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THE SEMIGROUP OF NON LEFT ZERO DIVISORS IN AN ALGEBRA, INFINITE POWERS OF MATRICES AND RELATED MATTERS.

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Dedicated to the memory of Gus Efroymson

1. A non left zero divisor (*nlzd*) $\alpha \neq 0$ in an associative finite dimensional algebra $\mathscr{A}$ is an element $\alpha \in \mathscr{A}$ such that $\alpha x = 0$, $x \in \mathscr{A}$ implies $x = 0$, but that $x\alpha = 0$ can be solved with $x \neq 0$. The *nlzd*'s form a semigroup $\mathscr{S}_\ell$. Examples of algebras with $\mathscr{S}_\ell$ not empty are given by the algebra $\mathscr{M}$ formed by the sets of $n \times n$ matrices M with entries in a field F, with rank $M = r < n$, of the form

$$M = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \cdot & \cdot \;\cdots & \cdot \\ a_{r1} & a_{r2} \cdots & a_{rn} \\ 0 & 0 \;\cdots & 0 \\ \cdot & \cdot & \cdot \\ 0 & 0 \;\cdots & 0 \end{pmatrix}$$

Let $M_r = (a_{ik})$ $(i, k = 1, \ldots, r)$. M is an *nlzd* in $\mathscr{M}$ iff $\det M_r \neq 0$. The following observations can be made:

1) While the matrix algebra $F^{n\times n}$ does not contain a *nlzd* every singular matrix in $F^{n\times n}$ gives rise to an algebra which contains *nlzd*'s, namely a conjugate of $\mathscr{M}$.

2) Every element in $\mathscr{S}_\ell$ has a right identity and a left inverse with respect to it.

3) In connection with $\mathscr{M}$ the hypersurface $\det(x_{ik}) = 0$, x_{ik} independent indeterminates, deserves investigation.

4) If $\det M_r \neq 0$ then M can be transformed to block diagonal form $\left(\begin{smallmatrix} M_r & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

5) Assume that there exists an irreducible subalgebra of $\{M_r\}$ such that all $\det M_r \neq 0$ apart from $M_r = 0$. For F the real number field this is the case only for $r = 1, 2, 4$ by Frobenius' theorem concerning as-

sociative finite dimensional division algebras, namely the reals, the complex numbers, the real quaternions. It is known that the norm forms of these algebras are sums of squares and come from determinants.

2. Some time ago the author studied elements λ in a real algebra with the property that $\lambda^n \to 0$ in the euclidean topology (this notation is adopted from Pontrjagin). In a sequel written jointly with John Todd, these elements were explored further and, in particular, complex matrices C with $C^n \to 0$ were examined. Later the author became interested in these again via a theorem by P. Stein. There such matrices are characterized by the existence of a positive definite matrix H such that $H - C H C^* > 0$. For 1×1 matrices this means: a complex number z with $|z| < 1$ is also given by the property $|z|^n \to 0$. This simple well known fact shows the connections between topological properties and positivity ideas. The author proved earlier that a Cayley transformation links Stein's theorem to Lyapunov's theorem concerning stable matrices. Further connections have been explored.

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