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PIECEWISE-RATIONAL RETRACTIONS ONTO CLOSED, CONVEX, SEMI-ALGEBRAIC SETS WITH INTERIOR-SYNOPSIS

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Dedicated to the memory of Gus Efroymson

Let $(K, <)$ be an ordered field, contained in a real closed order-extension field R. Let $X = (X_1, \ldots, X_n)$ be indeterminates and let $x = (x_1, \ldots, x_n) \in R^n$. A set $A \subseteq R^n$ is called semi-algebraic (abbreviated s.a.: more precisely, K-R-s.a.) if it is a finite union of finite intersections of sets (and of complements of sets) of the form $\{x \in R^n | f(x) > 0\}$, $f \in K[X]$. Similarly for subsets of R^m, $m \neq n$. If $A \subseteq R^n$ and $B \subseteq R^m$ are s.a., and if L is a subfield of R, then a function f: $A \to B$ will be called an L-function if f takes points of A with coordinates in L ("L-rational points") to points of B with coordinates in L; i.e., if $f(A \cap L^n) \subseteq L^m$.

DEFINITION. We shall call a function $f = (f_1, \ldots, f_m)$, from a (K-R-) s.a. set A in R^n to a (K-R-) s.a. set in R^m, (K-R-) piecewise-rational, abbreviated (K-R-) p.r., if we can decompose A into a finite number of (K-R-) s.a. sets W_i, $A = \bigcup_i W_i$, such that for each i and for $1 \leqq j \leqq m$, there is a rational function in $K(X)$ which agrees with f_j on W_i.

The absolute value function $x \mapsto |x|$ is a good example of a (continuous) $\mathbf{Q}$-R-p.r. function from R^1 to R^1. Of course, all rational functions are also p.r. Clearly, K-R-p.r. functions are L-functions, uniformly for all fields L between K and R (i.e., for $K \subseteq L \subseteq R$).

DEFINITION. A K-R-s.a. set S is a K-R-p.r.-neighborhood-retract if there exists an open K-R-s.a. neighborhood $U \supseteq S$ and a retraction r: $U \to S$ which is K-R-p.r.

We may as well require U to be regular (i.e., equal to the interior of its closure), since we can shrink it if necessary until it is regular, by triangulating U and S and subdividing.

Recall that an ordered field K is called Archimedean (over $\mathbf{Q}$) if for all $d \in K$ there exists $e \in \mathbf{Q}$ such that $d < e$ (e.g., $\mathbf{Q}$ and $\mathbf{R}$ are Archimedean). We can now state the main theorem.

RETRACTION THEOREM. *Let K be Archimedean. Let $W \subseteq R^n$ be a closed,* *convex K-R-s.a. set with interior. Then W is a K-R-p.r.-neighborhood-retract.*

The full proof of the retraction theorem is too long for this note; instead, we shall consider only the following special case. For $x \in R^n$, write $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$, and for $f \in K[X]$ write $\nabla f = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$, the gradient of f. Fix $f \in K[X]$, and let $W = \{x \in R^n \mid f(x) \geqq 0\}$, and assume furthermore that $W \neq \varnothing$ and that there exists an $\eta > 0$ such that for all $x \in \partial W$, $\|\nabla f(x)\|^2 > \eta$. Then even though such a W need not satisfy the convexity hypothesis of the retraction theorem, we still claim that W is a p.r.-neighborhood-retract. The partial proof we now give will illustrate the main idea of the proof of the full retraction theorem. This idea is that the retraction $r: U \to W$ should "push into W in the direction of the gradient of f."

Indeed, define $r: R^n \to R^n$ by $r(x) = \begin{cases} x & \text{if } x \in W, \\ x - (1/\eta)f(x)\nabla f(x) & \text{if } x \notin W. \end{cases}$

Note that r is continuous (since $\partial W \subseteq Z\{f\}$) and p.r. We shall show that there is an open neighborhood $U \supseteq W$ such that $r(U) \subseteq W$. To show that U exists, it suffices to show that for all $\varepsilon_1 > 0$ and for all (continuous, s.a.) paths b: $[0, \varepsilon_1) \to R^n$ starting in W (i.e., with $b(0) \in W$), there exists an $\varepsilon_2 > 0$ ($\varepsilon_2 < \varepsilon_1$) such that for all t, $t \in [0, \varepsilon_2)$ implies $r(b(t)) \in W$. We need only consider b with $b(0) \in \partial W$, since $r = id$ on W^0. For the same reason we need only consider b such that for all $t \in (0, \varepsilon_1)$, $b(t) \notin W$, i.e., $f(b(t)) < 0$. Therefore $L \leqq 0$, where $L = df(b(t))/dt|_{t=0}$. For the sake of brevity in this note, let us consider only those b such that $L < 0$. For such b we must show that there exists an $\varepsilon_2 > 0$ ($\varepsilon_2 < \varepsilon_1$) such that for all $t \in [0, \varepsilon_2)$, $r(b(t)) \in W$, i.e., $f(r(b(t))) \geqq 0$. This holds if $L > 0$ (and occasionally even if $L = 0$, though we shall not deal with this possibility here). That $L > 0$ follows from a direct computation which expresses L as a product of 2 negative quantities.

For more general W satisfying the hypotheses of the retraction theorem, the proof uses a stratification lemma, the primitive element theorem, a partition-of-unity argument, and a semi-algebraically parametrized version of the Heine-Borel theorem.

If the retraction r were not required to be p.r., but merely s.a. (i.e., to have s.a. graph in $R^n \times R^n$), then the theorem could be proved in one sentence, by triangulating W, even without the hypotheses of convexity, non-empty interior, and Archimedeanness; this was done in [**3**]. A novel feature of the retraction which we construct is that it does not satisfy $r(U - W) \subseteq \partial W$, i.e., r must "push in" through ∂W, unlike most retractions; this is unavoidable if r is to be p.r., since it must then be a K-function, while ∂W need not contain any points of K^n.

None of the hypotheses can be dropped in the retraction theorem. The first hypothesis, that K be Archimedean, is necessary, for let $K = \mathbf{Q}(T)$, T an indeterminate, ordered so that $T > \mathbf{Q}$. Then it is easy to check that there is no element of K, or even of $K(\sqrt{2})$, between the elements $\sqrt{T}$ and $\sqrt{2T}$ of R. Now let $W = \{x \in R^2 | x_1 \geqq 0$ and $\sqrt{T} x_1 \leqq x_2 - \sqrt{2} \leqq \sqrt{2T}\, x_1\}$ (see Figure 1). W is convex. W is prima facie R-R-s.a.; by Tarski-Seidenberg, it is in fact K-R-s.a. Also, W is closed and has dense interior. But the conclusion of the retraction theorem fails: Suppose r: $U \to W$ is a K-R-p.r. retraction, where U is a neighborhood of W (or even of a small piece of W near $(0, \sqrt{2}) \in R^2$); we derive a contradiction. U contains a line segment S on the X_2-axis, say $\{(0, x_2) | \sqrt{2} \leqq x_2 \leqq \sqrt{2} + \varepsilon\}$, where $\varepsilon > 0$ is so small that $r|_S$ is a rational map (r_1, r_2), $r_i \in K(X)$. Now r cannot map S constantly into $(0, \sqrt{2})$, or else $r_2(0, X_2) = \sqrt{2} \notin K(X)$. So $r(S)$ is a real algebraic curve c in W starting at $(0, \sqrt{2})$. Figure 1 shows that the slope m of the graph of c at $(0, \sqrt{2})$ is between $\sqrt{T}$ and $\sqrt{2T}$, hence $m \notin K(\sqrt{2})$. But a comparison of the formal fractional power series expansion $X_2 = f(X_1)$ of c with the Taylor expansions about $X_2 = \sqrt{2}$ of $r_i(0, X_2)$ $(i = 1, 2)$ gives $m \in K(\sqrt{2})$, a contradiction.

Thus we need the Archimedean hypothesis. Second, we cannot drop the hypothesis that W be convex $(n \geqq 2)$. To see this, let $K = \mathbf{Q}$ and $W = \{x \in R^2 | X_1^3 - (\sqrt{3} X_1 - (X_2 - \sqrt{2}))^2 \geqq 0\}$. Then Figure 2 shows that $m = \sqrt{3}$, while the argument in the above paragraph shows $m \in \mathbf{Q}(\sqrt{2})$, a contradiction.

Third, any s.a. retract must be closed, and fourth, we cannot drop the hypothesis that W have interior, or else it might have no points in K^n to which to map those points in $U \cap K^n$ (U, being open, must contain such points, since K is Archimedean over $\mathbf{Q}$, hence dense in $\bar{K} \subseteq R$). Thus, all the hypotheses in the retraction theorem are necessary.

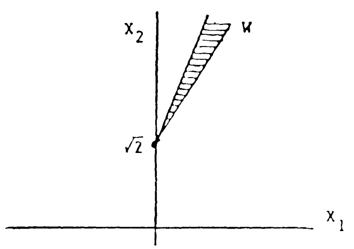

Figure 1

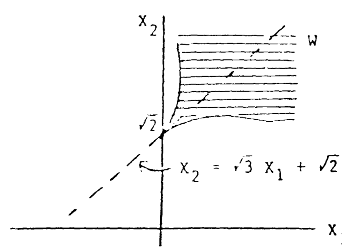

Figure 2

Potential Application to Hilbert's 17th Problem. Let $d > 0$ be even, let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$ be a multi-index, let $|\alpha| = \sum \alpha_i$, let $C = \langle C_\alpha \,|\, |\alpha| \leqq d\rangle$ be a sequence of indeterminates, and let $f \in \mathbf{Z}[C; X]$ be the general polynomial of degree d in X with coefficients C, i.e., $f = \sum_{|\alpha| \leq d} C_\alpha X^\alpha$, where $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Write $P_{nd} = \{c \in R^{\binom{n+d}{n}} \mid f(c; X)$ is positive semi-definite over R in $X\}$. In 1962 Kreisel asked [5] whether a "continuous" solution to Hilbert's 17th problem exists. More precisely, do there exist finitely many K-functions p_j: $P_{nd} \to R^+$ and r_j: $P_{nd} \times R^n \to R$ such that $f(c; X) = \sum_j p_j(c) r_j(c; X)^2$ for all $c \in P_{nd}$, with each r_j rational in X, and each summand $p_j r_j^2$ continuous simultaneously in c and x for $(c; x) \in P_{nd} \times R^n$?

The retraction theorem fits in as follows. The "finiteness theorem" [1] says there exist finitely many W_i such that $P_{nd} = \bigcup_i W_i$, where each W_i is a finite intersection of sets of the form $\{c \in R^{\binom{n+d}{n}} \mid g(c) \geqq 0\}$, some $g \in \mathbf{Q}[C]$. For each i we can use Stengle's Positivstellensatz to construct rational functions $p_j \in \mathbf{Q}(C)$ and $r_j \in \mathbf{Q}(C; X)$ as above, except that we have $p_j(c) \geqq 0$ only for $c \in W_i$, and not for all $c \in P_{nd}$. These functions on each W_i can be glued together into a globally continuous map with $p_j(c) \geqq 0$ for all $c \in P_{nd}$, by a partition-of-unity. This requires writing each W_i as a neighborhood-retract. In [3] we did this with s.a. retractions; the resulting p_j and r_j were only s.a. in c, not rational or p.r., and thus they were K-functions only for real closed K. In [4] we constructed, for each countable subfield K of $\mathbf{R}$, a retraction which was a K-function, resulting in p_j and r_j being K-functions for these special K. To answer Kreisel's question for arbitrary K, it would suffice for the p_j and r_j to be rational in C, hence K-functions; in [2] we showed this is impossible, even without continuity requirements. Fortunately, it would also suffice for them to be $\mathbf{Q}$-R-p.r., and this would be achieved if the W_i could be chosen to be $\mathbf{Q}$-R-p.r.-neighborhood-retracts. Since $\mathbf{Q}$ is Archimedean, and since those W_i without interior may be ignored, the retraction theorem would finish the answer to Kreisel's question, provided the W_i could be chosen to be convex. While this last is still unsettled, there is hope, since P_{nd} is obviously convex.

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