTRY 44(i). *As x tends to ∞,*

$$\varphi(x) \sim \sum_{k=0}^\infty \frac{(-1)^k k!}{x^{k+1}}.$$

Entry 44(i) was established by Euler, and a rigorous discussion of it can be found in Hardy's book [**33**, pp. 26, 27]. Ramanujan also stated this result in Chapter 4 [**64**, vol. 2, p. 44], [**12**, p. 102].

For Entry 44 (ii), we quote Ramanujan [**64**, vol. 2, p. 153].

ENTRY 44(ii). *$\varphi(x)$ lies between $1/x$ and $1/(x+1)$ and very nearly equals $\sqrt{\varphi(x+1)/x}$.*

PROOF. Letting n tend to 0 in the corollary of §42, we find that, for $x > 0$,

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^k}{k!k}=\lim_{n\to 0}\left(\frac{1}{n}-\frac{\Gamma(n)}{x^n}\right)+e^{-x}f(x) \tag{44.2}$$
$$=\gamma+\operatorname{Log} x+e^{-x}f(x),$$

where

$$f(x)=\frac{1}{x}+\frac{1}{1}+\frac{1}{x}+\frac{2}{1}+\frac{2}{x}+\cdots. \tag{44.3}$$

Comparing (44.1) and (44.2), we deduce that $f(x)=\varphi(x)$.

Now, from (44.3), it is immediate that $\varphi(x)<1/x$. Secondly, if

$$F=x+\frac{2}{1}+\frac{2}{x}+\frac{3}{1}+\frac{3}{x}+\cdots,$$

we can write (44.3) as

$$\varphi(x)=\frac{1}{x+1/(1+1/F)}=\frac{1}{x+\dfrac{F}{1+F}}>\frac{1}{x+1}.$$

Thus, Ramanujan's upper and lower bounds for $\varphi(x)$ are established.

Squaring the asymptotic series from Entry 44(i), we find that, as x tends to ∞,

$$\varphi^2(x)\sim\frac{1}{x^2}-\frac{2}{x^3}+\frac{5}{x^4}-\frac{16}{x^5}+\cdots.$$

On the other hand, also from Entry 44(i), as x tends to ∞,

$$\frac{\varphi(x+1)}{x}\sim\frac{1}{x(x+1)}-\frac{1}{x(x+1)^2}+\frac{2}{x(x+1)^3}+\cdots$$
$$=\frac{1}{x^2}\left(1-\frac{1}{x}+\frac{1}{x^2}-\frac{1}{x^3}\right)-\frac{1}{x^3}\left(1-\frac{2}{x}+\frac{3}{x^2}\right)+\frac{2}{x^4}\left(1-\frac{3}{x}\right)+0\left(\frac{1}{x^6}\right)$$
$$=\frac{1}{x^2}-\frac{2}{x^3}+\frac{5}{x^4}-\frac{10}{x^5}+0\left(\frac{1}{x^6}\right).$$

Thus, the initial three terms of the asymptotic expansions for $\varphi^2(x)$ and $\varphi(x+1)/x$ agree. Hence, Ramanujan's approximation for $\varphi(x)$ is reasonable.

ENTRY 44(iii). *For $x>0$,*

$$\varphi(x)=\frac{1}{x}+\frac{1}{1}+\frac{1}{x}+\frac{2}{1}+\frac{2}{x}+\frac{3}{1}+\frac{3}{x}+\cdots$$
$$=\frac{1}{x+1}-\frac{1^2}{x+3}-\frac{2^2}{x+5}-\frac{3^2}{x+7}-\cdots.$$

PROOF. The former continued fraction was established in the course of proving Entry 44(ii) (see (44.3)). To obtain the latter continued fraction, apply Entry 14.

The second continued fraction above was first derived by Tschebyscheff **[73]**.

ENTRY 44 (iv). *Let x be any complex number exterior to* $(-\infty, 0]$, *and let n be a natural number. Then*

$$\varphi(x) = \sum_{k=0}^{n-1} \frac{(-1)^k k!}{x^{k+1}} + \frac{(-1)^n n!}{x^n}\left(\frac{1}{x+n+1} - \frac{n+1}{x+n+3} - \frac{2(n+2)}{x+n+5} - \frac{3(n+3)}{x+n+7} - \cdots\right).$$

PROOF. Integrating by parts n times, we find that

$$\begin{aligned}\varphi(x) &= \sum_{k=0}^{n-1} \frac{(-1)^k k!}{x^{k+1}} + (-1)^n\, n! \int_0^\infty \frac{e^{-t}}{(x+t)^{n+1}}\, dt \\ &= \sum_{k=0}^{n-1} \frac{(-1)^k k!}{x^{k+1}} + \frac{(-1)^n}{x^n} \int_0^\infty \frac{e^{-t} t^n}{x+t}\, dt,\end{aligned} \tag{44.4}$$

where we have used the equality [**57**, p. 219]

$$\frac{1}{\Gamma(b)} \int_0^\infty \frac{e^{-t} t^{b-1}}{(1+t)^a}\, dt = \frac{1}{\Gamma(a)} \int_0^\infty \frac{e^{-t} t^{a-1}}{(1+t)^b}\, dt, \qquad \mathrm{Re}(a), \mathrm{Re}(b) > 0.$$

However, for $x \notin (-\infty, 0]$, [**57**, p. 219, eq. (12)], [**38**, p. 148, eq. (11.17)],

$$\begin{aligned}&\frac{1}{n!} \int_0^\infty \frac{e^{-t} t^n}{x+t}\, dt \\ &= \frac{1}{x+n+1} - \frac{n+1}{x+n+3} - \frac{2(n+2)}{x+n+5} - \frac{3(n+3)}{x+n+7} - \cdots.\end{aligned} \tag{44.5}$$

Substituting (44.5) into (44.4), we deduce the proposed identity.

COROLLARY 1. *Let* $H_n = \sum_{k=1}^n 1/k$. *Then if* $x > 0$,

$$\sum_{k=1}^\infty \frac{H_k x^k}{k!} = e^x(\mathrm{Log}\, x + \gamma) + \varphi(x).$$

Corollary 1 is also given by Ramanujan in Chapter 4 [**64**, vol. 2, p. 44]. See [**12**, p. 103] for a proof.

Our formulation of Corollary 2 corrects that given by Ramanujan [**64**, vol. 2, p. 153].

COROLLARY 2. *For* $|h| < 1$ *and* $n > 0$, *define* $f(h, n)$ *by*

$$\int_0^{n(1-h)} \frac{1-e^{-t}}{t}\, dt = \gamma + \mathrm{Log}\, n + e^{-n} \varphi(n) - e^{-n} f(h, n). \tag{44.6}$$

Then

$$f(h, n) = \sum_{k=1}^{\infty} \frac{1}{k}\left(e^n - \sum_{j=0}^{k-1} \frac{n^j}{j!}\right)h^k.$$

PROOF. First, if $h = 0$, we see from Entry 44 that $f(0, n) = 0$.

For brevity, set $g(h) = f(h, n)$. Clearly, we shall be finished if we can show that

$$g^{(k+1)}(0) = k!\left(e^n - \sum_{j=0}^{k} \frac{n^j}{j!}\right), \qquad k \geqq 0. \tag{44.7}$$

First, differentiating (44.6), we find that

$$e^{-n}g'(h) = \frac{e^{nh-n} - 1}{h - 1}. \tag{44.8}$$

Setting $h = 0$ in (44.8), we deduce (44.7) in the case $k = 0$. For $k > 0$, we apply Leibniz's rule to (44.8) to find that

$$\begin{aligned} e^{-n}g^{(k+1)}(h) &= \sum_{j=0}^{k} \binom{k}{j} \frac{d^j}{dh^j}\left(\frac{1}{h-1}\right) \frac{d^{k-j}}{dh^{k-j}}(e^{nh-n} - 1) \\ &= \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j j!}{(h-1)^{j+1}} \left\{n^{k-j}e^{nh-n} - \delta_j\right\}, \end{aligned}$$

where $\delta_k = 1$ and $\delta_j = 0$, $0 \leqq j \leqq k - 1$. Thus,

$$g^{(k+1)}(0) = k!e^n - \sum_{j=0}^{k} \binom{k}{j} j!n^{k-j}.$$

Equality (44.7) now follows upon replacing j by $k - j$ above.

Ramanujan concludes section 44 by recording the values $\varphi(1) = .5963474$ and $\varphi(1/2) = .9229106$. From (44.1),

$$\varphi(1) = e\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!k} - \gamma\right)$$

and

$$\varphi(1/2) = \sqrt{e}\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^k k!k} - \gamma - \operatorname{Log} 2\right).$$

Using calculated values for $\gamma, e, \sqrt{e}$, and Log 2 [**1**, pp. 2, 3] and 11 and 9 terms, respectively, from the two sums above, we can readily verify that Ramanujan's calculations are correct.

ENTRIES 45(i), (ii). *Consider the continued fraction*

$$\frac{1}{1} + \frac{x}{1} + \frac{x}{1} + \frac{2x}{1} + \frac{2x}{1} + \frac{3x}{1} + \frac{3x}{1} + \cdots + \frac{(n-1)x}{1} + \frac{nx}{1}.$$

Then in the notation of (1.4), *for* $n \geqq 1$,

(45.1)
$$B_{2n} \equiv B_{2n}(x) = \sum_{k=0}^{n} \frac{(-n)_k^2 x^k}{k!}$$

and

(45.2)
$$B_{2n-1} \equiv B_{2n-1}(x) = \sum_{k=0}^{n-1} \frac{(-n)_k^2}{k!}\left(1 - \frac{k}{n}\right)x^k.$$

PROOF. We shall induct on n. For $n = 1$, both (45.1) and (45.2) are easily seen to be correct.

We shall thus assume that both (45.1) and (45.2) are true up to a specific positive integer n. By (1.4),

$$\begin{aligned} B_{2n+1}(x) &= B_{2n}(x) + nxB_{2n-1}(x) \\ &= \sum_{k=0}^{n} \frac{(-n)_k^2 x^k}{k!} + \sum_{k=1}^{n} \frac{(-n)_{k-1}^2 (n - k + 1)x^k}{(k-1)!}. \end{aligned}$$

But, for $1 \leqq k \leqq n$,

$$\begin{aligned} \frac{(-n)_k^2}{k!} + \frac{(-n)_{k-1}^2(n-k+1)}{(k-1)!} &= \frac{(-n)_{k-1}^2(n-k+1)}{k!}\{(n-k+1)+k\} \\ &= \frac{(-n-1)_k^2}{k!}\left(1 - \frac{k}{n+1}\right). \end{aligned}$$

Hence, we have established (45.2) with n replaced by $n + 1$.

By (1.4) and the proof just completed above,

$$\begin{aligned} B_{2n+2}(x) &= B_{2n+1}(x) + (n+1)xB_{2n}(x) \\ &= \sum_{k=0}^{n+1} \frac{(-n-1)_k^2}{k!}\left(1 - \frac{k}{n+1}\right)x^k + (n+1)\sum_{k=1}^{n+1} \frac{(-n)_{k-1}^2}{(k-1)!}x^k. \end{aligned}$$

But, for $1 \leqq k \leqq n + 1$,

$$\begin{aligned} &\frac{(-n-1)_k^2}{k!}\left(1 - \frac{k}{n+1}\right) + (n+1)\frac{(-n)_{k-1}^2}{(k-1)!} \\ &= \frac{(-n-1)_k^2}{k!}\left(1 - \frac{k}{n+1} + \frac{k}{n+1}\right) = \frac{(-n-1)_k^2}{k!}. \end{aligned}$$

Hence, (45.1) is established with n replaced by $n + 1$.

We have slightly rearranged the ordering of the formulas in §46.

ENTRY 46(i). *For* $|x| < 1$, *set*

(46.1)
$$\Gamma(x+1) = \sum_{k=0}^{\infty} \frac{A_k(-x)^k}{k!}.$$

Define $\varphi_n(x)$ *to be the constant term in the Laurent expansion of* $x^p\Gamma(1-p)/p^n$, $0 < |p| < 1$, *where* n *is a nonnegative integer. Then*

$$\varphi_n(x) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}A_{n-k}\operatorname{Log}^k x. \tag{46.2}$$

Furthermore, define $\psi_n(x)$, $n \geqq 0$, *by*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{k^n k!} = \varphi_n(x) + (-1)^{n-1}e^{-x}\psi_n(x). \tag{46.3}$$

Then, for $n \geqq 1$,

$$\psi_n(x) - \psi_n'(x) = \frac{\psi_{n-1}(x)}{x}. \tag{46.4}$$

PROOF. First, for $|p| < 1$, by (46.1),

$$\begin{aligned}\frac{x^p\Gamma(1-p)}{p^n} &= \frac{1}{p^n}\sum_{k=0}^{\infty}\frac{p^k\operatorname{Log}^k x}{k!}\sum_{j=0}^{\infty}\frac{A_j p^j}{j!}\\ &= \frac{1}{p^n}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\sum_{k=0}^{r}\binom{r}{k}A_{r-k}\operatorname{Log}^k x\right)p^r.\end{aligned}$$

Equality (46.2) is now immediate.

Using (46.2) in (46.3) and differentiating both sides with respect to x, we find that, for $n \geqq 1$,

$$\begin{aligned}&(-1)^n e^{-x}\psi_n(x) + (-1)^{n-1}e^{-x}\psi_n'(x)\\ &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^{k-1}}{k^{n-1}k!} - \frac{1}{n!}\sum_{k=1}^{n}\binom{n}{k}A_{n-k}\,k\,\frac{\operatorname{Log}^{k-1}x}{x}\\ &= \frac{1}{x}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^k}{k^{n-1}k!} - \frac{1}{(n-1)!}\sum_{k=0}^{n-1}\binom{n-1}{k}A_{n-1-k}\operatorname{Log}^k x\right)\\ &= \frac{1}{x}(-1)^{n-2}e^{-x}\psi_{n-1}(x).\end{aligned}$$

The proof of (46.4) is now complete.

ENTRY 46(ii). *For* $n \geqq 1$,

$$A_n = \sum_{k=1}^{n}\frac{(n-1)!}{(n-k)!}S_k A_{n-k}, \tag{46.5}$$

where A_k *is defined by* (46.1), $S_1 = \gamma$, *and* $S_k = \zeta(k)$, $k \geqq 2$, *where* ζ *denotes the Riemann zeta-function.*

PROOF. Entry 46(ii) is a reformulation of a well-known result that can be found in Luke's book [**47**, p. 27]. Namely, if $\Gamma(x+1) = \sum_{k=0}^{\infty}b_k x^k$, $|x| < 1$, then, for $n \geqq 1$,

$$nb_n = \sum_{k=1}^{n}(-1)^k S_k b_{n-k}. \tag{46.6}$$

Translating the recursion formula (46.6) in terms of the coefficients A_k, we readily obtain (46.5).

We state Entry 46(iii) as recorded by Ramanujan. Afterward, we discuss the accuracy of his numerical calculations.

ENTRY 46(iii). *In the notation* (46.6),

$$\begin{aligned} b_1 &= -.5772156649, \\ b_2 &= .9890560173, \\ b_3 &= -.9074790803, \\ b_4 &= .9817280965. \end{aligned}$$

Furthermore, if we write

$$\Gamma(x+1) = 1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 \frac{x^4}{1+\theta_x x}, \tag{46.7}$$

then

$$\begin{aligned} \theta_0 &= 1.00027, \\ \theta_1 &= 51/52, \\ \theta_2 &= 77/82, \\ \theta_6 &= 5/68, \\ \theta_7 &= -1/38 \end{aligned}$$

"nearly".

The coefficient b_1 is equal to γ, and the numerical value that is given is correct. The given values for b_2, b_3, and b_4 do not seem to be correct. We have employed (46.6) along with values of S_k given in [**1**, p. 811] and have found that

$$\begin{aligned} b_2 &= .9890559953, \\ b_3 &= -.9074790762, \\ b_4 &= .9817280865. \end{aligned}$$

Evidently, we are to interpret θ_x to be that unique number yielding an equality in (46.7). The values given by Ramanujan are rational approximations. The value for θ_0 is enigmatic, because, for $x = 0$, θ_0 is not well defined. In the table below, we give the calculated values of the right side of (46.7) using Ramanujan's determinations and also our determinations of b_2, b_3, and b_4.

x	θ_x	$\Gamma(x+1)$	Ramanujan's value	our value
1	51/52	1	.999990949	.999990967
2	77/82	2	1.999702292	1.999702625
6	5/68	720	719.9611865	719.9612493
7	−1/38	5040	2623.541808	2623.542013

Thus, the values for θ_1, θ_2, and θ_6 give good approximations, but the value for θ_7 certainly does not.

We are very grateful to Henri Cohen for motivating the proof of Entry 46(iv) below. In particular, he informed us of formula (46.20).

ENTRY 46(iv). *If n is a nonnegative integer, then*

$$\psi_n(x) = \frac{x}{\left(x + \frac{n}{2} + \frac{5n+10}{6x} + \frac{41n+58}{10} + \cdots\right)^{n+1}}. \tag{46.8}$$

PROOF. From (46.4), it is clear that $\psi_n(x)$ can be expressed as a power series in $1/x$. Putting

$$\psi_n(x) = \sum_{k=0}^{\infty} \frac{a_k(n)}{x^k}, \qquad n \geqq 0,$$

we then write (46.4) in the form

$$\sum_{k=0}^{\infty} \frac{a_k(n)}{x^k} + \sum_{k=2}^{\infty} \frac{(k-1)a_{k-1}(n)}{x^k} = \sum_{k=1}^{\infty} \frac{a_{k-1}(n-1)}{x^k}, \tag{46.9}$$

where $n \geqq 1$. It follows immediately that $a_0(n) = 0$ if $n \geqq 1$, $a_1(n) = 0$ if $n \geqq 2$, and

$$a_k(n) + (k-1)a_{k-1}(n) = a_{k-1}(n-1), \tag{46.10}$$

for $k \geqq 2$ and $n \geqq 1$. Now assume that, up to some fixed integer $k-1$, $a_{k-1}(n) = 0$ if $n \geqq k$. Thus, $a_{k-1}(n-1) = 0$ if $n \geqq k+1$. It follows from (46.10) and our inductive assumption that $a_k(n) = 0$ if $n \geqq k+1$. Hence, we shall rewrite (46.9) in the form

$$\sum_{k=0}^{\infty} \frac{b_k(n)}{x^{n+k}} + \sum_{k=1}^{\infty} \frac{(n+k-1)b_{k-1}(n)}{x^{n+k}} = \sum_{k=0}^{\infty} \frac{b_k(n-1)}{x^{n+k}}.$$

Hence, for $n \geqq 1$,

$$b_0(n) = b_0(n-1) \tag{46.11}$$

and, for $k, n \geqq 1$,

$$b_k(n) + (n+k-1)b_{k-1}(n) = b_k(n-1). \tag{46.12}$$

From the definition (46.3) of $\psi_n(x)$, it is easy to see that $\psi_0(x) \equiv 1$. Hence, by (46.11) and induction, we find that

$$b_0(n) = 1, \qquad n \geqq 0. \tag{46.13}$$

Next, in (46.12), let $k = 1$ and replace n by j. Since $b_0(j) = 1$, $j \geqq 0$, we find that

$$b_1(j) + j = b_1(j-1), \qquad j \geqq 1. \tag{46.14}$$

Summing both sides of (46.14) for $1 \leqq j \leqq n$ and recalling that $b_1(0) = a_1(0) = 0$, we deduce that $b_1(n) + \sum_{j=1}^{n} j = 0$, or

$$b_1(n) = -\frac{1}{2}n(n+1). \tag{46.15}$$

Put $k = 2$ and $n = j$ in (46.12) to obtain the equality

$$b_2(j) + (j+1)b_1(j) = b_2(j-1). \tag{46.16}$$

Sum both sides of (46.16) on j, $1 \leqq j \leqq n$. Using the fact that $b_2(0) = 0$ as well as (46.15), we find that

$$\begin{aligned} b_2(n) &= \frac{1}{2}\sum_{k=1}^{n}(j^3 + 2j^2 + j) \\ &= \frac{1}{24}n(n+1)(n+2)(3n+5). \end{aligned} \tag{46.17}$$

Lastly, we set $k = 3$ and $n = j$ in (46.12) and find that

$$b_3(j) + (j+2)b_2(j) = b_3(j-1). \tag{46.18}$$

Summing both sides of (46.18) for $1 \leqq j \leqq n$ and employing (46.17), we find that

$$\begin{aligned} b_3(n) &= -\frac{1}{24}\sum_{j=1}^{n} j(j+1)(j+2)^2(3j+5) \\ &= -\frac{1}{48}n(n+1)(n+2)^2(n+3)^2, \end{aligned} \tag{46.19}$$

after a lengthy calculation. (Formulae for summing $\sum_{1 \leqq j \leqq n} j^k$, $1 \leqq k \leqq 5$, may be found in [**32**, pp. 1, 2].)

In conclusion, from (46.13), (46.15), (46.17), and (46.19), we have demonstrated that, for x sufficiently large,

$$\begin{aligned} \psi_n(x) = \frac{1}{x^n}\bigg(1 - \frac{n(n+1)}{2x} &+ \frac{n(n+1)(n+2)(3n+5)}{24x^2} \\ &- \frac{n(n+1)(n+2)^2(n+3)^2}{48x^3} + \cdots\bigg). \end{aligned} \tag{46.20}$$

Now, by (46.8), we wish to prove that

$$\begin{aligned} \{x^{-1}\psi_n(x)\}^{-\frac{1}{n+1}} &= x\{x^n\psi_n(x)\}^{-\frac{1}{n+1}} \\ &= x + \frac{n}{2\,+}\,\frac{5n+10}{6x\,+}\,\frac{41n+58}{10\,+}\cdots \\ &= x\left(1 + \frac{\frac{n}{2x}}{1\,+}\,\frac{\frac{5n+10}{12x}}{1\,+}\,\frac{\frac{41n+58}{60x}}{1\,+}\cdots\right). \end{aligned}$$

In fact, it will be slightly more convenient to show that the reciprocals of the expressions above are equal. Hence, we shall prove that

$$\text{(46.21)}\quad \{x^n\psi_n(x)\}^{\frac{1}{n+1}} = \frac{1}{1}\,+\,\frac{\frac{n}{2x}}{1}\,+\,\frac{\frac{5n+10}{12x}}{1}\,+\,\frac{\frac{41n+58}{60x}}{1}\,+\cdots.$$

In order to establish (46.21), we shall first compute the power series for $\{x^n\psi_n(x)\}^{1/(n+1)}$ in powers of $1/x$. By (46.20) and the binomial theorem, we find that, for x sufficiently large,

$$\begin{aligned}\{x^n\psi_n(x)\}^{\frac{1}{n+1}} = 1 &+ \frac{1}{n+1}\left(-\frac{n(n+1)}{2x} + \frac{n(n+1)(n+2)(3n+5)}{24x^2}\right.\\ &\left.- \frac{n(n+1)(n+2)^2(n+3)^2}{48x^3} + \cdots\right)\\ &- \frac{n}{2(n+1)^2}\left(-\frac{n(n+1)}{2x} + \frac{n(n+1)(n+2)(3n+5)}{24x^2} + \cdots\right)^2\\ &+ \frac{n(2n+1)}{6(n+1)^3}\left(-\frac{n(n+1)}{2x} + \cdots\right)^3 + \cdots.\end{aligned}$$

We now compute the coefficients $c_1(n)$, $c_2(n)$, and $c_3(n)$ of $1/x$, $1/x^2$, and $1/x^3$, respectively. Clearly, $c_1(n) = -n/2$. Secondly,

$$c_2(n) = \frac{n(n+2)(3n+5)}{24} - \frac{n^3}{8} = \frac{n(11n+10)}{24}.$$

Thirdly,

$$\begin{aligned}c_3(n) &= -\frac{n(n+2)^2(n+3)^2}{48} + \frac{n^3(n+2)(3n+5)}{48} - \frac{n^4(2n+1)}{48}\\ &= -\frac{n(9n^2+20n+12)}{16}.\end{aligned}$$

Hence,

$$\text{(46.22)}\quad \{x^n\psi_n(x)\}^{\frac{1}{n+1}} = 1 - \frac{n}{2x} + \frac{n(11n+10)}{24x^2} - \frac{n(9n^2+20n+12)}{16x^3} + \cdots.$$

We now employ Entry 17 to compute the continued fraction representation (46.21). In the notation of Entry 17, by (46.22), $A_1 = n/2$, $A_2 = n(11n+10)/24$, and $A_3 = n(9n^2+20n+12)/16$. First,

$$\text{(46.23)}\qquad a_1 = A_1 = \frac{n}{2}.$$

Secondly,

$$P_2 = a_1(a_1 + a_2) = A_2 = \frac{n(11n+10)}{24}.$$

Using (46.23) and solving for a_2, we readily find that

$$a_2 = \frac{5n + 10}{12}. \tag{46.24}$$

Lastly,

$$P_3 = a_1a_2(a_1 + a_2 + a_3) = A_3 - a_1A_2 = \frac{n(9n^2 + 20n + 12)}{16} - \frac{n^2(11n + 10)}{24}.$$

Solving for a_3 and employing (46.23) and (46.24), we find, after a mild calculation, that

$$a_3 = \frac{41n + 58}{60}. \tag{46.25}$$

Employing (46.23)–(46.25) in Entry 17, we complete the proof of (46.21).

EXAMPLE. Let

$$F(x) = \int_0^x \frac{1 - e^{-t}}{t}\, dt.$$

Then

$$\lim_{x\to\infty}\left(\int_0^x \frac{F(t)}{t}\, dt - \frac{1}{2}F^2(x)\right) = \frac{\pi^2}{12}.$$

PROOF. First, from Entry 44,

$$\frac{1}{2}F^2(x) = \frac{1}{2}\gamma^2 + \frac{1}{2}\operatorname{Log}^2 x + \gamma \operatorname{Log} x + o(1), \tag{46.26}$$

as x tends to ∞.

Next, integrating by parts twice and using Entry 44, we find that, as x tends to ∞,

$$\begin{aligned}\int_0^x \frac{F(t)}{t}\,dt &= F(x)\operatorname{Log} x - \int_0^x \frac{1 - e^{-t}}{t}\operatorname{Log} t\, dt \\ &= (\gamma + \operatorname{Log} x)\operatorname{Log} x + o(1) - \frac{1}{2}(1 - e^{-x})\operatorname{Log}^2 x \\ &\quad + \frac{1}{2}\int_0^x e^{-t}\operatorname{Log}^2 t\, dt.\end{aligned} \tag{46.27}$$

Combining (46.26) and (46.27), we deduce that

$$\begin{aligned}\int_0^x \frac{F(t)}{t}\,dt - \frac{1}{2}F^2(x) &= -\frac{1}{2}\gamma^2 + \frac{1}{2}\int_0^\infty e^{-t}\operatorname{Log}^2 t\, dt + o(1) \\ &= -\frac{1}{2}\gamma^2 + \frac{1}{2}\Gamma''(1) + o(1),\end{aligned} \tag{46.28}$$

as x tends to ∞.

It is well known that [**1**, p. 256], [**64**, vol. 2, p. 89], [**10**, Entry 26]

$$\operatorname{Log} \Gamma(x+1) = -\gamma x + \sum_{k=2}^{\infty} \frac{\zeta(k)(-x)^k}{k}, \qquad |x| < 1.$$

Hence, after two differentiations,

$$\frac{\Gamma''(x+1)}{\Gamma(x+1)} - \psi^2(x+1) = \sum_{k=2}^{\infty} (-1)^k (k-1)\zeta(k) x^{k-2},$$

and so $\Gamma''(1) = \psi^2(1) + \zeta(2) = \gamma^2 + \pi^2/6$. Substituting the value for $\Gamma''(1)$ found above into (46.28) and letting x tend to ∞, we complete the proof.

ENTRY 47. *If* $n > 0$, *then*

$$\int_0^\infty e^{-x}(1+x/n)^n dx = 1 + \frac{n}{1+}\frac{1(n-1)}{3+}\frac{2(n-2)}{5+}\frac{3(n-3)}{7+}\cdots \tag{47.1}$$

$$= 2 + \frac{n-1}{2+}\frac{1(n-2)}{4+}\frac{2(n-3)}{6+}\frac{3(n-4)}{8+}\cdots \tag{47.2}$$

$$= \frac{e^n\Gamma(n+1)}{n^n} - \frac{2n}{2+}\frac{3n}{3+}\frac{4n}{4+}\frac{5n}{5+}\cdots. \tag{47.3}$$

PROOF. In (21.2), let $x = \gamma/n$ and $\beta = -n$. Thus, for $n, \gamma > 0$,

$$\begin{aligned} &{}_2F_1(1-n, 1; \gamma+1; -\gamma/n) \\ &= \frac{\gamma}{\gamma+}\frac{(1-n)\gamma/n}{1+}\frac{1(1+\gamma/n)}{\gamma+}\frac{(2-n)\gamma/n}{1+}\frac{2(1+\gamma/n)}{\gamma+}\cdots. \end{aligned} \tag{47.4}$$

Now, for $n, \gamma > 0$ [**4**, p. 4],

$$\begin{aligned} {}_2F_1(1-n, 1; \gamma+1; -\gamma/n) &= \gamma\int_0^1 (1-t)^{\gamma-1}(1+t\gamma/n)^{n-1} dt \\ &= \int_0^\gamma (1-u/\gamma)^{\gamma-1}(1+u/n)^{n-1} du. \end{aligned} \tag{47.5}$$

Thus, letting γ tend to ∞ in (47.4) and (47.5), we find that

$$\int_0^\infty e^{-u}(1+u/n)^{n-1} du = \frac{1}{1+}\frac{(1-n)/n}{1+}\frac{1/n}{1+}\frac{(2-n)/n}{1+}\frac{2/n}{1+}\cdots.$$

Integrating by parts once, adding 1 to both sides, and writing the right side above in an equivalent form, we see that

$$\begin{aligned} \int_0^\infty e^{-u}(1+u/n)^n du &= 1 + \frac{n}{n+}\frac{1-n}{1+}\frac{1}{n+}\frac{2-n}{1+}\frac{2}{n+}\cdots \\ &= 1 + \frac{n}{1+}\frac{n-1}{3+}\frac{2(n-2)}{5+}\frac{3(n-3)}{7+}\cdots, \end{aligned}$$

by Entry 14. This completes the proof of (47.1).

Secondly, let $x = \gamma/n$ and $\beta = 1 - n$ in (21.2). Then, for $n, \gamma > 0$,

$$\frac{n-1}{n}\,{}_2F_1(2-n,1;\gamma+1;-\gamma/n)$$
$$= \frac{(n-1)\gamma/n}{\gamma} + \frac{(2-n)\gamma/n}{1} + \frac{1(1+\gamma/n)}{\gamma} + \frac{(3-n)\gamma/n}{1} + \frac{2(1+\gamma/n)}{\gamma} + \cdots.$$

Now proceed as above and let γ tend to ∞ to find that

$$\frac{n-1}{n}\int_0^\infty e^{-t}(1 + t/n)^{n-2}\,dt$$

$$\begin{aligned} &= \frac{(n-1)/n}{1} + \frac{(2-n)/n}{1} + \frac{1/n}{1} + \frac{(3-n)/n}{1} + \frac{2/n}{1} + \cdots \\ (47.6)\qquad &= \frac{n-1}{n} + \frac{2-n}{1} + \frac{1}{n} + \frac{3-n}{1} + \frac{2}{n} + \frac{4-n}{1} + \cdots \\ &= \frac{n-1}{2} + \frac{n-2}{4} + \frac{2(n-3)}{6} + \frac{3(n-4)}{8} + \cdots, \end{aligned}$$

by Entry 14.

If $n = 1$, (47.2) is elementary. Assuming that $n \neq 1$ and integrating by parts twice, we find that

$$\frac{n-1}{n}\int_0^\infty e^{-t}(1 + t/n)^{n-2}\,dt = -2 + \int_0^\infty e^{-t}(1 + t/n)^n\,dt.$$

Substituting the formula above into (47.6), we establish (47.2).

Thirdly, setting $x = t - n$, we find that

$$\begin{aligned} \int_0^\infty e^{-x}(1 + x/n)^n\,dx &= \frac{e^n}{n^n}\int_n^\infty e^{-t}t^n\,dt \\ &= \frac{e^n\Gamma(n+1)}{n^n} - \frac{e^n}{n^n}\int_0^n e^{-t}t^n\,dt \\ &= \frac{e^n\Gamma(n+1)}{n^n} - \sum_{k=0}^\infty \frac{n^{k+1}}{(n+1)_{k+1}} \\ &= \frac{e^n\Gamma(n+1)}{n^n} + 1 - {}_1F_1(1; n+1; n), \end{aligned}$$

where in the penultimate line we employed (43.2). Applying Corollary 2 in §21, we complete the proof of (47.3).

In essence, Entry 47 is due to Nielsen **[50]**, **[51]**. Equality (47.1) may be derived from [**51**, p. 46, eq. (6)]. Equality (47.2) can be deduced from [**51**, p. **47**, eq. (11)]. Lastly, equality (47.3) can be proved using [**50**, p. 219, eq. (8)].

ENTRY 48. *As n tends to* ∞,

$$\int_0^\infty e^{-x}(1+x/n)^n\,dx = \frac{e^n\Gamma(n+1)}{2n^n} + \frac{2}{3} - \frac{4}{135n} + \frac{8}{2835n^2} + \frac{16}{8505n^3} - \frac{8992}{3^8\cdot 5^2\cdot 7\cdot 11n^4} + \cdots. \tag{48.1}$$

The asymptotic expansion given above was first proposed by Ramanujan [**62**], [**63**, pp. 323, 324] in an ultimately famous problem in the Journal of the Indian Mathematical Society. In addition to Ramanujan's (formal) solution, later proofs were given by Watson [**75**] and Szegö [**71**]. In fact, the last displayed term on the right side of (48.1) has not been recorded by any of the aforementioned authors. Because no new ideas are involved and because the calculation is extremely laborious, we hope that the reader will trust us when we inform him or her that we have, indeed, verified that Ramanujan's last determined term is correctly given.

COROLLARY. *Define* $\theta = \theta_n$ *by*

$$\sum_{k=0}^{n-1}\frac{n^k}{k!} + \frac{n^n}{n!}\theta = \frac{e^n}{2}. \tag{48.2}$$

Then

$$\theta \approx \theta^* = \theta_n^* \equiv \frac{4+15n}{8+45n}. \tag{48.3}$$

PROOF. It is easy to show that [**62**], [**63**, p. 324]

$$\theta = \frac{e^n\Gamma(n+1)}{2n^n} + 1 - \int_0^\infty e^{-x}(1+x/n)^n\,dx, \tag{48.4}$$

and so (48.1) may be reformulated as

$$\theta = \frac{1}{3} + \frac{4}{135n} - \frac{8}{2835n^2} + \cdots, \tag{48.5}$$

as n tends to ∞. On the other hand,

$$\frac{4+15n}{8+45n} = \frac{1}{3} + \frac{4}{135n} - \frac{32}{6075n^2} + \cdots, \tag{48.6}$$

as n tends to ∞. Thus, θ^* is a fairly good approximation to θ.

A result analogous to (48.5) has been obtained by Copson [**17**] for e^{-n}. More precisely, if φ_n is defined by

$$e^{-n} = \sum_{k=0}^{n-1}\frac{(-n)^k}{k!} + \frac{(-n)^n}{n!}\varphi_n,$$

then

$$\varphi_n = \frac{1}{2} + \frac{1}{8n} + \frac{1}{32n^2} + \cdots,$$

as n tends to ∞.

Generalizations of Ramanujan's and Copson's theorems have been established by Buckholtz [**13**] and Paris [**54**]. The commentary in Szegö's Collected Papers [**72**, pp. 151, 152] provides a good summary of the literature on generalizations and related problems. Another proof of Ramanujan's result (48.5) as well as some related results may be found in Knuth's book [**39**, pp. 112–117].

Ramanujan concludes §48 with the following table.

n	θ_n	θ_n^*
0	.50000	.50000
1/2	.37750	.37705
1	.35914	.35849
3/2	.35146	.35099
2	.34726	.34694
∞	.33333	.33333

Of course, when $n = 0$, it is trivial that $\theta_0 = \theta_0^* = 1/2$. From (48.5) and (48.6), it is clear that $\theta_\infty = \theta_\infty^* = 1/3$. The proposed values for θ_1, θ_1^*, θ_2, θ_2^*, $\theta_{1/2}^*$, and $\theta_{3/2}^*$ are easily corroborated by using the definitions of θ_n and θ_n^* given in (48.2) and (48.3). It remains to examine the values of $\theta_{1/2}$ and $\theta_{3/2}$.

In order to calculate $\theta_{1/2}$ and $\theta_{3/2}$, we shall employ (48.4) and the continued fraction (47.3). Hence,

$$(48.7)\quad \theta_n = 1 - \frac{e^n\Gamma(n+1)}{2n^n} + \frac{2n}{2} + \frac{3n}{3} + \frac{4n}{4} + \frac{5n}{5} + \cdots, \qquad n > 0.$$

In the notation of (1.3) and (1.4), when $n = 1/2$,

$$A_k = (k+1)A_{k-1} + \frac{1}{2}(k+1)A_{k-2}, \qquad k \geqq 1,$$

and

$$B_k = (k+1)B_{k-1} + \frac{1}{2}(k+1)B_{k-2}, \qquad k \geqq 1.$$

By successive calculations, we eventually find that $A_5/B_5 = .4106925$, $A_6/B_6 = .4106857$, $A_7/B_7 = .4106862$. Thus,

$$\frac{2/2}{2} + \frac{3/2}{3} + \frac{4/2}{4} + \frac{5/2}{5} + \cdots = .410686.$$

Since $(1/2)\sqrt{(e\pi)/2} = 1.033182838$, we conclude from (48.7) that Ra manujan's proposed value for $\theta_{1/2}$ is correct.

If $n = 3/2$, again, from (1.3) and (1.4),

$$A_k = (k+1)A_{k-1} + (3/2)(k+1)A_{k-2}, \qquad k \geqq 1,$$

and

$$B_k = (k+1)B_{k-1} + (3/2)(k+1)B_{k-2}, \qquad k \geqq 1.$$

Iterated calculations yield $A_7/B_7 = .972952$, $A_8/B_8 = .972930$, $A_9/B_9 = .972933$. Proceeding as above, we find that $\theta_{3/2} = .35145$, which differs slightly from the value given by Ramanujan.

Ramanujan [**62**], [**63**, p. 324] conjectured, probably partially on the basis of his calculations above, that θ_n always lies between 1/2 and 1/3. This conjecture was proved by both Watson [**75**] and Szegö [**71**].

ENTRY 49. *Define* $\theta = \theta_n$ *by*

$$\gamma + \operatorname{Log} n + \sum_{k=1}^{\infty} \frac{n^k}{k!k} = e^n\left(\sum_{k=0}^{n-2} \frac{k!}{n^{k+1}} + \frac{(n-1)!}{n^n}\,\theta\right),$$

where γ *denotes Euler's constant. Then, as n tends to* ∞,

$$\theta = \frac{2}{3} + \frac{4}{135n} + \frac{8}{2835n^2} + \cdots.$$

We are very grateful to Frank W. J. Olver for providing us the following solution based upon material from his book [**53**].

PROOF. First, observe that, for $n > 0$,

$$\sum_{k=1}^{\infty} \frac{n^k}{k!k} = \int_0^n \frac{e^t - 1}{t}\,dt. \tag{49.1}$$

By combining (49.1) with a familiar formula for γ [**53**, p. 40], we readily find that

$$\gamma + \operatorname{Log} n + \sum_{k=1}^{\infty} \frac{n^k}{k!k} = PV\int_{-\infty}^{n} \frac{e^t}{t}\,dt \equiv Ei(n),$$

where $n > 0$. Olver has calculated an asymptotic series for $Ei(n)$, and in the notation of [**53**, p. 529, eq. (4.06)], $\theta = C_{n-1}(n)$. By [**53**, p. 529, formula (4.07)],

$$\theta = C_{n-1}(n) \sim \sum_{k=0}^{\infty} \frac{\gamma_k(1)}{(n-1)^k}, \tag{49.2}$$

as n tends to ∞, where the first three values for $\phi_k(1)$ are given by (see [**53**, p. 530]) $\gamma_0(1) = 2/3$, $\phi_1(1) = 4/135$, and $\phi_2(1) = -76/2835$. Putting these values in (49.2), we deduce that

$$\begin{aligned}\theta &= \frac{2}{3} + \frac{4}{135(n-1)} - \frac{76}{2835(n-1)^2} + \cdots \\ &= \frac{2}{3} + \frac{4}{135n}\left(1 + \frac{1}{n}\right) - \frac{76}{2835n^2} + 0\left(\frac{1}{n^3}\right),\end{aligned}$$

from which the proposed asymptotic expansion follows.

For much of the theory of $Ei(n)$, see Nielsen's book **[51]**.

W. N. Everitt has kindly provided the bibliographic information about Wilson in our introduction. As the reader has seen, G. E. Andrews, H. Cohen, and F. W. J. Olver have made valuable contributions to this paper. R. A. Askey has made several helpful comments, especially on the continued fractions involving hypergeometric series. We are very grateful to each of them.

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