ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 4, Fall 1984

TOWARDS A CLASSIFICATION OF REAL ALGEBRAIC SURFACES

R. SILHOL

Dedicated to the Memory of Gus Efroymson

In [2] we continue to develop the methods used in [1] for the study of real algebraic varieties. The idea underlying these methods is essentially the following. Let X be a complex variety (variety = projective integral scheme) such that $X(\mathbf{C})$ (the set of complex points of X) admits an anti-holomorphic involution σ. Consider the category $\mathscr{C}$ whose objects are such pairs (X, σ). Then $\mathscr{C}$ is equivalent to the category $\mathscr{C}'$ of projective geometrically integral schemes over $\mathbf{R}$.

Reference [2] is devoted to the application of this idea to the study of real algebraic surfaces, using what is known of the classification of complex surfaces. For technical reasons we have restricted in this first attempt to surfaces admitting a (real) rational or elliptic fibering.

We define a real fibering in the following way. Let X be a real smooth surface (i.e., a smooth projective integral scheme of dimension 2 over $\mathbf{R}$) we say that X admits a (real) fibering if there exists a real curve B and a morphism (of real schemes) $\pi: X \to B$ such that the generic fiber $\pi^{-1}(\eta)$ is a smooth curve over the residue field $K(\eta)$ of the generic point η of B. The fibering will be called rational (resp. elliptic) if the curve $\pi^{-1}(\eta)$ is a rational (resp. elliptic) curve. We will also say that X is B-minimal if, for any other smooth surface X' fibered over B, a birational B-morphism (that is a morphism compatible with the fiberings) $f: X \to X'$ is necessarily an isomorphism of surfaces fibered over B.

For surfaces admitting a rational fibering results are fairly complete, and we can give a complete classification of real B-minimal surfaces admitting a rational fibering (i.e., ruled surfaces). This classification also enables us to describe the topological nature of $X(\mathbf{R})$ (the real part) for such surfaces. The main result in this direction is the following theorem.

THEOREM. *Let X be a real smooth surface ruled over B. Then if X is B-minimal $X(\mathbf{R})$ is topologically the disjoint union of t tori, k Klein bottles and s spheres with: $t + k \leqq r$ (where r is the number of connected com-*

ponents of $B(\mathbf{R})$) *and* $2s = \text{rank } NS(\bar{X}) - 2$ (*where* $NS(\bar{X})$ *is the Néron-Severi group of* $\bar{X} = X \times_{\mathbf{R}} \mathbf{C}$).

For real elliptic surfaces the situation is more complex and results are not as complete. We have concentrated in **[2]** on the local theory of such surfaces, that is the study of the real structure of the singular fibers.

We have obtained a complete classification of such fibers. The theorems giving this classification are much too long to be given here (Th. (4.1), (4.2) and (4.3) of **[2]**) but we will give here a few examples indicating the nature of the information that we obtain.

Let $\pi: X \to B$ be a real elliptic fibering, and consider the fibering obtained by complexification, i.e.,

$$\bar{\pi}: \bar{X} \to \bar{B} \text{ (where } \bar{X} = X \times_{\mathbf{R}} \mathbf{C} \text{ and } \bar{B} = B \times_{\mathbf{R}} \mathbf{C}).$$

Let $b \in B(\mathbf{R})$ and suppose that the complex fiber $\bar{\pi}^{-1}(b) = 2D_0 + D_1 + D_2 + D_3 + D_4$ is of type I_0^* in Kodaira's language (type c_4 in Néron's language.) Then in terms of the antiholomorphic involution σ we must be in one of the following situations:

(a) for all i $\sigma(D_i) = D_i$ (D_i considered as a divisor on $\bar{X}$)
(b) $\sigma(D_0) = D_0$, $\sigma(D_{2i}) = D_{2i}$ and $\sigma(D_1) = D_3$:
(c) $\sigma(D_0) = D_0$, $\sigma(D_i) \neq D_i$ for $i \neq 0$;
(d) the fiber has no real points.

We have the following additional information. Let U be a small neighbourhood of b in $B(\mathbf{R})$ and $\alpha: U \to]-a, a[$ a real analytic isomorphism. Choose ε sufficiently small; then for all $x \neq 0$ $x \in]-\varepsilon, \varepsilon]$ the real elliptic curve $(\alpha \circ \pi)^{-1}(x)$ has two real connected components in case (a) and only one in case (b). In case (c) $(\alpha \circ \pi)^{-1}(x)$ (resp. $(\alpha \circ \pi)^{-1}(-x)$) has no real points for $-\varepsilon < x < 0$ and two real connected components for $0 < x < \varepsilon$. In case (d) for all x in $]-\varepsilon, \varepsilon[$ the corresponding fiber has no real point.

To give another example consider the case when the fiber above the point b is of type II, IV, IV* or II* in Kodaira's language (type c_1, c_3, c_6 or c_8 in Néron's language.) Then the fibers above all points in a sufficently small neighbourhood of b in $B(\mathbf{R})$ have exactly one real connected component.

We obtain similar information on the behaviour of the real part of the fibers above points in a small neighbourhood of b in $B(\mathbf{R})$ for all singular fibers, so in practice the classification we obtain allows one to determine the topological type of $X(\mathbf{R})$ once one knows the nature of the singular fibers.

Conversly the results we have obtained allows us to construct explicit examples for certain situations. For example we have constructed a (real) K_3 surface such that $X(\mathbf{R})$ is connected, non-orientable and of Euler characteristic -18. We have also studied various types of Enriques surfaces. In fact the method is such that we can construct an algebraic surface X such that $X(\mathbf{R})$ has any smooth and compact topological type given a priori.

REFERENCES

1. R. Silhol, *Real abelian varieties and the theory of Comessatti*, Math. Z. **181** (1982), 345–364.

2. ———, Real algebraic surfaces with rational or elliptic fiberings to appear in Math. Z. (1984).

UNIVERSITÉ D'ANGERS FACULTÉ DES SCIENCES, DEPT. DE MATHS. 2 BLD. LAVOISIER 49045 ANGERS CEDEX FRANCE