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CONSTRUCTING REAL PRIME DIVISORS USING NASH ARCS

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Dedicated to the memory of Gus Efroymson

Let $A = R[x_1, \ldots, x_n]$ be the affine coordinate ring of a variety V defined over the real closed field R. We denote the closed real points of V by $X \subset R^n$ and the simple points of X by $X_0 \subset X$. A geometric preorder P on the function field $K = R(x_1, \ldots, x_n)$ is a preorder corresponding to an (open) semialgebraic subset of X_0—in other words, there is an open semialgebraic set $U \subset X_0$ such that $f \in A \cap P$ precisely if $f \geqq 0$ on U.

Fix a geometric order P on K. If $B \subset K$ is any subring and $I \subset B$ is an ideal, we say that I is convex if $f \in I$ whenever $0 \leqq f \leqq g$ and $g \in I$. Here "$f \leqq g$" means $g - f \in P$. A valuation ring $(B, \mathfrak{m}) \subset K$ is said to be a real prime divisor if there is a domain $C \subset K$ of finite type over R and a minimal convex prime $\mathfrak{q} \subset C$ such that B is the localization $C_{(\mathfrak{q})}$. The theorem motivating this work is the following.

THEOREM *Let $\mathfrak{p} \subset A$ be a convex prime. Then there is a real prime divisor $(B, \mathfrak{m}) \subset K$ with $\mathfrak{m} \cap A = \mathfrak{p}$.*

Set $r = \text{tr.deg.}_R K$. In order to prove this theorem we construct $(r - 1)$ functions $\xi_1, \ldots, \xi_{r-1} \in K$ and a total order $Q \subset K$ containing:

(A) P,

(B) $h^2(\xi_1, \ldots, \xi_{r-1}) - C_h^2$ for every non-zero polynomial $h \in R[T_1, \ldots, T_r]$ (pure polynomial ring) and some constants $C_h \in A \sim \mathfrak{p}$ depending on h, and

(C) $g^2 - f^2 h^2(\xi_1, \ldots, \xi_{r-1})$ for every $h \in R[T_1, \ldots, T_{r-1}]$, $g \in A \sim \mathfrak{p}$, and $f \in \mathfrak{p}$.

Once we know that such an order exists, it is a routine matter to show that the convex hull of the ring $A_{(\mathfrak{p})}[\xi_1, \ldots, \xi_{r-1}] \subset K$ in the order Q is our desired real prime divisor. Thus the hard part is defining $\xi_1, \ldots, \xi_{r-1}$ and proving the existence of Q.

Once the ξ_i are defined, Q exists providing that given any finite collection of inequalities from (A), (B), and (C) we may find a point $p \in U$ at which all the inequalities are fulfilled. Our definition of the ξ_i uses power series associated to nash arcs contained in U ending in $X(\mathfrak{p})$, the real zeroes of $\mathfrak{p}$. An example followed by a few general remarks will best serve to illustrate our methods and results.

Let $A = R[X, Y, Z]$ (so $X_0 = X = R^3$) and let U be defined by the following inequalities: $Y^2 < ZX < Y^2 + Y^3$, so P is generated by $\sum R(X, Y, Z)^2$, $ZX - Y^2$, and $Y^2 + Y^3 - ZX$. Let $\mathfrak{p} = \langle X, Y\rangle$, so $X(\mathfrak{p})$ is the Z-axis. Given any $a, z \in R$ with $z > 2$, the power series

$$X(t) = t^2$$
$$Y(t) = \sqrt{z}\, t - t^2 + at^3 \qquad (1)$$
$$Z(t) = z$$

define a nash arc $\gamma_{(z,a)}(t)$ lying in U for small positive t with $\gamma_{(z,a)}(0) = z \in X(\mathfrak{p})$. Let $\xi_2 = Z$ and $\xi_1 = (1/X)((1/X)((Y^2/X) - z)^2 - 4z)^{\lambda}$. Then $\xi_1(X(t), Y(t), Z(t)) = 16z(2a\sqrt{z} + 1)^2 +$ higher order terms.

Now, if $h(T_1, T_2)$ is given, then either $h \in R[T_2]$, in which case we set $C_h = (1/2)h(Z)$, or $h \notin R[T_2]$, in which case $C_h = 1$. Given finitely many non-zero h_μ; $g_\mu \in A \sim \mathfrak{p}$, and $f_\mu \in \mathfrak{p}$, we find a point $(z_0, 0, 0) \in X(\mathfrak{p})$ such that all $h_\mu(T_1, z_0) \in R[T_1]$ are non-constant and all $g_\mu(z_0, 0, 0)$ are non-zero. The power series $h_\mu(\xi_1(X(t), Y(t), Z(t), \xi_2(X(t), Y(t), Z(t)))$ have first terms $h_\mu(16\, z_0(2a\sqrt{z_0} + 1)^2, z_0)$. These are non-constant polynomials in a, so we may find $a_0 \in R$ such that they are all greater than 2. Since $g_\mu(z) \neq 0$ and $f_\mu(z) = 0$ for all μ, we may find a small positive t such that (A), (B), (C) are satisfied at the point $\gamma_{(z_0, a_0)}(t)$.

We now summarize the steps of our general procedure, most of which were illustrated by our example.

STEP 1 (Not in example). Let $r = \text{tr.deg.}_R K$. Construct a finite algebraic projection $\pi\colon X \to R^r$ with $\pi(X(\mathfrak{p}))$ contained in $E = \{(p_1, \ldots, p_r) \in R^r \mid p_1 = \cdots = p_s = 0\}$, where $s = \operatorname{codim} \mathfrak{p}$. Shrink U so that π^{-1} is nash on $\pi(U)$ but $\overline{\pi(U)}$ contains an open semialgebraic subset of E. This reduces to the "smooth" case as in the example.

STEP 2. Choose a nash wing in $\pi(U)$ ending in E. This wing has coefficients which are nash functions of $p_{s+1}, \ldots, p_s$. The existence of this wing follows from a nash curve selection lemma and from the characterization of the real closure of a function field K with respect to a total order Q as the ring of germs of nash functions on a model of K with respect to the directed set of open subsets of the ultrafilter of semialgebraic sets corresponding to Q.

STEP 3. Observe that the power series associated to this wing may be truncated and that arbitrary m-th terms may be added for some m.

STEP 4. Using the fact that the nash function coefficients satisfy polynomials over R, construct $\xi_1, \ldots, \xi_{r-1}$ as rational functions whose power series start with constant terms which are non-trivial polynomials in the coefficients of the m-th terms.

STEP 5. Apply an argument similar to that in the example to find an appropriate $(0, \ldots, 0, p_{s+1}, \ldots, p_r) \in E$, together with coefficients for the m-th terms, so that finitely many pre-assigned inequalities from (A), (B), and (C) are satisfied at $\pi^{-1}(\gamma(t))$ for the associated power series $\gamma(t)$ and some small positive t.

We remark that the nash theory we use is valid over any real closed field—Cantor or not—and that the nash wing selection lemma we prove is a nice generalization of the classical curve selection lemma. We state this result as follows, although we really prove a more useful parametrized version. We do not investigate questions pertaining to differentiability of the wing at its boundary.

NASH WING SELECTION. *Let $Z \subset R^n$ be a semialgebraic set. let $F \subset R^n$ be a non-empty irreducible algebraic set of dimension d with $Z \cap F$ Zariski-dense in F. Then there are an open semialgebraic subset $H \subset R^d$, a non-empty interval* $(0, \varepsilon) \subset R$, *and a semialgebraic injection ω: $H \times [0, \omega) \to R^n$ such that*

(i) $\omega(H \times \{t\}) \subset Z$ *if* $0 < t < \varepsilon$,

(ii) $\omega(H \times \{0\}) \subset F$ *is an open semialgebraic subset of F, and*

(iii) ω *is a nash isomorphism on* $H \times (0, \varepsilon)$ — i.e., ω *is nash with nash inverse on its image.*

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