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CHAIN CONDITIONS IN ENDOMORPHISM RINGS

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1. Introduction. Sometimes, a general theorem for a class of abelian groups can be proved by first showing that the property lives in the endomorphism rings. Then some theorems are proved about the groups using ring-theoretic properties of their endomorphism rings. This strategy was successfully used in [2], [3], and [4].

Whereas in [3] and [4], only torsion-free abelian groups of finite rank are considered, there is no bound on the ranks in [2]. However, this generalization requires the introduction of chain conditions on left ideals of the endomorphism ring $E(A)$ of the group A.

The central condition is one of these. It requires that every essential left ideal of $E(A)$ contains a central, regular element. If A has finite rank, this condition is equivalent to $E(A)$ being semi-prime. In general, if A satisfies the central condition, then $E(A)$ is a Goldie-ring, i.e., it has the ascending chain condition for left annihilators and finite left Goldie-dimension. These rings have been of interest in module-theory for the last few years since semi-prime Goldie-rings are exactly the rings with a semi-simple, Artinian left quotient ring.

In this paper, abelian groups A satisfying the central condition are considered from the point of view that they are a special class of Goldie-rings (Theorem 5.2). The requirement that $E(A)$ is a Goldie-ring is more natural than the central condition if A is not torsion-free. Therefore, this paper concentrates mostly on Goldie-groups, i.e., on abelian groups whose endomorphism ring is a left Goldie-ring.

Because the defining conditions of a Goldie-group behave quite differently with respect to decompositions in direct sums, they are studied in separate sections. In §2, it is shown that direct summands of an abelian group A such that $E(A)$ has the ascending chain condition for left annihilators have this property too (Proposition 2.1). Furthermore, an order-inverting, one-to-one correspondence between the left annihilators in $E(A)$ and certain subgroups of A is given (Theorem 2.5), as well as some applications of it.

In §3, abelian groups A are considered such that $E(A)$ has finite, left

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Goldie-dimension. Sufficient conditions are given such that a direct summand of such a group has this property too. However, an example shows that this does not hold in general.

The combination of the results of these sections allows to prove the following theorem that describes the structure of a Goldie-group up to a torsion-free reduced direct summand.

THEOREM 4.1. *For an abelian group A, the following are equivalent*:

a) *A is a Goldie-group*;

b) *$A \cong B \oplus T \oplus D \oplus E$ where B is a torsion-free reduced Goldie-group, T is finite, D is a divisible, torsion-free group of finite rank, and E is a divisible torsion group of finite rank such that*

i) *$T_p \neq 0$ implies B/pB is finite,*

ii) *$E_p \neq 0$ implies $B = pB$, and*

iii) *$D \oplus E \neq 0$ implies that B has finite rank.*

However, a satisfactory, more or less explicit description of the torsion-free reduced summand seems very hard to achieve. This is mostly due to the fact that Goldie-groups are closed neither under direct summands nor under finite direct sums. Nevertheless, in some cases, it is possible to obtain further results.

THEOREM 5.1. *Let A be a Goldie-group such that $E(A)$ is semi-prime. If A is non-singular as an $E(A)$-module, then the functor* $\mathrm{Hom}_{\mathbf{Z}}(A, -)$ *preserves direct sums of copies of A.*

In this setting, the condition that A is a non-singular $E(A)$-module is the same as requiring that every regular element of $E(A)$ is a monomorphism. If, in addition, A satisfies the central condition, and $E(A)$ is a prime ring, then further information can be obtained [**2**, Theorem 3.6].

In this context, the question arises whether the endomorphism rings of these groups have finite rank over their center. The results of this paper allow one to construct an example showing that this does not hold in general.

Most of the notation used here is standard. In particular, if G is an abelian group, then its torsion subgroup is denoted by $T(G)$ or TG if there is no possibility of confusion.

This research was initiated by parts of my dissertation written at New Mexico State University in 1981/82. I want to use this opportunity to thank my advisor, Prof. Dr. D. M. Arnold, for his patience and helpful ideas without which this paper would not have been completed.

2. Endomorphism rings with the ACC for left annihilators. In the following, the word ring always denotes an associative ring with identity. For a non-empty subset S of a ring R, the set ann $(S) = \{r \in R: rS = 0\}$

is a left ideal of R. It is called the left annihilator of S in R. If ann $(S_1) \subseteq$ ann (S_2), then it is possible to assume that the S_i's are right ideals of R with $S_2 \subseteq S_1$. R satisfies the ascending chain condition (ACC) for left annihilators if every strictly ascending chain of left annihilators has finite length.

In this section, abelian groups are considered whose endomorphism ring satisfies the ACC for left annihilators. In this and the following sections, if $G = A \oplus B$, then e_A and e_B denote idempotents of $E(G)$ with $e_A(G) = A$, $e_B(G) = B$, and $e_A + e_B$ is the identity map on G. $E(A)$ is identified with the subring $e_A E(G) e_A$ of $E(G)$.

PROPOSITION 2.1. *If A is an abelian group whose endomorphism ring satisfies the* ACC *for left annihilators, then every direct sum decomposition of A has only finitely many non-zero summands, and the endomorphism ring of each of these summands has the* ACC *for left annihilators.*

PROOF. Suppose $A = \oplus_{i<\omega} A_i$, where ω is the first infinite ordinal number. For every $n < \omega$, define $D_n = \oplus_{i>n} A_i$, and choose idempotents $e_i\colon A \to A_i$ and $f_n\colon A \to D_n$ in $E(A)$ coming from the given decomposition of A. Then

$$\operatorname{ann}(f_n) = E(A)e_1 \oplus \cdots \oplus E(A)e_n.$$

Because of the ACC for left annihilators, $e_i = 0$ for almost all $i < \omega$. Furthermore, $E(A_i) = e_i E(A) e_i$ implies that $E(A_i)$ satisfies the ACC for left annihilators.

On the other hand, if A and B are abelian groups whose endomorphism rings satisfy the ACC for left annihilators, then, in general, $A \oplus B$ does not satisfy this condition. An example is $A = Z$ and $B = Z(p^\infty)$. Nevertheless, it is possible to obtain

PROPOSITION 2.2. *Let $G = A \oplus B$ be an abelian group such that B is fully invariant in G, and* Hom(G, B) *is a noetherian left $E(B)$-module. If $E(A)$ satisfies the* ACC *for left annihilators, then $E(G)$ does too.*

PROOF. Suppose, $\{\operatorname{ann}(S_i)\}_{i<\omega}$ is an ascending chain of left annihilators in $E(G)$. Since B is fully invariant in G, $e_A$ann (S_i) is the left annihilator of $e_A S_i e_A$ in the ring $e_A E(G) e_A$. By the assumptions, this ring satisfies the ACC for left annihilators. Therefore, $e_A\operatorname{ann}(S_i) = e_A\operatorname{ann}(S_{i+1})$ for almost all i.

Moreover, $\{e_B\operatorname{ann}(S_i)\}$ is an ascending chain of $E(B)$-submodules of Hom(G, B). Since this module is noetherian, $e_B\operatorname{ann}(S_i) = e_B\operatorname{ann}(S_{i+1})$ for all but finitely many i.

This proves the proposition since, as abelian groups,

$$\operatorname{ann}(S_i) = e_A\operatorname{ann}(S_i) \oplus e_B\operatorname{ann}(S_i).$$

In [4], left ideals of $E(A)$ have been considered that annihilate subsets of the group A. A one-to-one, order-inverting correspondence between these left ideals and certain subgroups of A proved to be a useful tool. It is possible to give a similar correspondence between left annihilators of subsets of $E(A)$ and a class of subgroups of A.

If S is a subset of $E(A)$, then let $B_S = \sum_{s \in S} s(A)$ be the image subgroup of S. If U is a subgroup of A, then define $I_U = \{f \in E(A): U \subseteq \ker(f)\}$. I_U is a left ideal of $E(A)$. It is straightforward to prove

LEMMA 2.3. *If A is an abelian group, and S is a subset of $E(A)$, then* $\operatorname{ann}(S) = I_{B_S}$.

On the family of all image subgroups of A, define an equivalence relation by $U \sim V$ if and only if $I_U = I_V$. An equivalence class is denoted by $[U]$.

In this case, if U is an image subgroup of A, then $\bar{U} = \sum\{V: V \sim U\}$ is an image subgroup of A which is equivalent to U. Therefore, every equivalence class contains a largest element. Such an image subgroup is called closed. If S is a subset of $E(A)$, then let $\Phi(\operatorname{ann}(S))$ be $\overline{B_S}$.

LEMMA 2.4. *Φ defines a one-to-one order-inverting correspondence between left annihilators in $E(A)$ and closed image subgroups of A.*

PROOF. By Lemma 2.3, the map Φ is well-defined, and a bijection. Moreover, if $U_1 \subseteq U_2$ are closed subgroups of A, then $I_{U_1} \supseteq I_{U_2}$ are left annihilators in $E(A)$ with $\Phi(I_{U_i}) = U_i$.

Conversely, if $\operatorname{ann}(S_1) \subseteq \operatorname{ann}(S_2)$, then one can assume that $S_1 \supseteq S_2$. Thus $\overline{B_{S_1}} \supseteq \overline{B_{S_2}}$.

THEOREM 2.5. *An abelian group A satisfies the descending chain condition for closed subgroups if and only $E(A)$ has the* ACC *for left annihilators.*

Obviously, if $U \subseteq V$ are image subgroups, then $\bar{U} \neq \bar{V}$ if the natural map $0 \to \operatorname{Hom}(A/V, A) \to \operatorname{Hom}(A/U, A)$ of left ideals of $E(A)$ is not surjective. The last theorem has several consequences which will be important in the following.

COROLLARY 2.6. *Let A be an abelian group such that $E(A)$ satisfies the* ACC *for left annihilators. The torsion subgroup TA of A is a direct summand of A which is isomorphic to a direct sum of a finite group and a divisible torsion group of finite rank.*

PROOF. Let D be the largest divisible subgroup of TA, and write $A = B \oplus D$. By Proposition 2.1, D has finite rank and $E(B)$ satisfies the ACC for left annihilators.

Suppose, TB is unbounded. Then there is a sequence $\{n_i\}_{i<\omega}$ of non-zero integers such that $n_{i+1} = p_i n_i$ where p_i is a prime of $\mathbf{Z}$, and $n_i(B/n_{i+1}B) \neq 0$ for all i.

Consider a basic subgroup of the unbounded, reduced torsion group TB. Then, for all n, $B = \langle a_1 \rangle \oplus \cdots \oplus \langle a_n \rangle \oplus B_n$ and $B_n = \langle a_{n+1} \rangle \oplus B_{n+1}$ where $a_{n+1} \neq 0$. Therefore, $\overline{I_{B_{n+1}}} \supsetneq \overline{I_{B_n}}$. Since B_n is an image subgroup of B, this implies $\overline{B_n} \supsetneq \overline{B_{n+1}}$. By Theorem 2.5, this contradicts the fact that A has the ACC for left annihilators.

Consequently, TB is bounded. Hence, it is a direct sum of finite groups. Since bounded groups are pure-injective, TB is a direct summand of B. By Proposition 2.1, TB is finite.

COROLLARY 2.7. *Suppose, A is an abelian group such that $E(A)$ satisfies the ascending chain condition for left annihilators. If D is the largest divisible subgroup of TA, and its largest p-subgroup D_p is non-zero, then $A/TA = p(A/TA)$.*

PROOF. By Proposition 2.6, TA is a direct summand of A, say $A = A' \oplus TA$. Suppose, $A' \neq pA'$. Then $p^n A' \supsetneq p^{n+1}A'$ for all $n < \omega$. Since TA contains a subgroup isomorphic to $\mathbf{Z}(p^\infty)$, this induces a descending chain of closed subgroups of infinite length.

The last result of this section can be viewed as a partial converse of the last corollary.

PROPOSITION 2.8. *Let $G = B \oplus T$ be an abelian group such that T is finite, TB is a non-zero divisible group of finite rank such that $B = pB$ whenever $(TB)_p \neq 0$, and B has finite torsion-free rank. Then $E(G)$ satisfies the* ACC *for left annihilators.*

PROOF. In the first step, it is assumed that $T = 0$. Write $G = C \oplus D$, where C is torsion-free of finite rank and D is torsion divisible of finite rank. Let $e \in E(G)$ be an idempotent with $e(G) = D$ and $(1 - e)(G) = C$.

If $\{\mathrm{ann}(S_i)\}_{i<\omega}$ is an ascending chain of left annihilators in $E(G)$, then $(1 - e)\ \mathrm{ann}(S_i)$ is the left annihilator of $(1 - e)S_i$ in $E(C)$, since $\mathrm{Hom}(D, C) = 0$. Since C is torsion-free of finite rank, $E(C)$ trivially is a Goldie-ring. Therefore, $(1 - e)\ \mathrm{ann}(S_i) = (1 - e)\ \mathrm{ann}(S_{i+1})$ for almost all $i < \omega$.

On the other hand, $e(\mathrm{ann}(S_i))$ is an $E(D)$-submodule of the $E(D)$-module $eE(G) = \mathrm{Hom}(G, D)$. Since $D = \oplus_{j=1}^{n} \mathbf{Z}(p_j^\infty)^{n_j}$ with primes $p_i \neq p_j$ for $i \neq j$, and $n, n_j < \omega$, the ring $E(D)$ is isomorphic to the product of $n_j \times n_j$-matrix-rings R_j over the p_j-adic integers J_{p_j}. Let $e_1, \ldots, e_n$ be central idempotents in $E(D)$ such that $R_j = e_j E(D) e_j$. The e_j's are just the projections on the p_j-components D_{p_j} of D. Every $E(D)$-moudle M has a natural decomposition $M = \oplus_{j=1}^{n} e_j M$. In particular, $e_j eE(G) = \mathrm{Hom}(G, D_{p_j})$.

Since $G = pG$ for every prime p with $D_p \neq 0$, and C is torsion-free of finite rank, $\mathrm{Hom}(G, D_p)$ is a torsion-free J_p-module of finite rank.

Moreover, $e_j e(\mathrm{ann}(S_i))$ is a pure J_{p_j}-submodule of $e_j eE(G)$ since if, for $x \in e_j eE(G)$ and $r \in J_{p_j}$, one has that rx is an element of $e_j e(\mathrm{ann}(S_i))$, then, for every $s \in S$, $0 = rxs$ and xs is an element of $e_j eE(G)$. Since $e_j eE(G)$ is a torsion-free J_{p_j}-module, $xs = 0$, i.e., $x = e_j ex \in e_j e(\mathrm{ann}\ (S_i))$ Therefore, the chain $\{e_j e(\mathrm{ann}(S_i))\}$ has finite length.

This proves that $E(B)$ has the ascending chain condition for left annihilators since

$$\mathrm{ann}(S_i) = e(\mathrm{ann}(S_i)) \oplus (1 - e)\ \mathrm{ann}(S_i)$$

as abelian groups.

In the second step, $TE(G)$ is finite since both T and $E(B)/mE(B)$ for every non-zero integer m are finite. If n is a non-zero integer with $nTE(G) = 0$, then $nE(G) \subseteq E(B)$.

Suppose $\{\mathrm{ann}(S_i)\}_{i<\omega}$, again, is a strictly ascending chain of left annihilators in $E(G)$. Without loss of generality, one can assume that the S_i's are right ideals of $E(G)$. Since $T(\mathrm{ann}(S_i))$ is contained in the finite group $TE(G)$, one can assume that $T(\mathrm{ann}(S_i)) = T(\mathrm{ann}(S_{i+1}))$ for all i.

If $n(\mathrm{ann}(S_i)) = n(\mathrm{ann}(S_{i+1}))$ for some $i < \omega$, then pick $y \in \mathrm{ann}(S_{i+1})\backslash\mathrm{ann}(S_i)$. For some $x \in \mathrm{ann}\ (S_i)$, one has $nx = ny$, i.e., $y - x \in T(\mathrm{ann}(S_{i+1})) = T(\mathrm{ann}(S_i))$. Hence, $y \in \mathrm{ann}(S_i)$. However, this contradicts the choice of y.

In the ring $E(B)$, consider the left annihilator of nS_i. By the first step, $E(B)$ has the ACC for left annihilators. Therefore, it is possible to assume that the sets nS_i and nS_{i+1} have the same left annihilator in $E(B)$. Denote this annihilator by M.

Because $nE(G) \subseteq E(B)$ and $\mathrm{ann}_{E(B)}(nS_i)$ equals $[\mathrm{ann}_{E(G)}(nS_i)] \cap E(B)$, one has $n[\mathrm{ann}(S_i)] \subseteq M$. Also, $nMS_i = MnS_i = 0$ implies $n^2M \subseteq n[\mathrm{ann}(S_i)]$. This gives

$$n^2M \subseteq n(\mathrm{ann}\ (S_1)) \subsetneq \cdots \subsetneq n(\mathrm{ann}\ (S_i)) \subsetneq \cdots \subsetneq M.$$

Consequently, $M/(n^2M)$ is infinite.

On the other hand, M is a pure subgroup of $E(B)$. Since $E(B)$ has a torsion-free additive group, $M/(n^2M)$ is isomorphic to a subgroup of $E(B)/(n^2E(B))$. But the latter group is finite, and $M/(n^2M)$ is infinite. This contradiction shows that $E(G)$ satisfied the ACC for left annihilators.

3. Finite dimensional endomorphism rings and generalizations. A module over a ring R has finite left Goldie-dimension if every direct sum of non-zero submodules has only finitely many summands. R itself is finite dimensional if it is as a left R-module. If M is a finite dimensional R-module, then there exists a smallest integer n such that every direct sum of non-zero submodules has at most n summands. n is called the Goldie-dimension of M.

LEMMA 3.1. *Let $G = A \oplus B$ be an abelian group such that*
i) *$E(A)$ and $E(B)$ have finite left Goldie-dimension,*
ii) Hom (B, A) *is a finite dimensional left $E(A)$-module, and*
iii) Hom (A, B) *is a finite dimensional left $E(B)$-module.*
Then, $E(G)$ has finite left Goldie-dimension.

PROOF. Let $\oplus_{i\in I}J_i$ be a direct sum of left ideals of $E(G)$. e_AJ_i is an $E(A)$-submodule of the finite dimensional left $E(A)$-module $e_AE(G) \cong E(A) \oplus \mathrm{Hom}\,(B, A)$. Consequently, $e_AJ_i = 0$ for almost all $i \in I$. By symmetry, the same holds for e_BJ_i. As abelian groups, $J_i = e_AJ_i \oplus e_BJ_i$. Therefore, $E(G)$ has finite left Goldie-dimension.

However, the class of abelian groups having a left finite dimensional endomorphism ring is not closed under direct summands. Before an example of such a group is given, sufficient conditions on a direct summand of a group with a finite dimensional endomorphism ring are proved that guarantee that the summand has this property too.

THEOREM 3.2. *Let $G = A \oplus B$ be an abelian group such that $E(G)$ has finite left Goldie dimension. $E(A)$ is a finite dimensional ring if either A is fully invariant in G, or $\bigcap\ \{\ker(f): f \in \mathrm{Hom}(B, A)\} = 0$.*

PROOF. Let $\oplus_{i\in I}J_i$ be a direct sum of non-zero left ideals of $E(A)$.

If A is fully invariant in G, then Hom $(A, B) = 0$, and hence $E(A) = e_AE(G)e_A = E(G)e_A$. Thus, $J_i = J_ie_A$ is a left ideal of $E(G)$, and the sum of the J_ie_A's is direct. Since $E(G)$ is a finite dimensional ring, $J_ie_A = 0$ for all but finitely many i.

On the other hand, if $\bigcap\{\ker(f)\colon f \in \mathrm{Hom}(B, A)\} = 0$, then define $K_i = E(G)J_ie_A$. The K_i's are left ideals of $E(G)$. It it is shown that their sum is direct, then the theorem follows since $E(G)$ has finite Goldie-dimension and $K_i = 0$ implies $J_i = 0$.

Suppose, one has chosen elements $x_i \in K_i$ such that $x_i = 0$, for almost all i, and $\sum_{i\in I} x_i = 0$. Then, $\sum_{i\in I} e_Ax_i = 0$ and $\sum_{i\in I} e_Bx_i = 0$.

Since $e_Ax_i \in J_i$, one has $e_Ax_i = 0$, for all $i \in I$. Furthermore, for every $f \in \mathrm{Hom}\,(B, A)$, one has $\sum_{i\in I} fe_Bx_i$ is zero. Since $fe_Bx_i \in J_i$, for all $i \in I$, and the sum of the J_i is direct, $fe_Bx_i = 0$. By the assumptions on Hom(B, A), this implies $e_Bx_i = 0$ for all $i \in I$. This is only possible if all x_i's are zero, i.e., the sum of the K_i's is direct.

In the following, an example is given that, in general, a direct summand of an abelian group with a left finite dimensional endomorphism ring does not have this property. Because of Corner's Theorem [7, Theorem 110.1], it is enough to find a left finite dimensional ring R with a reduced, countable, torsion-free additive group containing an idempotent e such that eRe is not finite dimensional. Corner's Theorem guarantees the existence of a group G with $R \cong E(G)$. Then $e(G)$ is a direct summand of G with $E(e(G)) \cong eRe$.

R is constructed following Chatters and Hajarnavis [**6**, Examples 1.22 and 8.22], but modifications are necessary to apply Corner's result.

Let x and y be independent variables, and p be a fixed prime of $\mathbf{Z}$. If K is the polynomial ring $\mathbf{Z}[y]$, then $s\colon K \to K$, with $s(y) = y^2$, defines a ring-monomorphism. Let R_1 be the set of all polynomials in x with coefficients in K written on the right of the powers of x. The elements of R_1 are added as usually, but the multiplication is defined by $kx^i = x^i s^i(k)$ for all $k \in K$. This makes R_1 as ring without zero-divisors. Moreover, p is in the center of R_1.

LEMMA 3.3. *R_1 has infinite left Goldie-dimension. Its right dimension is* 1.

PROOF. Consider the left ideals R_1x and R_1xy of R_1. Suppose $u \in R_1x \cap R_1xy$, and write $u = \sum x^i k_i$. The nonzero k_i have only even powers in y, since $yx = xy^2$ and $u \in R_1x$. Siminarly, these powers in y have to be odd because of $yxy = xy^3$. This is only possible if $u = 0$. An induction shows that the sum of the left ideals R_1xyx^i is direct, i.e., R_1 has infinite left Goldie-dimension.

On the other hand, let $0 \neq a$, $b \in R_1$. Write $a = \sum_{i=0}^n x^i a_i$ and $b = \sum_{i=0}^n x^i b_i$. To show that $aR_1 \cap bR_1 \neq 0$, it is enough to find non-zero polynomials $c = \sum_{j=0}^m x^j c_j$ and $d = \sum_{j=0}^m x^j d_j$ with $ac = bd$ and $m \geqslant n$. However, this is equivalent to the existence of a non-zero solution of the linear equation-system $\sum_{i+j=k}(s^j(a_i)c_j - s^j(b_i)d_j) = 0$ $(k = 0, \ldots, m+n)$ in the variables c_j and d_j over the quotient field of K. This system has $n + m + 1$ equations and $2(m + 2)$ variables. Since $n + m + 1 < 2(m + 2)$, such a solution exists.

By this result, R_1 has a right quotient ring Q, i.e., there is a ring $Q \supseteq R_1$ such that every element of R_1 which is not a zero-divisor, i.e., every regular element, is a unit in Q, and every element of Q can be written as ac^{-1} with a, $c \in R_1$ and c regular. Since all non-zero elements of R_1 are regular, Q is a division algebra. Let R_2 be the subring of Q consisting of all ac^{-1} with $a \in R_1$ and $c \in R_1 \backslash pR_1$. It is straightforward to prove the following lemma.

LEMMA 3.4. *Every non-zero one-sided ideal of R_2 has the form R_2p^n for some $n < \omega$. Moreover, $Q = \mathbf{Q} \otimes_{\mathbf{Z}} R_2$.*

If R is defined to be

$$\begin{vmatrix} R_1 & 0 \\ R_2 & R_2 \end{vmatrix},$$

then R clearly satisfies the conditions of Corner's Theorem.

THEOREM 3.5. *R is left Goldie-ring containing an idempotent e such that eRe does not have finite left Goldie-dimension.*

PROOF.

$$\begin{vmatrix} R_1 & 0 \\ R_2 & 0 \end{vmatrix} \text{ and } \begin{vmatrix} 0 & 0 \\ 0 & R_2 \end{vmatrix}$$

are one-dimensional left ideals of R. Therefore, R has left Goldie-dimension 2. By Proposition 2.2, with G being an abelian group such that $R = E(G)$ and

$$A = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} G \text{ and } B = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} G,$$

one has that R satisfies the ACC for left annihilators.

Finally, if

$$e = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix},$$

then $eRe \cong R_1$ is not finite dimensional.

In the following, an abelian group G is called a Goldie-group if $E(G)$ is a left Goldie-ring. By the previous example, direct summands of Goldie-groups are not, in general, Goldie-groups. However, if U is a subgroup of the Goldie-group G, and $G/U \cong \oplus_{i\in I} U_i$, then the left ideal $\mathrm{Hom}(G/U, G)$ of $E(G)$ decomposes in left ideals as $\prod_{i\in I} \mathrm{Hom}\,(U_i, G)$. Since $E(G)$ has finite left Goldie-dimension, $\mathrm{Hom}(U_i, G) = 0$ for almost all $i \in I$. Call an abelian group, G, whose endomorphism ring satisfies the ACC for left annihilators, quasi-Goldie if $G/U = \oplus_{i\in I} U_i$ implies $\mathrm{Hom}(U_i, G) = 0$, for all but finitely many $i \in I$.

In contrast to Goldie-groups, the class of quasi-Goldie-groups is closed under direct summands. If $G = A \oplus B$ is a quasi-Goldie-group, and $U \subseteq A$ with $A/U = \oplus_{i\in I} U_i$, then $G/(U \oplus B) \cong \oplus_{i\in I} U_i$ implies $\mathrm{Hom}(U_i, G) = 0$ for almost all $i \in I$. But this is only possible if A is a quasi-Goldie-group.

Theorem 3.5 and the previous result show that the class of Goldie-groups is properly contained in the class of quasi-Goldie-groups.

LEMMA 3.6. *Let A be a quasi-Goldie-group whose largest divisible subgroup $D(A)$ is non-zero. Then A has finite torsion-free rank.*

PROOF. Suppose A has infinite torsion-free rank. By Proposition 2.1, $D(A)$ is countable. Choose a free subgroup F of A of infinite rank. It contains a subgroup F_1 such that $F/F_1 = \oplus_\omega D(A)$. Since the latter group is divisible, there is some subgroup V of A with $A/V \cong \oplus_\omega D(A)$. Because $\mathrm{Hom}(D(A), A) \neq 0$, this contradicts the fact that A is quasi-Goldie.

If a quasi-Goldie-group A has a non-zero torsion subgroup, then write $TA = T \oplus D$ where D is the largest divisible subgroup of TA. By Corollary 2.6, T is finite.

LEMMA 3.7. *Let A be a quasi-Goldie-group. If $T_p \neq 0$, then $(A/TA)/p(A/TA)$ is finite.*

PROOF. $(A/TA)/p(A/TA)$ is a direct sum of cyclic groups of order p. Because of $T_p \neq 0$, one has $\mathrm{Hom}(Z/pZ, A) \neq 0$. Thus, $(A/TA)/p(A/TA)$ has to be finite, since A is quasi-Goldie.

4. The structure of Goldie-and quasi-Goldie-groups. Combining the results of the previous two sections will allow us to describe the structure of quasi-Goldie- and Goldie-groups up to a torsion-free reduced summand. A further discussion of this summand will be given in the next section.

THEOREM 4.1. *For an abelian group A, the following are equivalent:*

a) *A is a (quasi-) Goldie-group;*

b) $A = B \oplus T \oplus D \oplus E$ where B is a torsion-free reduced (quasi-) Goldie-group, T is finite, D is a divisible torsion-free group of finite rank, and E is a divisible torsion group of finite rank such that

i) *$T_p \neq 0$ implies B/pB is finite,*

ii) *$E_p \neq 0$ implies $B = pB$, and*

iii) *$D \oplus E \neq 0$ implies that B has finite rank.*

PROOF. a) $\Rightarrow$ b). By the results proved till now, it is only left to show that B is a Goldie-group if A is. This is trivial if $D \oplus E \neq 0$. If A is reduced, then choose a non-zero integer n with $nT = 0$.

If $I_1 \oplus \cdots \oplus I_m$ is a direct sum of non-zero left ideals of $E(B)$, then $I_i n e_B$ is a left ideal of $E(A)$ for all i. The sum of these ideals is direct, and hence, m has to be less than the Goldie-dimension of $E(A)$.

b) $\Rightarrow$ a). First, $E(A)$ has the ACC for left annihilators. If $E = 0$, this follows from Proposition 2.2, while the case $E \neq 0$ has been settled by Proposition 2.8.

If B is a quasi-Goldie-group, let U be a subgroup of A with $A/U = \oplus_{i \in I} U_i$.

In the first step, assume $E = 0$. Then,

$$\begin{aligned}\mathrm{Hom}(A/U, A) &= \textstyle\prod_{i \in I} \mathrm{Hom}(U_i, A)\\ &= (\textstyle\prod_{i \in I} \mathrm{Hom}(U_i, B \oplus D)) \oplus (\textstyle\prod_{i \in I} \mathrm{Hom}(U_i, T)).\end{aligned}$$

Therefore, $T(\mathrm{Hom}(A/U, A)) = \prod_{i \in I} \mathrm{Hom}(U_i, T)$ is contained in $T(E(A)) = \mathrm{Hom}(A, T)$. The conditions in b) imply that the latter group is finite. Hence, $\mathrm{Hom}(U_i, T) = 0$ for almost all $i \in I$.

Since $B \oplus D$ is torsion-free, it follows that

$$\mathrm{Hom}(U_i, B \oplus D) = \mathrm{Hom}(U_i/TU_i, B \oplus D).$$

Choose a subgroup V of A with $A/V = \oplus_{i\in I}(U_i/TU_i)$. T is the torsion subgroup V, and there is a subgroup W of V such that $V = T \oplus W$. By [7, Lemma 100.3], A contains a subgroup $H \supseteq W$ such that $A = H \oplus T$. Consequently,

$$A/V \cong H/W \cong \oplus_{i\in I}(U_i/TU_i).$$

If $D = 0$, then H is a quasi-Goldie-group by assumption, while if $D \neq 0$, then H has finite torsion-free rank. In either case, Hom $(U_i/TU_i, B \oplus D) = 0$ for almost all i.

In the second step, assume $E \neq 0$. Then, as in the proof of Proposition 2.8, write $E = \oplus_{j=1}^{m} \mathbf{Z}(p_j^\infty)^{m_j}$. Consequently, $\mathrm{Hom}(U_i, \mathbf{Z}(p_j^\infty)^{m_j})$ is a J_{p_j}-submodule of $\mathrm{Hom}(A, Z(p_j^\infty)^{m_j})$. But the torsion submodule of the latter is finite, and it has finite torsion-free J_{p_j}-rank. Therefore, $\mathrm{Hom}(U_i, E) = 0$ for all $i \in I\backslash J$ for some finite subset J of I.

Choose a subgroup C of A such that $A/C = \oplus_{i\in I\backslash J} U_i$. By the choice of J, E is a subgroup of C, and there is a subgroup X of A such that $A = E \oplus X$.

Moreover, $C = E \oplus (C \cap X)$. But $X = B \oplus T \oplus D$ is a quasi-Goldie-group by the first step, and $X/(C \cap X) \cong A/C$. Therefore, $\mathrm{Hom}(U_i, X) = 0$ for almost all $i \in I\backslash J$. Thus, A is quasi-Goldie.

On the other hand, if B is a Goldie-group, then there is nothing to prove if $E = 0$ by Lemma 3.1. If $E \neq 0$, then a similar rank argument as before is used.

A class of Goldie-rings is particular important. A ring R is semi-prime if, for every non-zero left ideal I of R the ideal I^2 is non-zero. The semi-prime left Goldie-rings are exactly the rings with a semi-simple Artinian left quotient-ring.

Observe that if $G = A \oplus B$ is an abelian group such that A is fully invariant in G and $\mathrm{Hom}(B, A) \neq 0$, then $E(G)$ is not semi-prime. This result and Theorem 4.1 imply

COROLLARY 4.2. *For an abelian group A, the following are equivalent*:

a) *A is a (quasi-) Goldie-group and $E(A)$ is semi-prime.*

b) *There is a finite group T which is the direct sum of cyclic groups of prime order such that either*

i) *there is a torsion-free reduced (quasi-) Goldie-group B with $E(B)$ semi-prime and the property that $T_p \neq 0$ implies $B = pB$, and $A = B \oplus T$, or*

ii) *there is a torsion-free divisible group D of finite rank such that $A = D \oplus T$, or*

iii) *there is a divisible torsion group E of finite rank such that $T_p \neq 0$ implies $E_p = 0$, and $A = E \oplus T$.*

COROLLARY 4.3. *Let A be an abelian group. If its largest divisible subgroup is non-zero, then the following are equivalent:*

a) *A is a Goldie-group;*

b) *A is a quasi-Goldie-group; and*

c) *A has finite torsion-free rank.*

5. Torsion-Free Goldie-Groups. The fact that Goldie-groups are closed neither under direct summands nor finite direct sums puts the structure-theory of torsion-free reduced Goldie-groups in chaos. However, Goldie-groups arise quite naturally in many applications, e.g., in **[2]**. Therefore, it is of interest to single out classes of Goldie-groups that have properties which prove to be useful tools.

One of these conditions is self-smallness. An abelian group A is self-small if the functor Hom$(A, -)$ preserves direct sums of copies of A. In this context, abelian groups A become important such that $Ia \neq 0$ for every $0 \neq a \in A$ and every essential left ideal I of $E(A)$. If A is a Goldie-group, and $E(A)$ is semi-prime, then the last condition is equivalent to saying that every regular element of $E(A)$ is a monomorphism. Abelian groups satisfying this property will be called non-singular over their endomorphism rings.

THEOREM 5.1. *Let A be a Goldie-group such that $E(A)$ is a semi-prime ring. If A is non-singular over $E(A)$, then A is self-small.*

PROOF. In **[5]**, Arnold and Murley showed that A is self-small if every strictly decreasing chain of annihilators of subsets of A has finite length.

Suppose $I_1 \supsetneq \cdots \supsetneq I_n \supsetneq \cdots$ is a decreasing chain of such ideals, say $I_i = \{f \in E(A): X_i \subseteq \ker(f)\}$, where X_i is a subset of A.

Since $E(A)$ is a semi-prime left Goldie-ring, it has a left quotient ring Q which is semi-simple, Artinian. Consequently, the induced chain $QI_1 \supseteq \cdots QI_n \supseteq \cdots$ of left ideals of Q becomes stationary.

If $QI_n = QI_{n+1}$, then pick $f \in I_n$. Because of $f \in QI_{n+1}$, there are c, $r \in E(A)$ and $g \in I_{n+1}$ such that $f = c^{-1}rg$ and c is regular. Since the regular elements are monomorphisms, $cf(X_{n+1}) = rg(X_{n+1}) = 0$ implies $f(X_{n+1}) = 0$, i.e., $f \in I_{n+1}$. Thus, A is self-small.

For a subclass of the class of Goldie-groups, it is possible to give a structure-theory. An abelian group A satisfies the central condition if every essential left ideal contains a central regular element. In a first step, it is shown that abelian groups satisfying the central condition are Goldie-groups.

THEOREM 5.2. *If A is an abelian group which satisfies the central condition, then $E(A)$ is a semi-prime right and left Goldie-ring.*

PROOF. If $J = \oplus_{i \in I} J_i$ is a direct sum of non-zero left ideals of $E(A)$, then one can assume that J is essential. Therefore, J contains a central regular element c. $E(A)c$ is an essential left ideal of $E(A)$. Consequently, I has to be finite.

Moreover, if $0 \neq r \in E(A)$, then ann(r) cannot be essential in $E(A)$. By [**6**, Lemma 1.14], $E(A)$ has the ACC for left annihilators.

Finally, let N be a left ideal of $E(A)$ with $N^2 = 0$. Choose a left ideal K of $E(A)$ such that $N \oplus K$ is essential. Then $N \oplus K$ contains a central regular element c.

Consider $cN \subseteq N$. Since c is central, cN is a left ideal of $E(A)$. Moreover, $cN \subseteq N^2 \oplus NK \subseteq K$. Thus, $cN = 0$. Now, c is regular implies that $N = 0$.

Therefore, $E(A)$ is a semi-prime left Goldie-ring whose semi-simple, Artinian left quotient-ring Q is generated by central elements because of the central condition. If I is an essential right ideal of $E(A)$, then IQ is an essential right ideal of Q. This implies $Q = IQ$. Therefore, I contains a central regular element. The rest of the theorem follows by symmetry.

Combining the results of the last two theorems, it is possible to prove the following. In this context, a ring R is called prime if every non-zero two-sided ideal is essential.

THEOREM 5.3. [**2**, Theorem 3.6]. *For a torsion-free group A, the following are equivalent*:

a) *A satisfies the central condition, and is non-singular over $E(A)$ which is a prime ring*; *and*

b) *The center R of $E(A)$ is an integral domain with quotient field F. As an R-module, A is torsion-free and quasi-isomorphic to B^m where B is a torsion-free R-module whose R-quasi-endomorphism ring $F \otimes_R E_R(B)$ is a division algebra.*

Herein, two torsion-free R-modules M and N are quasi-isomorphic if there is $0 \neq r \in R$ and a submodule U of M such that $N \cong U$ and $rM \subseteq U$.

It should be remarked that if A satisfies the central condition then the center of the quotient-ring of $E(A)$ is a product of a finite number of fields. Since these fields are, in general, non-isomorphic, the definition of quasi-isomorphism would become rather clumsy in a more general setting. This is the reason why only prime rings have been considered in Theorem 5.3.

Rather than giving the proof which can be found in [**2**], it seems more important to give an example that shows the size of the class of groups described by Theorem 5.3. Naturally, the question arises as to whether it contains only the groups whose endomorphism ring has finite rank over its center. That this is not so is shown by

EXAMPLE 5.4. If R_2 is the ring which was defined before Lemma 3.3, then there exists an abelian group A with R as the endomorphism ring of A. The ring ring $\mathbf{Q} \otimes_{\mathbf{Z}} E(A)$ is a division algebra which has infinite dimension over its center.

PROOF. It is enough to show that the center of $\mathbf{Q} \otimes_{\mathbf{Z}} R_2$ is isomorphic to $\mathbf{Q}$. Let q be a central element of this division algebra. There are polynomials $f(x)$, $g(x) \in R_1$ with $q = f(x)g(x)^{-1}$. Thus, for every $t \in \mathbf{Z}$, one has $f(x)x^t g(x) = g(x)x^t f(x)$. Write $f(x) = \sum_{i=0}^{n} x^i a_i$ and $g(x) = \sum_{j=0}^{m} x^j b_j$, where the a_i's and b_i's are polynomials in y with a_n, $b_m \neq 0$. Then one obtains

$$(1) \qquad \sum_{i+j=k} s^{(t+i)}(b_j)a_i = \sum_{i+j=k} s^{(t+j)}(a_i)b_j,$$

with $k = 0, \ldots, n + \mathrm{m}$.

Choose $k = n + m$, and obtain

$$(2) \qquad s^{(t+n)}(b_m)a_n = s^{(t+m)}(a_n)b_m.$$

Comparing the degrees of this polynomials in $\mathbf{Z}[y]$ gives $2^{t+n}\deg(b_m) + \deg(a_n) = 2^{t+m}\deg(a_n) + \deg(b_m)$.

Therefore, $\deg(a_n) = \deg(b_m)$. If $\deg(a_n) \neq 0$, then $n = m$. Over the algebraic closure of Q, the polynomials a_n and b_n can be written as $a_n = a\prod_{i=1}^{n'}(y - c_i)^{n_i}$ and $b_n = b\prod_{j=1}^{m'}(y - d_j)^{m_j}$ where $c_i \neq c_{i'}$ $(d_j \neq d_{j'})$ for $i \neq i'$ $(j \neq j')$. Substitute this in (2) for a_n and b_n. Then

$$\prod_{j=1}^{m'}(y^{2(n+t)} - d_j)^{m_j} \prod_{i=1}^{n'}(y - c_i)^{n_i} = \prod_{i=1}^{n'}(y^{2(n+t)} - c_i)^{n_i} \prod_{j=1}^{m'}(y - d_j)^{m_j}.$$

Suppose there is some index i such that $c_i \notin \{0, 1\}$. Then all the powers $c_i^{2(n+t)}$ are different. Choose $t < \omega$ large enough so that $c_i^{2(n+t)} \neq c_j$ for $j \in \{1, \ldots, n'\}$. Consequently, for some $j \in \{1, \ldots, m'\}$ one has $c_i = d_j$ and $n_i \leqq m_j$. By symmetry, equality holds.

Thus, after cancelling the factors with $c_i \notin \{0, 1\}$, one has to consider an equation of the form

$$(y^{2(n+t)m_1}(y^{2(n+t)} - 1)^{m_2} y^{n_1}(y - 1)^{n_2} = (y^{2(n+t)n_1}(y^{2(n+t)} - 1)^{n_2} y^{m_1}(y - 1)^{m_2}.$$

Therefore, $n_1 = m_1$ and $n_2 = m_2$. Consequently, $a_n = rb_n$ for some rational number r.

On the other hand, if $\deg(a_n) = \deg(b_n) = 0$, then a_n, b_n are non-zero integers, and an argument similar to before, applied to $f(x)y^t g(x) = g(x)y^t f(x)$, shows $n = m$ and $a_n = rb_n$.

Let i_0 be the largest integer in $\{0, \ldots, n\}$ such that $a_{i_0} \neq rb_{i_0}$. Choose in (1) $k = n + i_0$. The choice of i_0 implies

$$s^{(t+i_0)}(b_n)(a_{i_0} - rb_{i_0}) = s^{(t+n)}(a_{i_0} - rb_{i_0})b_n.$$

The same calculations as before show that either $n = i_0$ or $a_{i_0} = rb_{i_0}$. Both results give a contradiction.

Consequently, $f(x)g(x)^{-1} \in \mathbf{Q}$. Thus, the center of $\mathbf{Q} \otimes_{\mathbf{Z}} R_2$ is isomorphic to $\mathbf{Q}$.

By Lemma 3.4, R_2/I is torsion for every essential left ideal I of R_2. Therefore, every essential left ideal of R_2 contains a non-zero integer multiple of the ring identity, and thus, R_2 satisfies the central condition. Therefore, this paper concludes with the consideration of torsion-free abelian groups A such that $E(A)/I$ is torsion for every essential left ideal I of $E(A)$. By Theorem 5.2, $E(A)$ is a semi-prime right and left Goldie-ring, and $\mathbf{Q}E(A) = \mathbf{Q} \otimes_{\mathbf{Z}} E(A)$ is its quotient ring.

THEOREM 5.5. *For a torsion-free abelian group A, the following are equivalent:*

a) *$E(A)$ is a semi-prime left Goldie-ring and $E(A)/I$ is torsion for every essential left ideal of $E(A)$;*

b) *A is quasi-isomorphic to $\oplus_{i=1}^{n} B_i^{n_i}$ where n, $n_i < \omega$, and the B_i's are torsion-free abelian groups such that, for $i \neq j$, one has* Hom $(B_i, B_j) = 0$, *and $QE(B_i)$ is a division algebra for all $i = 1, \ldots, n$.*

PROOF. Since $\mathbf{Q}E(A)$ is a semi-simple, Artinian ring, there are central, orthogonal idempotents $e_1, \ldots, e_n \in \mathbf{Q}E(A)$ such that $1 = e_1 + \cdots + e_n$ and $e_i\mathbf{Q}E(A)e_i$ is a simple, Artinian ring. If m is a non-zero integer such that $me_i \in E(A)$, then let $f_i = me_i$. Clearly, $f_i(A)$ is a fully invariant subgroup of A for all i. Moreover, the sum of the $f_i(A)$ is direct, and it contains mA.

Consequently, $\mathbf{Q}E(f_i(A))$ is a simple ring, and $\mathbf{Q}E(A)$ is the product of the $\mathbf{Q}E(f_i(A))$. By Theorem 5.3, $f_i(A)$ is quasi-isomorphic to $B_i^{n_i}$ where B_i is a torsion-free abelian group such that $\mathbf{Q}E(B_i)$ is a division algebra. Since $f_i(A)$ is fully invariant in A, one has Hom $(B_i, B_j) = 0$ for $i \neq j$. This proves b).

For the converse, observe that b) guarantees that $\mathbf{Q}E(A)$ is isomorphic to $X_{i=1}^{n} \mathrm{Mat}_{n_i}(\mathbf{Q}E(B_i))$. Since $\mathbf{Q}E(B_i)$ is a division algebra, $\mathbf{Q}E(A)$ is a semi-simple, Artinian ring. Hence, $\mathbf{Q}I = \mathbf{Q}E(A)$ for every essential left ideal I of $E(A)$. This shows a).

REFERENCES

1. U. Albrecht, *Ideal conditions in endomorphism rings*, Ph.D. Dissertation, New Mexico State University, Las Cruces, 1982.

2. ———, *Endomorphism rings and A-projective torsion-free abelian groups, Abelian Group Theory*, Proceedings Honolulu, Springer Verlag, Berlin, Heidelberg, New York, 1983.

3. D.M. Arnold, *Finite Rank Torsion-Free Abelian Groups and Rings*, Springer Lecture Notes **931**, Springer Verlag, Berlin, Heidelberg, New York, 1982.

4. ——— and E.L. Lady, *Endomorphism rings and direct sums of torsion-free abelian groups*, Trans. Act. Amer. Math. Soc. **211** (1975), 225–237.

5. ——— and C.E. Murley, *Ablian groups A such that* $\mathrm{Hom}(A,-)$ *preserves direct sums of copies of A*, Pac. J. of Math. **56** (1975), 7–20.

6. A.W. Chatters and C.R. Hajarnavis, *Rings with Chain Conditions*, Research Notes in Mathematics **44**, Pitman Advanced Publishing Program, Boston, London, Melbourne, 1980.

7. L. Fuchs, *Infinite Abelian Groups Vol.* I/II, Academic Press, New York, London, 1970/73.

8. A.W. Goldie, *Rings with Maximum Conditions*, Lecture Notes, Yale University, 1964.

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