ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 4, Fall 1984

ORDERS ON REAL ALGEBRAIC SETS AND ANALYTIC GERMS

T. RECIO

Dedicated to the memory of Gus Efroymson

This abstract surveys some results about the structure of the order space Ω for the function field F of rational (or meromorphic) functions over an irreducible algebraic set (or an analytic germ) X in $\mathbf{R}^n$. These results are contained in several recent works of Alonso, Andradas, Gamboa, Ruiz and the author, developing some ideas of Professor D.W. Dubois, whose continuous encouragement and direction made possible this contribution.

A basic idea, which appeared first in a paper of Dubois and Recio [**4**], is the study of the dense orbit property (DOP) for the order space Ω of F; namely we say that F is DOP when for each $\alpha \in F$, the orbit of α under the action of the group Aut (F) of automorphisms of F, is dense in Ω with respect to the Harrison topology. This property is just a weak statement of "homogeneity" for Ω, i.e., the case when any two orders of Ω are isomorphic. The DOP condition was soon observed to be strictly weaker than the homogeneity, as is clear in the following.

PROPOSITION 1. [**4**]. *Let K be a* DOP *field, totally dense. Then any finitely generated purely transcendental extension $K(x_1, \ldots, x_n)$ is also* DOP.

In [**8**] appears a detailed study of the converse to this proposition, which is related to the possibility of extending, in a unique way, automorphisms of $K(x)$ to $R(x)$, where R denotes a real closure of K. For example, it is proved that every archimedean field with a unique ordering has this extension property. We have in this direction the following.

PROPOSITION 2. [**8**]. *Let K be a field with the extension property. Then*

i) *If K has a unique ordering, then K is totally dense if and only if $K(x)$ is* DOP.

ii) *If $K(x)$ (where x is one transcendental over K) is* DOP, *then K is also* DOP.

As a by-product of our work on this topic we obtained irreducible polynomials positive over a field with a unique ordering which are not sums of squares, completing in this way a well-known example of Dubois [3] concerning Hilbert's 17th problem. More aspects and geometrical applications of the DOP condition for purely transcendental extensions of real closed fields are developed in [4]. For the moment let us state here two results based on the techniques which originated there.

PROPOSITION 3. [1] *For each clopen H of $F = \mathbf{R}(x_1, \ldots, x_n)$, there exists a finite extension $E \supseteq F$, such that $H = \operatorname{im}(\varepsilon_{E/F})$, where $\varepsilon_{E/F}$ is the restriction from Ω_E to Ω_F.*

This proposition, which is obtained with the help of Bertini's theorem, is related to some work of Motzkin [9] and to the paper of Elman, Lam, and Wadsworth [5] which turned out to be basic for the final formulation of the DOP condition.

The geometrical techniques that are involved in the proof of the above proposition are interpreted algebraically by means of a bijection between the set of clopen sets of Ω, (the order space of the function field of an arbitrary algebraic set or analytic germ X in $\mathbf{R}^n$) and the set of semialgebraic subsets (or semianalytic germs) of X that are regularly closed in X^*, the locus of maximum dimension of X, namely

$$\rho\left(\bigcup_{i=1}^{s} H(f_1^i, \ldots, f_r^i)\right) = \bigcup_{i=1}^{s} \overline{\{f_1^i > 0, \ldots, f_r^i > 0\}^*}.$$

With this notation we may state our next proposition, which is due to Ruiz [10] in the analytic case.

PROPOSITION 4. (*General solution to Hilbert's 17th problem for algebraic sets or analytic germs*). *Let H be a clopen of Ω and M a semianalytic germ or semialgebraic subset intersecting X^*. Let $P(M)$ be the set of elements of the ring A of analytic or polynomial functions on X, greater than or equal to zero over M. Then the following are equivalent*:

i) $P(M) = \bigcap_{\alpha \in H} \alpha \cap A$,

ii) $\bar{M} = \rho(H)$.

This result introduces us into the study of the order space of the function fields of algebraic varieties or analytic germs in the general (i.e., not necessarily equal to $\mathbf{R}^n$) case. When X is algebraic of arbitrary dimension we have only a few results. For example, its function field F can not be DOP if the group of automorphisms of F is finite [6]. Also we obtain a sufficient criterion for being DOP, stated in terms of a kind of normality for algebraic extensions of ordered fields, which applies to hypersurfaces in $\mathbf{R}^{n+1}$ that are well spread over $\mathbf{R}^n$[8]. Moreover, by suitably choosing a smooth and compact model X of F, we may assume that all the orders of Ω_F are centered at regular points. When dim $X = 1$, i.e., for curves, we have the following.

PROPOSITION 5. ([**6**], [**7**]). *Assume X in a smooth compact model of F. Then*

1) *Given* $P \in X$, *there exist exactly two orders centered at P, corresponding to each of the half branches of X at P,* $c_i\colon (0, 1) \to X$, $\alpha_{c_i} = \{f \in F \mid f$ *defined and positive on* $c(t)$, *for* $t > 0$ *and small*$\}$, $i = 1, 2$.

2) *If genus* $(X) \leqq 1$, Ω_F *is homogeneous, and, a fortiori, it is* DOP.

3) *If genus* $(X) \geqq 2$, Ω_F *is not* DOP.

For surfaces (dim $X = 2$) the situation gets more difficult. Using the analytic version of the positivestellensatz [**10**] and adapting the local parametrization theorem of analytic germs to their order spaces [**10**] we may reduce the study of orders on algebraic (or analytic germs of) surfaces to the formal regular case $\mathbf{R}[[X, Y]]$. For this ring an explicit description of all its orders is obtained in [**2**]. As a consequence we may state

PROPOSITION 6. [**2**] *Let X be an algebraic (or an analytic germ of) surface. Then for each order* α *of its function field, there exists* $c\colon (0, 1) \to X$ *analytic, such that* $\alpha = \{f \in F \mid f$ *is defined and positive on* $c(t)$, *for all small positive* $t\}$.

Thus we may say that, as in the case of curves, orders on a surface are always centered at half branches. For higher dimensions we conjecture this result to be also true, but for the moment we only have the following final proposition.

PROPOSITION R. *The set of orders centered at half branches is dense in* Ω_F.

REFERENCE

1. C. Andradas and J. M. Gamboa, *A note on projections of algebraic varieties*, preprint.

2. M. E. Alonso, J. M. Gamboa and J. M. Ruiz, *On orderings in real surfaces*, preprint.

3. D. W. Dubois, *Note on Artin's solution of Hilbert's 17th problem*. Bull. Amer. Math. Soc. **73** (1967), 540–541.

4. ——— and T. Recio, *Order extensions and real algebraic geometry*. Contemporary Math. **8** (1982), 265–288.

5. R. Elman, T. Y. Lam and A. Wadsworth, *Orderings under field extensions*, J. reine ang. Math. **306** (1979), 6–27.

6. J. M. Gamboa, *El espacio de ordenes de un cuerpo de funciones algebraicas en una variable*, Disseration. Univ. Compl. Madrid, 1982.

7. ——— and E. Bujalance, *Automorphisms groups of algebraic curves of* R^n *of genus two*, preprint.

8. ——— and T. Recio, *Ordered fields with the dense orbits property*, preprint.

9. T. S. Motzkin, *The real solution set of a system of algebraic inequalities*, Inequalities II., Ac. Press (1970), 251–254.

10. J. M. Ruiz, *Central orderings in fields of real meromorphic function germs*, preprint.

DEPARTAMENTO DE ALGEBRA FACULTAD DE CIENCIAS UNIVERSIDAD DE SANTANDER SANTANDER, SPAIN