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EXTENSIONS OF SEMIDEFINITE FUNCTIONS

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Dedicated to the memory of Gus Efroymson

Gondard and Ribenboim [4] proposed the following problem: let $V \subseteq \mathbf{R}^n$ be an algebraic set, $f \in \mathbf{R}[x]$, $\mathbf{x} = (x_1, \ldots, x_n)$, such that $f/_V \geqq 0$. Does there exist $F \in \mathbf{R}[x]$, $F \geqq 0$ over $\mathbf{R}^n$, such that $F/_V = f/_V$? They gave a partial positive answer. We prove a stronger result.

THEOREM 1. *Let $V \subset \mathbf{R}^n$ be an algebraic set, $f \in \mathbf{R}[x]$ with $f/_V \geqq 0$ and $\{a \in \mathbf{R}^n : f(a) = 0\} \cap V_c = \emptyset$ (V_c is the locus of central points of V). Then, there exists $F \in \mathbf{R}[x]$ non negative over $\mathbf{R}^n$ such that $F/_V = f/_V$. Moreover, if $f/_V > 0$, F is positive over $\mathbf{R}^n$.*

In the same paper [4] it is proved that the answer to the problem of extending f, if $f/_V \geqq 0$, is negative in general. For example, let us take $V = \{y^2 - x^3 = 0\}$ and $f = x$ However, we have been able to prove the following theorem.

THEOREM 2. *If $f \in \mathbf{R}[x]$ and $f/_V \geqq 0$, there exists an odd positive integer m and $F \in \mathbf{R}[x]$ non negative over $\mathbf{R}^n$, verifying $F/_V = f^m/_V$*

We look at these problems in two different ways.
A) We restrict ourselves to the case where $V \subset \mathbf{R}^n$ is a curve.
B) Given $f \in \mathbf{R}[x]$ such that $f/_V \geqq 0$, does there exist
  i) A polynomial $F \in \mathbf{R}[x]$
  ii) A regular function F
  iii) A rational function F
such that $F \geqq 0$ (where F is defined) and $f/_V = F/_V$?
If we denote by $D(f)$ the "bad set" of f (see [3]), we prove the following.

PROPOSITION 3. *Condition* ii) *is equivalent to condition* iii), *and both are implied by $D(f) = \emptyset$.*

So, we are concerned with the existence of polynomial extensions (condition i)) or regular extensions (condition ii))

Since $\operatorname{codim}_V D(F) \geqq 2$ when $V \subset \mathbf{R}^n$ is a normal algebraic set, ([3]), we conclude the following proposition.

PROPOSITION 4. *The regular extension problem has an affirmative answer if $V \subset \mathbf{R}^n$ is a non-singular curve.*

The situation is not completely known for singular curves, but we obtain some negative answers for plane curves.

PROPOSITION 5. *The regular extension problem has negative answer in the following cases:*
1) *$V \subset \mathbf{R}^n$ is a curve with a point whose multiplicity is greater than two.*
2) *$V \subset \mathbf{R}^n$ is a curve with a non-ordinary double point.*

We do not know what happens in case V contains an ordinary double point.

Concerning the polynomial extension we know the following sufficient condition.

PROPOSITION 6. *Let $V \subset \mathbf{R}^n$ be a curve and $f \in \mathbf{R}[x]$ with $f/_V \geqq 0$. Let us assume that each point of $(f = 0) \cap V$ has principal maximal ideal in the coordinate ring $\mathbf{R}[V]$ of V. Then there exists $F \in \mathbf{R}[x]$ such that $F \geqq 0$ and $F/_V = f/_V$.*

Even for non-singular curves we do not know the answer to the polynomial extension problem. Consider for example $V = \{y^2 - x(x^2 + 1) = 0\}$, and $f = x$. However, a well known result **[2]** proves that the answer is affirmative when V is a conic.

For arbitrary dimension, the answer to the regular extension problem (and so, to the polynomial extension problem) is negative in general Consider for example $V = \{x^2 + y^2 + z^3 = 0\}$ and $f = -z$.

The problems on extensions are closely related with the problem posed by Lam in the conference Boulder.

Let $V \subset \mathbf{R}^n$ be an algebraic set and $f \in \mathbf{R}[x]$. Does $f/_V \geqq 0$ imply that f is a sum of squares in $\mathbf{R}[V]$?

We can also study: does $f/_V \geqq 0$ imply that f is a sum of squares of regular functions over V?

Of course, affirmative answers to those problems would give us affirmative answers to the extensions problems. But the converse is not true. For example (see **[5]**) consider the curve $V = \{y^2 = x(x^2 + 1)\}$. Then $f = x^3$ admits a polynomial extension but it is not a sum of squares in $\mathbf{R}[V]$.

Finally, some few remarks about this question in the local analytic case. Let $x_0 \subset \mathbf{R}^n_0$ be an irreducible analytic germ, $f \in 0[X_0]$ an analytic function germ such that $f/_{x_0} \geqq 0$. Is f a sum of squares in $0[X_0]$?

In the non-singular case the answer is affirmative if $\dim x_0 \leqq 2$ (see **[1]**) and negative otherwise.

In the singular case we prove that the answer is negative when $\dim X_0 = 1$, when $\dim X_0 > 1$, we know very little.

EXAMPLES. 1) $X_0 = \{z(x+y)(x^2+y^2) = x^4\}$ and $f = z(x+y)$. Since codim in X_0 of $D(f) = 1$, f is not a sum of squares in $0[x_0]$.

2) $X_0 = \{x^2 + y^3 + z^5 = 0\}$. Each $f \in 0[X_0]$ with $f/_{X_0} \geqq 0$ is a sum of two squares in $0[X_0]$.

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