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A POLYNOMIAL-TIME ALGORITHM FOR THE TOPOLOGICAL TYPE OF A REAL ALGEBRAIC CURVE —EXTENDED ABSTRACT

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Dedicated to the memory of Gus Efroymson

1. Introduction. Let $f(x, y, z)$ be a homogeneous polynomial with rational coefficients. Let C_f be the real projective curve defined by $f = 0$. It is well known [9] that if C_f is nonsingular, then it is a compact one-dimensional manifold, and so homeomorphic to a disjoint union of circles. A circle can have either a one-sided or two-sided imbedding in $\mathbf{R}P^2$; in the latter case it has both an interior (homeomorphic to a disk), and an exterior (homeomorphic to a Möbius strip). The two-sided components of C_f are called ovals. If f has even degree, then every component of C_f is an oval; if degree (f) is odd, every component except one is an oval.

Curves C_1 and C_2 have the same topological type if there is a homeomorphism φ: $\mathbf{R}P^2 \rightarrow \mathbf{R}P^2$ which maps C_1 onto C_2. Each oval of a nonsingular curve C_f is either inside or outside any other; the partial ordering of the ovals induced by this inclusion relation, together with the parity of the degree of f, determine the topological type of the curve.

We present an algorithm which, given $f(x, y, z)$ with rational coefficients, determines whether C_f is nonsingular, and if so, determines the ordering of its ovals.

2. Description of algorithm. We may assume that f is squarefree; (if not, we can replace f by its greatest squarefree divisor h, as $C_f = C_h$). The main step of the algorithm is construction of a cellular decomposition D_f of $\mathbf{R}P^2$ such that every component of C_f is a union of cells of D_f. The following description of D_f is produced: (1) a list of the pairs of adjacent cells (two cells are adjacent if their union is connected), and (2) a list of the cells contained in C_f. In the course of constructing D_f we determine if C_f has singularities, and if so, halt.

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Assuming C_f is nonsingular, the rest of the algorithm is straightforward. The reflexive transitive closure $\bar{R}$ of the adjacency relation is an equivalence relation; for a subset X of $\mathbf{R}P^2$, let $\bar{R}(X)$ denote the restriction of $\bar{R}$ to the cells of D_f which meet X. We construct the equivalence classes of $\bar{R}(C_f)$; (the union of) each class is a component of C_f. Let O be one of these components and K the corresponding class of $\bar{R}(C_f)$. We construct the equivalence classes of $\bar{R}$ (complement (O)); (the union of) each class is a component of complement (O). O is an oval if and only if there are two such classes; if there is only one, we do not process O further. Suppose there are two classes K_1 and K_2. Let V be the union of the cells of K_1 and let W be the union of the cells of K_2. We want to determine which of V and W is the interior (Int(O)) and which is the exterior (Ext(O)) of O. Now it can be shown that D_f gives $\mathbf{R}P^2$ the structure of a finite cell complex. As $O \cup V = \bar{V}$ and $O \cup W = \bar{W}$, $K \cup K_1$ and $K \cup K_2$ give $\bar{V}$ and $\bar{W}$ respectively the structure of subcomplexes of $\mathbf{R}P^2$. We can therefore compute the Euler characteristic χ of each of $\bar{V}$ and $\bar{W}$ using the formula

$$\chi = \alpha_0 - \alpha_1 + \alpha_2,$$

where α_i is the number of i-cells. But $\overline{\mathrm{Int}(O)}$ is homeomorphic to the closed disc and $\overline{\mathrm{Ext}(O)}$ is homeomorphic to the closed Möbius band. Thus $\chi(\overline{\mathrm{Int}(O)}) = 1$ and $\chi(\overline{\mathrm{Ext}(O)}) = 0$. Hence we can determine from χ which of $\bar{V}$ and $\bar{W}$ is $\overline{\mathrm{Int}(O)}$ and which is $\overline{\mathrm{Ext}(O)}$. After making this determination for all ovals of C_f, we know, for any oval, which cells of D_f are inside, which on, and which outside it. From this information the ordering of ovals follows.

The chief tool for constructing D_f is the cylindrical algebraic decomposition (cad) algorithm [2], [3], [4]. We use this algorithm to construct a cellular decomposition of the affine plane relative to the polynomial $g(x, y) = f(x, y, 1)$. (The algorithm would compute the discriminant $D(x)$ of $g(x, y)$, isolate the real roots of $D(x)$, and then "lift" the cells in the real line which are determined by the roots of $D(x)$ to cells in the plane which are determined by the locus of $g(x, y)$. Note that $D(x)$ is not the zero polynomial as $f(x, y, z)$, and hence $g(x, y)$, is squarefree.) Regarding the affine plane as a subset of $\mathbf{R}P^2$, we extend to a cellular decomposition of $\mathbf{R}P^2$ relative to $f(x, y, z)$ by appropriately partitioning the line at infinity into cells. Before applying the cad algorithm, we perform a linear change of coordinates of $\mathbf{R}P^2$, if necessary, to ensure that the curve C_f has only simple intersections with the line at infinity, and that C_f does not contain the point $[0, 1, 0]$. These properties facilitate the passage from the affine to the projective plane.

3. Concluding remarks. It can be shown that the computing time of our algorithm is $O(p(n, d))$, for some polynomial function p of the degree n of f and the size d of its coefficients. Polotovskii [7] gave a topological type algorithm for curves of even degree, but did not establish a bound for it. His approach is quite different from ours; he examines the curves $f(x, y, z) + \varepsilon z^n$, $(n = \text{degree}(f))$, for various small values of ε. As noted by Fuks and Delzell [5], one could get a topological type algorithm from Tarski's decision procedure for elementary algebra and geometry [8], but such an algorithm would have an exponential computing time bound. We have recently learned of an independently developed topological type algorithm by Gianni and Traverso [6], which has some resemblance to our method, but does not make use of cell complexes.

Because the time of our method depends almost entirely on the time required by the cad algorithm, and because the cad algorithm has recently been implemented [1], our algorithm appears to have some practical value. It could be used, for example, to study examples relating to Hilbert's 16th problem [9].

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