ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 4, Fall 1984

HILBERT'S PROBLEM 16 (B)

J.-P. FRANCOISE

Dedicated to the memory of Gus Efroymson

Hilbert's Problem 16 (B) is related to interesting questions of real analytic geometry and dynamical systems. Let L be the canonical fibre line bundle on $P^2_{\mathbf{C}}$ and m be a positive integer. A Pfaff algebraic form (P.A.F.) of degree m on $P^2_{\mathbf{C}}$ is an algebraic section of $T(P^2_{\mathbf{C}})^* \otimes L^{\otimes -(m+1)}$. Let $E^3_{\mathbf{C}}$ be the affine space of dimension 3; a P.A.F. is equivalent to the data of a 1-form $\omega = \omega_1 dx_1 + \omega_2 dx_2 + \omega_3 dx_3$, where ω_i are homogeneous of degree $m + 1$ and $\sum_{i=1}^3 x_i \omega_i = 0$. A P.A.F. defines a foliation with singularities of $P^2_{\mathbf{C}}$ whose leaves are open Riemann surface. They are called leaves of the P.A.F. An algebraic ordinary differential equation of degree m (A.O.D.E.) is a vector field $X = f(x, y)\, \partial/\partial x + g(x, y)\partial/\partial y$ on $E^2_{\mathbf{R}}$ whose components are two polynomials of degree m. The flow of X defined by equations

$$\text{(i)} \qquad \frac{dx}{dt} = f(x, y), \qquad \frac{dy}{dt} = g(x, y)$$

determines a foliation with singularities of $E^2_{\mathbf{R}}$. A limit cycle (L.C.) is a periodic solution of (i) isolated in the set of periodic solutions. After complexification and compactification, an A.O.D.E. gives a P.A.F. whose leaves are invariant by the involution $(x_1, x_2, x_3) \to (\bar{x}_1, \bar{x}_2, \bar{x}_3)$ of $P^2_{\mathbf{C}}$. Now we are ready to state the main.

PROBLEM. What is the relation between the number of L.C. and the degree m?

We have found a partial solution, i.e.,

THEOREM (Françoise-Pugh [2]). *For any integer m and real T, there exists $b(m, T)$ such that the number of* L.C. *of period less than T of an* A.O.D.E. *of degree m is less than $b(m, T)$.*

A key theorem in the local-real study is Bautin's theorem generalized to arbitrary m in [1]. Let $\mathscr{B}(m)$ be the set of A.O.D.E. of degree m with $f(x, y) = \lambda x - y + \cdots$, and $g(x, y) = x + \lambda y + \cdots$.

THEOREM (Bautin). *Let* $X_0 \in \mathscr{B}(2)$ *be a center at* $O \in E^2_{\mathbf{R}}$; *then the number of* L.C. *which may appear in an arbitray small neighborhood of* $O \in E^2_{\mathbf{R}}$ *for an arbitrary perturbation* $X_0 \to X \in \mathscr{B}(2)$ *is less than* 3.

We used the analytic geometry statement that if A, B are subanalytic sets, and f: $A \to B$ is a sub-analytic proper morphism, then for any point $y_0 \in B$, there exist an integer N and a neighborhood $U(y_0)$ such that the number of connected components of $f^{-1}(y)$, $y \in U(y_0)$ is less than N.

Of course we had to appeal to dynamical systems, whose methods begin with Poincaré-Bendixson, and yield proofs that periodic solutions may accumulate on: i) singular points, ii) periodic solution, and iii) graphics. A graphic is a union of singular points and adherent trajectories. L.C. cannot accumulate on a periodic solution because a periodic solution has an analytic first-return map. A graphic or a singular point limit of periodic solutions has a first-return map which may fail to be analytic a priori. Hence there is no simple way to prove that an A.O.D.E. has a finite number of L.C. Let us say that an A.O.D.E. is generic if its flow is Kupka-Smale. Then a uniform bound for the number of L.C. of generic A.O.D.E. of fixed degree m implies the existence of a uniform bound for all A.O.D.E. of degree m (Pugh). A similar idea in the complex version was one of the key ideas of Petrowski-Landis's methods.

REFERENCES

1. J.-P. Francoise, *Cycles limites, étude locale,* Preprint I.H.E.S., 83 M 13.

2. J.-P. Francoise et C. C. Pugh, *Déformations de cycles limites*, Preprint I.H.E.S., 82 M 62.

35 ROUTE DE CHARTRES, 91440 BURES SUR YVETTE, FRANCE