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TOPOLOGY AND REAL ALGEBRAIC GEOMETRY

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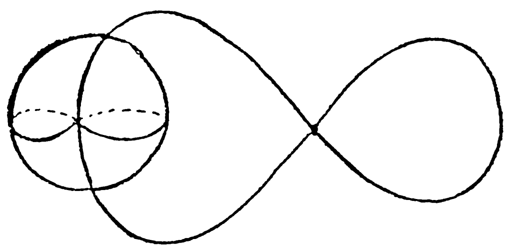
Dedicated to the Memory of Gus Efroymson

The problem of solving real polynomial equations long fascinated people. Starting with a finite number of points on the real line one can get many complicated spaces as zero sets of polynomials. For example every closed P.L. manifold occurs as an algebraic set [3]. The main goal of understanding the topology of algebraic sets is to topologically characterize the image of the forgetful functor

$$\{\text{Algebraic sets}\} \to \{\text{Topological spaces}\}.$$

Rather than discussing the history and development of this problem we simply refer the reader to the surveys ([5], [7]) and instead describe the program which I developed jointly with H. King towards solving this problem.

Lets take an algebraic set X in $\mathbf{R}^3$ given by $(2x^2 + 2y^2 - 1)((x - 1)^4 - (x - 1)^2 + y^2)^2 + 2z^2 = 0$ X looks like this.



By resolving singularities of X or just by inspecting, we see that X is obtained by a disjoint union of smooth manifolds by doing some identifications. More specifically, $X = \bigcup_{i=0}^{2} V_i/\sim$ where V_0, V_1, V_2 are a point, a circle, and a 2-sphere respectively; $\sim$ indicates that the equator of V_2 is folded onto V_1, and the two points of V_1 are folded onto V_0 (i.e., identified).

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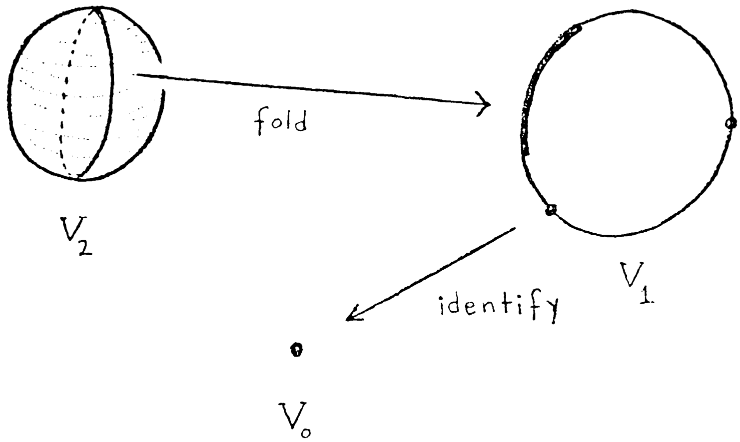


After inspecting many such examples one begins to see the emergence of a beautiful topological structure on algebraic sets, namely, smooth manifolds glued by special fold-maps! Hence the goal is to show that all algebraic sets are characterized by such structures.

DEFINITION. A topological resolution tower $\mathscr{F} = \{V_i, V_{ji}, p_{ji}\}$ is a collection of smooth manifolds V_i, $i = 0, 1, \ldots, n$, union of codimension one smooth submanifolds (sheets) in general position $V_{ji} \subset V_i$, $j = 0, 1, \ldots, i-1$, and maps $p_{ji}\colon V_{ji} \to V_j$ satisfying the following.

(1) $p_{ji}(V_{ji} \cap V_{ki}) \subset V_{kj}$, for $k < j$.

(2) $p_{kj} \circ p_{ji}|_{V_{ji} \cap V_{ki}} = p_{ki}|_{V_{ji} \cap V_{ki}}$, for $k < j$.

(3) $p_{ji}^{-1}(\bigcup_{m \leq k} V_{mj}) = V_{ji} \cap \bigcup_{m \leq k} V_{mi}$ also if S is a sheet of V_{kj} then $p_{ji}^{-1}(S) = V_{ji} \cap \bigcup_{m \leq k}$ (some sheets of V_{mi}).

(4) p_{ji} is smooth on each sheet of V_{ji} and $p_{ji}\colon V_{ji} - \bigcup_{k<j} V_{ki} \to V_j - \bigcup_{k<j} V_{kj}$ is a locally trivial fibration.

(5) For every $y \in V_{ji}$ let $y_i = y$, $y_j = p_{ji}(y)$. Then there are local coordinates $\theta_\alpha\colon (U_\alpha, 0) \to (V_\alpha, y_\alpha)$ for $\alpha = i, j$, where $U_\alpha \subseteq \mathbf{R}^{n_\alpha} = \mathbf{R}^{c_{\alpha 0}} \times \mathbf{R}^{c_{\alpha 1}} \times \cdots \times \mathbf{R}^{c_{\alpha\alpha}}$ is an open subset such that:

A) Each coordinate function of $\theta_j^{-1} \circ p_{ji} \circ \theta_i(x)$ is in the form $\varphi(x) \prod_{k=1}^{n_i} x_k^{s_k}$ where $\varphi(x)$ is some smooth function with $\varphi(0) \neq 0$.

B) If $(x_{k1}, \ldots, x_{kc_{\alpha k}})$ are the coordinates of $\mathbf{R}^{c_{\alpha k}} \subset \mathbf{R}^{n_\alpha}$ then $\theta_\alpha^{-1}(V_{k\alpha}) = \{x \in \mathbf{R}^{n_\alpha} \mid \prod_{j=1}^{c_{\alpha k}} x_{kj} = 0\} \cap U_\alpha$, and it is ϕ if $c_{\alpha k} = 0$.

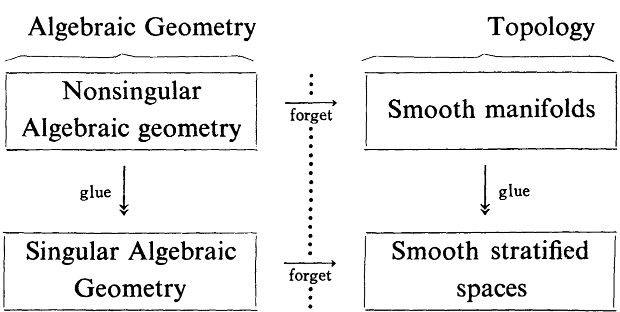
Even though (5) doesn't seem to be a topological condition, actually it is; it just says that the maps p_{ji} are allowed to have only fold-like singularities. Now we will call a topological resolution tower $\mathscr{F} = \{V_i, V_{ji}, p_{ji}\}$ an algebraic resolution tower if each V_i is a nonsingular algebraic set, the V_{ji} are unions of nonsingular algebraic sets in general position, and the $p_{ji}: V_{ji} \to V_j$ are entire rational functions. We call the identification space

$$|\mathscr{F}| = \bigcup_{i=0}^{n} V_i / x \sim p_{ji}(x), \qquad \text{for } x \in V_{ji}$$

the realization of the resolution tower $\mathscr{F}$. One can show that $|\mathscr{F}|$ is a smooth Thom-stratified space with strata equal to the $V_i - \bigcup_{j<i} V_{ji}$. Furthermore, if $\mathscr{F}$ is an algebraic resolution tower, then it turns out that $|\mathscr{F}|$ is an algebraic set with the above stratification. Hence we have the following commutative diagram

$$\begin{array}{ccc} \left\{\begin{array}{c}\text{Algebraic}\\ \text{Resolution towers}\end{array}\right\} & \xrightarrow{\tau'} & \left\{\begin{array}{c}\text{Topological}\\ \text{Resolution towers}\end{array}\right\} \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma'} \\ \left\{\text{Algebraic sets}\right\} & \xrightarrow{\tau} & \left\{\begin{array}{c}\text{Realization of Topological}\\ \text{resolution towers}\end{array}\right\} \end{array}$$

where γ and γ' are realization maps $\mathscr{F} \rightsquigarrow |\mathscr{F}|$, and τ, τ' are the natural forgetful maps. The main result of [6] says that γ is onto. γ' is onto by definition. To be precise, τ is not a well defined map but τ or is defined; since γ is onto, we loosely talk about τ. Hence showing that τ' is onto would slove the main problem of topologically characterizing algebraic sets, i.e., it would make τ onto. We want to emphasize that the surjectivity of γ reduces the understanding of singular algebraic sets to the understanding on nonsingular algebraic sets. Philosophically the above diagram should be seen as relating algebraic geometry to topology as follows.



Once surjectivity of τ' is established, then the problem of finding algebraic structures on general stratified spaces would reduce to an algebraic topology problem. Namely, what are the combinatorial topological invariants that characterize the image of the forgetful functor τ'' where

$$\left\{\begin{matrix}\text{Realization of Topological}\\ \text{resolution towers}\end{matrix}\right\} \xrightarrow{\tau''} \left\{\begin{matrix}\text{Stratified}\\ \text{spaces}\end{matrix}\right\}?$$

To prove that τ' is onto, one has to show that any topological resolution tower $\mathscr{F} = \{V_i, V_{ji}, p_{ji}\}$ comes from an algebraic resolution tower. By [2] the V_i can be made nonsingular algebraic sets so that the $V_{ji} \subset V_i$ are all nonsingular algebraic subsets for $j = 0, 1, 2, \ldots, i-1$. The difficulty is to make the maps $p_{ji}: V_{ji} \to V_j$ rational maps. This problem runs into obstructions, if to begin with, one has not chosen "correct" algebraic structure on the V_i's. This obstruction arises when the chosen algebraic structure on V_i does not possess enough algebraic subsets to generate all the $\mathbf{Z}_2$-homology of V_i. Therefore we can only show that τ' is onto up to dimension 6. But we believe that τ' is onto in general.

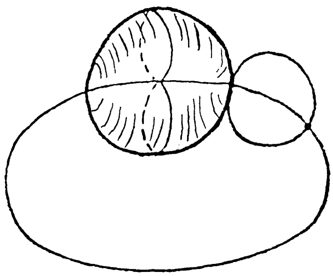
Sullivan [8] observed that any algebraic set has to be an Euler space (i.e., a smooth stratified space such that the link of each stratum has even Euler characteristic). It turns out that in dimension $\leqq 2$ the image of τ coincides with Euler spaces ([1], [4], [5]) so we have:

$$\left\{\begin{matrix}\text{Algebraic sets}\\ \text{of dimension } \leqq 2\end{matrix}\right\} = \left\{\begin{matrix}\text{Euler spaces}\\ \text{of dimension } \leqq 2\end{matrix}\right\}.$$

In general, starting with a Thom-stratified Euler space X of dimension n, one can form an obstruction theory to decide whether X lies in the image of τ''. First by enlarging the stratification we can assume that each i-dimensional stratum X_i has a trivial normal bundle in X. (X_i can be empty). $X = \bigcup_{i=0}^{n} X_i$ and for non-empty strata we have $X_i \subset \bar{X}_j$ if $i < j$. Assume that the complement of $\bar{X}_k$ has a realization of a topological tower structure. Then the link $\sum_{n-k-1}$ of each component of X_k has this structure. Let η_i denote the i-dimensional cobordism group of realizations of topological resolution towers, defined in the obvious way. Then the induced elements $[\sum_{n-k-1}] \in \eta_{n-k-1}$ are the obstructions to extending the structure to the complement of $\bar{X}_{k-1}$. This is not obvious; it follows from the nontrivial fact that $\sum_{n-k-1}$ bounds if and only if cone $(\sum_{n-k-1})$ has this structure. If one is willing to change the stratification of X in order to put in these structures, then one gets homology obstructions (following [9]). We hope to make these more precise in a future paper. For example in dimension 3 the only obstructions are the obstructions to extending the structure to the vertices of X^3. It turns out that $\eta_2 = \mathbf{Z}_2^{16}$, hence sixteen characteristic numbers β_ε, $\varepsilon \in \mathbf{Z}_2^4$ determine this cobordism group. Hence for each component of the 0-stratum $x_0 \in X_0$, if we can associate the characteristic numbers $\beta_\varepsilon(x_0) \in \mathbf{Z}_2$ of the link of x_0, then

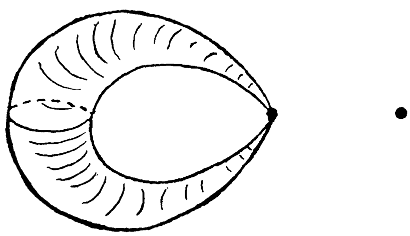
$$\left\{\begin{matrix}\text{3-dimensional}\\ \text{Algebraic sets}\end{matrix}\right\} = \left\{\begin{matrix}\text{3-dimensional}\\ \text{Euler spaces } X^3\end{matrix}\,\middle|\,\begin{matrix}\beta_\varepsilon(x_0) = 0, \text{ for all}\\ x_0 \in X_0 \text{ and } \varepsilon\end{matrix}\right\}.$$

The obstructions $\beta_\varepsilon(x_0)$ are easily computable once the stratification of X is given; in fact some combinations of $\beta_\varepsilon(x_0)$'s are topological invariants (i.e., independent of the stratification). From this one gets the amusing corollary that the Euler space X^3 = Suspension (Y^2), where

$Y^2 =$ 

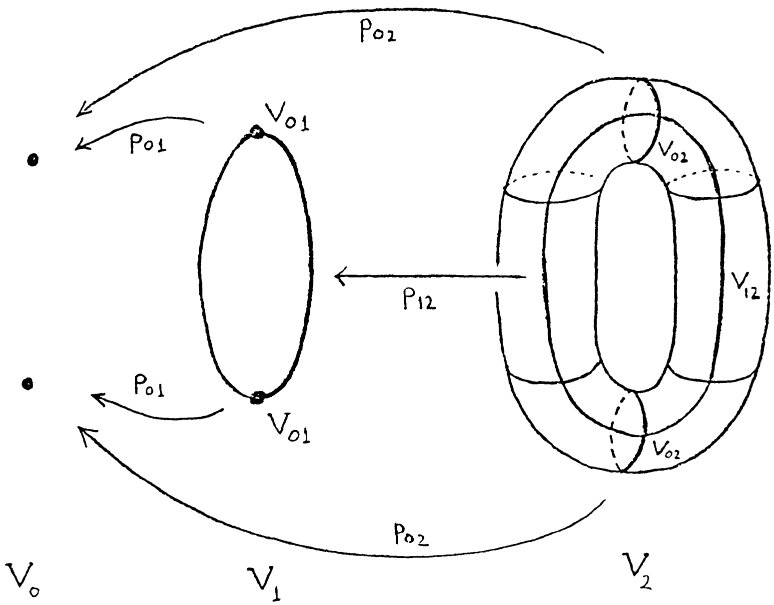
cannot be homeomorphic to an algebraic set.

Finally we discuss an example to give an idea how to recognize realization of topological tower structures in particular cases. Let X^3 = Suspension(Y^2) where Y^2 is the disjoint union of pinched torus and a point shown here.



We claim that $X = |\mathscr{F}|$ for some $\mathscr{F}$, hence X is homeomorphic to an algebraic set with this stratification. To see this, glue two copies $(S^1 \times B^2)_\pm$ of $S^1 \times B^2$ to $T^2 \times I$ along the two boundary components of $T^2 \times I$ where T^2 is a 2-torus. Call this manifold $V_2 = (S^1 \times B^2)_- \cup (T^2 \times I) \cup (S^1 \times B^2)_+$. Let $V_{12} = (a \times B^2)_- \cup (a \times S^1 \times I) \cup (a \times B^2)_+$ where $a \in S^1 (V_{12} \approx S^2)$, and let $V_{02} = (S^1 \times b)_- \cup (S^1 \times b)_+$ where $b \in$ interior (B^2). Let $V_1 = S^1$, V_0 = two points, and let V_{01} be two points of V_1.

Let p_{01} be the identification map, let p_{02} collapse the two components of V_{02} to the two points of V_0, let p_{12} fold the 2-sphere V_{12} onto an interval connecting two components of V_{01} on V_1. p_{12} collapses the circles



$a \times S^1 \times t$ to the points of V_1. Then it is easily seen that $X^3 = |\mathscr{F}|$ where $\mathscr{F} = \{V_i, V_{ji}, p_{ji}\}_{i=0}^2$. In fact X^3 has an A-structure [3], which is a particular case of this structure.

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Note (1984). Now H. King and the author have proved that, after a topological resolution, every $\mathbf{Z}_2$-cycle can be made algebraic. See the preprint *A resolution theorem* for $\mathbf{Z}_2$*-cycle of algebraic sets* for the implications.—