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ON THE PIERCE-BIRKHOFF CONJECTURE

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Dedicated to the memory of Gus Efroymson

1. Introduction. In 1956, Birkhoff and Pierce [1] asked the question of characterizing the "ℓ-rings" and "f-rings" free on n generators, and conjectured that they should be rings of continuous functions on R^n, piecewise polynomials. The precise question known as the "Pierce-Birkhoff conjecture" is: given $h: \mathbf{R}^n \to \mathbf{R}$ continuous, piecewise polynomial, is h definable with polynomials by means of the operations sup and inf?

In a paper of Henriksen and Isbell [5] we can find explicit formulas showing that the set of such functions is closed under addition and multiplication, and so is a ring. We will call that ring ISD (Inf and Sup-definable).

Here we give a proof in the case $n = 2$ and make a study for the general case. G. Efroymson proved also this result independently and in a somewhat different way.

2. General Presentation. Given $P_1, \ldots, P_r \in \mathbf{R}[X_1, \ldots, X_n]$, let A_i be the semialgebraic subset of $\mathbf{R}^n$ defined by $h = P_i$. The point is to show that for any pair (i, j), there exists $e_{ij} \in$ ISD such that $e_{ij/A_j} \geqq P_{j/Aj}$ and $e_{ij/A_i} \leqq P_{i/A_i}$: if we get such functions, we have $h = \sup_j(\mathrm{Inf}_i(e_{ij}, P_j))$ and we are done.

So, let us complete the set $\{P_i - P_j\}_{i,j}$ in a separating family $\{Q_1, \ldots, Q_s\}$ [2] [4], which we can suppose made with irreducible polynomials.

All the functions considered being continuous, it is enough to work with the open sets of the partition which are the $\{x \in \mathbf{R}^n / \bigwedge_{i=1}^{s} Q_i \varepsilon_i\ 0\}$ with ε_i strict inequalities [such a set of disjoint open sets whose union is dense in R^n will be called "open partition" of $\mathbf{R}^n$]. Let us call again $(A_i)_{i=1}^{p}$ these open sets:

We get three possibilities for the pair (A_i, A_j):
1) $\bar{A}_i \cap \bar{A}_j = \phi$
2) $\mathrm{codim}(\bar{A}_i \cap \bar{A}_j) = 1$

3) $\operatorname{codim}(\bar{A}_i \cap \bar{A}_j) \geqq 2$
and we give a special treatment for each case.

3. First case $\bar{A}_i \cap \bar{A}_j = \phi$. By the definition of a separating family we get a polynomial Q such that $Q(\bar{A}_i) < 0$ and $Q(\bar{A}_j) > 0$. The Lojasiewicz inequality (or positive stellensatz) gives us then a polynomial R such that $R(\bar{A}_j) \geqq 1$ and $R(\bar{A}_i) < 0$. In the case $P_i - P_j$ has the same sign (say positive) on A_i and A_j, $e_{ij} = (P_i - P_j)R + P_j$ is the function we need. (If $P_i - P_j$ changes sign, no problem).

4. Second case codim $\bar{A}_i \cap \bar{A}_j = 1$. One of the Q_i's is sign changing between A_i and A_j and so is zero on $\bar{A}_i \cap \bar{A}_j$: as it is irreducible, $Q_k = 0$ is the equation of $\bar{A}_i \cap \bar{A}_j$. But $P_i - P_j$ is also zero on $\bar{A}_i \cap \bar{A}_j$, so if $x_0 \in \bar{A}_i \cap \bar{A}_j$ and if U is a semialgebraic neighborhood of x_0, we get $x_0 \in Z_t(Q_k) \cap U \subset Z(P_i - P_j)$ (here $Z_t(Q_k)$ is the set of transversal zeros of Q_k and $Z(P_i - P_j)$ the set of zeros of $P_i - P_j$). According to the "transversal zeros theorem" [3], we have $(P_i - P_j)(x) = \lambda(x)Q_k(x)$. Suppose $Q_k(A_j) > 0$, $e_{ij} = |\lambda|\, Q_k + P_i$ has the needed property.

Before taking up the third case we prove the next proposition.

PROPOSITION 5. *Given a function h: $\mathbf{R}^n \to R$, continuous and piecewise polynomial, and given a direction D in $\mathbf{R}^n$, there exists an open partition of $\mathbf{R}^n$ in cylinders of direction D such that on each cylinder, h coincides with an* ISD *function.*

SKETCH OF PROOF. Let Z be the coordinate in the direction D and $x = (x_1, \ldots, x_{n-1})$ the others (after linear change of coordinates).

Let $P(\mathbf{x}, z) \in \mathbf{R}[X_1, \ldots, Z]$. There exists an open semi-algebraic partition of R^{n-1}, $(B_i)_{i=1}^{s}$, such that the zeros $\xi_j(\mathbf{x})$ of P lying over B_i are continuous semialgebraic functions $B_i \to \mathbf{R}$, and such that the sign of $P(x, z)$ in $B_i \times \mathbf{R}$ depends only on the sign of the $Z - \xi_j(\mathbf{x})$ ("Saucissonnage" of Cohen [4]). We have then by induction on $d_z^0 P$ that the function defined on $B_i \times \mathbf{R}$ as zero everywhere except between two given consecutive zeros of P, where it takes the value $P(x, z)$, [i.e., an alternation of P] is ISD.

Then an appropriate open partition of $\mathbf{R}^n$ in cylinders can be found for which the alternations of the $(P_i - P_j)_{ij}$ are ISD. Using the transversal zeros theorem, we get the proposition.

6. Suppose $n = 2$ and codim $\bar{A}_i \cap \bar{A}_j = 2$ and h: $\mathbf{R}^2 \to \mathbf{R}$ such that $h/A_i = P_i$, $h/A_j = P_j$. $\bar{A}_i \cap \bar{A}_j$ is a finite set of points, and eventually refining our partition we can suppose it is a single point c. Let us take two different directions ox and oy, $c = (x_0, y_0)$. We want to separate out a piece A_i' of A_i from a piece A_j' of A_j: if they are in a same "cylinder", we can apply proposition 5; If not, $(x - x_0)$ and $(y - y_0)$ are sign-changing between A_i' and A_j' and $P_i - P_j = A(x, y)(x - x_0) + B(x, y)(y - y_0)$, and a function such as $\varepsilon_1 |A(x,y)|(x - x_0) + \varepsilon_2 |B(x, y)|(y - y_0)[\varepsilon_i = \pm 1]$ gives the result.

7. Remarks. 1) There are domains of the plane for which the continuous piecewise polynomial functions are not ISD. Take the set

$$E = \{(x, y) \in \mathbf{R}^2 / x \leqq 0 \text{ or } y \leqq 0 \text{ or } y \geqq x^2\}$$

and define h on E such that $h(x, y) = x$ if $x \geqq 0$ and $y \geqq x^2$, and $h(x, y) = 0$ elsewhere. Now h cannot be ISD on E, or else it could be extended to an ISD function on R^2 and then to a piecewise polynomial function on R^2. But that is not possible.

2) The method of §6 suggests the idea that a variety V of codimension more than 2 in $\mathbf{R}^n$ could have its ideal generated by "cylindric" polynomials (in fact such a variety V is always the intersection of all the cylinders containing V). But that is not true. At the conference Efroymson suggested to me to study the twisted quintic $x = t^3$, $y = t^4$, $z = t^5$. Once computed (by Houdebine) it turned out to be a counterexample.

Reference

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