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SYSTEMS OF REAL VARIETIES

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Dedicated to the memory of Gus Efroymson

The orderings of a function field $F/\mathbf{R}$ can be described as ultra-filters of Zariski-dense semialgebraic sets of a given complete model of F ([3], p. 234). There are other interesting structures of F, e.g., orderings (resp. signatures) of higher level, which cannot be determined on a certain variety. We are forced then to consider systems of models of $F/\mathbf{R}$. In doing this we are mainly interested in the discrete valuations of F which have a real residue field, for we shall use them to present a geometric interpretation of signatures of higher level and a stronger form of the Local-Global-Principle for weakly isotropic quadratic forms.

In the following X denotes a regular, complete, r-dimensional variety and F its function field which is assumed to be formally real. This implies that the set $X(\mathbf{R})$ of real, closed points is not empty.

We shall try to describe signatures of higher level considering the vanishing orders of functions with respect to real prime divisors and their signs on certain ultrafilters.

Let $\beta_1 = \beta_1(X)$ be the set of pairs (Y, ψ), where Y is a regular complete $\mathbf{R}-$variety and ψ: $Y \to X$ a birational $\mathbf{R}$-morphism. This set can be ordered in the following way.

Given $(Y_1, \phi_1), (Y_2, \phi_2) \in \beta_1$ define $(Y_1, \phi_1) \geqq (Y_2, \phi_2)$ if there exists a birational $\mathbf{R}$-morphism ψ: $Y_1 \to Y_2$ representing $\phi_2^{-1} \circ \phi_1$. Let β denote the set of pairs $(Y, \phi) \in \beta_1$ such that ϕ is a composition of monoidal transformations with regular centers.

PROPOSITION 1. i) *β_1 is a directed set.*

ii) *β is cofinal.*

iii) *The spaces $Y(\mathbf{R}), (Y, \phi) \in \beta$, form an inverse system and $\varprojlim Y(\mathbf{R}) \cong M_F$, where M_F denotes the space of $\mathbf{R}$-valued places of F.*

DEFINITION. A character χ: $K^* \to \mu$ of a field K is called a signature of K, if the kernel of χ is additively closed.

A signature σ of F induces an $\mathbf{R}$-valued place, thus we can define the center of σ on a complete model as the center of this place. The theory of signatures of a field is studied in great detail in [2].

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Now let Y be a regular complete $\mathbf{R}$-variety, $P \in Y(\mathbf{R})$, and $\mathbf{R}\text{-Div}_P$ the local real divisor group in P (i.e., the free abelian group generated by the minimal non-zero prime ideals of O_P with formally real residue field). Let U be an ultrafilter of Y in the sense of [3] with center P and let σ be a signature of F with center P. The following proposition is readily checked.

PROPOSITION 2. *The signature σ induces a character $\tilde{\sigma}$ on $\mathbf{R}\text{-Div}_P$, defined by $\tilde{\sigma}((\pi)) = \sigma(\pi)\,\mathrm{sign}_U(\pi)$.*

It is now of interest to investigate which characters of $\mathbf{R}\text{-Div}_P$ come from the signatures in this way.

PROPOSITION 3. *Let $(Y_1, \phi_1) \geqq (Y_2, \phi_2) \in \beta$ with morphism $\phi\colon Y_1 \to Y_2$ and let P_1, P_2 be points in $Y_1(\mathbf{R})$ resp. $Y_2(\mathbf{R})$ with $\phi(P_1) = P_2$. If D_1 is a prime divisor (considered as a subvariety) of Y_1 through P_1 and if D_2 is a real prime divisor of Y_2 through P_2 with $\phi(D_1) \subset D_2$, then D_1 is also real.*

This statement fails if β is replaced by β_1. We now choose an ultrafilter U on X. Then U defines a unique ultrafilter $U_Y = \phi^{\#}\,U$ on every $(Y, \phi) \in \beta$. Its center is denoted by P_Y. The system $\{(\mathrm{P}_Y) | (Y, \phi) \in \beta\}$ represents a place $\wp$ (according to prop. 1 iii). Let $W_\wp$ be its value group. Proposition 3 implies that the character groups $(\mathbf{R}\text{-Div}_{P_Y})^*$ form an inverse system.

THEOREM 4. *The map $\sigma \mapsto \tilde{\sigma}$ induces a bijection between the set of signatures compatible with $\wp$ and* $\lim(R\text{-Div}_{P_Y})^*$ *and the latter is isomorphic to $W_\wp^*$.*

For every function $f \in F$ one can find a complete model Y such that every prime divisor through P_Y dividing f is real. If $\mathrm{div}(f) = D_1^{r_1} \cdot \cdots \cdot D_n^{r_n}$ and σ is a signature corresponding to the character χ then $\sigma(f) = \mathrm{sign}_{U_Y'}(f) \cdot \Pi\chi(D_i)^{r_i}$.

COROLLARY. *A function f has an n-divisible value with respect to each real prime divisor of $F/\mathbf{R}$ (see [7], p. 88) iff $\sigma(f) \in \{\pm 1\}$ for every signature $\sigma\colon F^* \to \mu(2n)$. In this case f is contained in $H_F^* \cdot \sum F^n$, where H_F denotes the "real holomorphy ring" of F [6].*

Let X be a surface and $P \in X(\mathbf{R})$. The multiplicity of P with respect to a curve D will be denoted by $m(P, D)$.

THEOREM 5. *If χ is a character of $\mathbf{R}\text{-Div}_P$ induced by a signature then we can find a divisor $C_0 \in \mathbf{R}\text{-Div}_P$ with $m(P, C_0) = 1$ and an element $b \in \mu$ satisfying*

$(*)$ $\chi(C) = b^{m(P,C)}$ for every prime divisor $C \in \mathbf{R}\text{-Div}_P$ which has no common tangent with C_0.

Conversely given any $C_0 \in \mathbf{R}\text{-Div}_P$ and $b \in \mu$ there exists a character χ induced by a signature satisfying $(*)$.

The Local-Global-Principle of Bröcker and Prestel says that a quadratic form q over a field K is weakly isotropic, (i.e., a multiple of q is isotropic) if q is weakly isotropic over the henselization of every real valuation of K and if it is indefinite with respect to every ordering of K. The main tool of the proof is the concept of "semiorderings" of K. A form $\langle a_1, \ldots, a_n \rangle$ is weakly isotropic if it is indefinite with respect to every semiordering. Now we assume that $\{a_1, \ldots, a_n\}$ is contained in a semiordering S of F. According to the Local-Global-Principle we may assume that S is compatible with a real valuation v (in the sense of **[4]**).

Using **[5]**, main theorem II, we obtain a regular complete model Y such that the support of the divisors of the a_i has only normal crossings. One can construct then a real prime divisor w of $F/\mathbf{R}$ and a semiordering which is compatible with w and contains $\{a_1, \ldots, a_n\}$. So we obtain the following theorem.

THEOREM 6. *The set of semiorderings of F which are compatible with a real divisor of $F/\mathbf{R}$ is a dense subset of the space of semiorderings.*

COROLLARY. *A quadratic form q over F is weakly isotropic iff it is indefinite with respect to every ordering of F and weakly isotropic in the henselization of every real prime divisor of $F/\mathbf{R}$.*

One can ask, by which other classes of valuations the class of the theorem may be replaced. In view of this we say that a class of valuations satisfies the Local-Global Principle, if a form which is weakly isotropic in the henselization of every valuation $v \in D$, is already weakly isotropic in F.

THEOREM 7. *If $F/\mathbf{R}$ is a function field of transcendence degree $n > 1$, the class of real prime divisors is the minimal class of valuations satisfying the Local-Global Principle. If the transcendence degree of $F/\mathbf{R}$ equals 1, then a minimal class doesn't exist.*

For example, the class of rank one, discrete, zero-dimensional real valuations doesn't suffice. For, as every semiordering compatible with such a valuation is a usual ordering, Theorem 6 necessarily fails. On the other hand, it follows from **[1]**, th.3.1, that the orderings compatible with such a valuation are dense in the space of orderings of F.

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