ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 4, Fall 1984

A LOWER BOUND FOR THE COMPLEXITY OF SEPARATING FUNCTIONS

GILBERT STENGLE

Dedicated to the memory of Gus Efroymson

1. Introduction. The following theorem of Mostowski [7] [3] [4] [2] [5] [8] is of fundamental importance for the algebraic theory of semialgebraic sets.

THEOREM 1. *Let X and Y be closed disjoint semialgebraic subsets of a real affine variety. Then there exists a function of the form $f = \sum P_k\sqrt{Q_k}$, where P_k and Q_k are regular functions and $P_k > 0$, such that f is positive on X and negative on Y.*

In this notice we state and outline the proof of the following complementary negative result.

THEOREM 2. *The family of functions of the form $f = \sum P_k\sqrt{Q_k}$ containing at most $M + 1$ summands, where the P_k and Q_k are polynomials and $Q_k >$ 0, fails to separate disjoint pairs of closed semialgebraic sets in R^{1+2M}.*

For $M = 0$ this reduces to the simple assertion that polynomials fail to separate closed disjoint pairs of semialgebraic sets in the plane. For general M it gives a kind of lower bound for the complexity of any separating subfamily of the family of Theorem 1.

2. A complex of groups. The proof of Theorem 2 exploits algebraic properties of certain groups of sign distributions. These are subgroups of the group of units of the ring of sign distributions as developed in [1]. Let V be a real variety in $S^N = R^{N+1}/R^+$. Let SD^* be the ring of chains of semialgebraic subsets of V with integer coefficients, ZSD^* the ideal of chains supported on subsets of positive codimension. Let $SD = SD^*/ZSD^*$. In $SD[R^N]$ define $SG_0 = \{\pm 1\}$. For each nonzero homogeneous polynomial a let $\langle a\rangle$ be the image in SD of $\{a > 0\} - \{a < 0\}$. If a is a general (nonhomogenous) polynomial let $\langle a\rangle$ be the sign distribution of the homogeneous part of highest degree. Define SG_{k+1} recursively to be the group generated by the sets $SG_k\{a > 0\} + SG_k\{a < 0\}$ as a ranges over the nonzero homogeneous polynomials. Let ρ be the reflection $x \to -x$

and let $g \to g^{\rho}$ denote the induced action on sign distributions. Let δ be the operator on SD: $\delta g = gg^{\rho}$.

PROPOSITION 3. $\delta SG_{k+1} \subset SG_k$.

REMARKS.

1. Since $\delta^2 = 1$ the SG_k equipped with δ form a complex.

2. These sign distributions on R^{N+1} can be regarded either as N-dimensional distributions on the sphere at infinity or as special $(N + 1)$-dimensional distributions defined by positively homogeneous conditions on R^{N+1}.

We also assign sign distributions to elements of quadratic extensions of the polynomial ring by the formula

$$\langle a + b\sqrt{q}\rangle = \langle a\rangle \{d > 0\} - \langle b\rangle\{d < 0\}$$

provided $q > 0$, where d is the homogeneous part of $a^2 - qb^2$ of highest degree. Using this recursively we can define $\mathscr{S}_k$ to be the group generated by sign distributions of the form $\langle \sum_0^k a_j\sqrt{b_j}\rangle$ where $b_j > 0$. We then can show

PROPOSITION 4. $\mathscr{S}_k \subset SG_{2^k}$.

If we define $k(f)$ to be the smallest k such that $\langle f\rangle \in \mathscr{S}_k$ we obtain what appears to be a well-structured partial measure of the complexity of f. Since the $\mathscr{S}_k$ are multiplicative groups, with respect to this measure multiplications are trivial by fiat, and $k(f)$ mainly reflects additive properties of f. The paramount significance in the real case of additive properties as opposed to multiplicative appears already in Descartes' venerable rule of signs and more recently in the striking recent work of Khovansky [6].

3. Proof of Theorem 2 (sketch). Let X be the subset of R^N where all coordinates are greater than or equal to 1, Y the subset where at least one coordinate is nonpositive. Let g_N be the sign distribution which is negative on the positive orthant and positive on its complement. Then the sign distribution of a function a which is negative on X and positive on Y is precisely g_N. We define mappings from $SG_k[R^N]$ to $SG_{k-1}[R^{N-1}]$ by $q[g] = [gg^{\rho}]_{x_N=1}$. Then it can be shown that q is well-defined and that $q[g_N] = g_{N-1}$. We can then arrive at a contradiction if we assume that $N = 2^M + 1$ but that the separating function a has only $M + 1$ summands. The latter condition, by Proposition 4, implies that $\langle a\rangle$ belongs to SG_{2^M} and hence $q^{2^M}\langle a\rangle$ belongs to SG_0. The former condition implies however that $q^{2^M}\langle a\rangle = q^{2^M}(g_{2^M+1}) = g_1$ which is not an element of $SG_0[R]$.

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DEPARTMENT OF MATHEMATICS, CHRISTMAS-SANCON HALL, LEHIGH UNIVERSITY
BETHLEHEM PA 18015