ORDERED FIELDS SATISFYING ROLLE'S THEOREM

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Dedicated to the memory of Gus Efroymson

Let F be an ordered field. F is called a Rolle field if whenever a polynomial over F has roots a and b in F, then its formal derivative has a root in F between a and b.

THEOREM. *F is a Rolle field if and only if F is a Henselian field with real closed residue field and with a value group divisible by all odd integers.*

The above criterion for a Rolle field is independent of the ordering on the field. Indeed, if a polynomial over a Rolle field F has roots a and b in F, then its derivative has a root in F which is between a and b simultaneously with respect to all the orderings on F. Something similar, but much weaker, can be proved for arbitrary fields. The fields which satisfy Rolle's theorem for all rational functions turn out to be exactly the real closed fields; the situation for "integral" functions over F is unclear to us.

Rolle fields have many nice properties, e.g., they are hereditarily pythagorean and superpythagorean. The following result, along with the "reduced theory" of quadratic forms, shows that the Witt ring of F, modulo its nil radical, can be explicitly described as a subdirect product of the Witt rings of certain Rolle field extensions of F.

PROPOSITION. *The minimal algebraic extensions of F which are Rolle fields are precisely the maximal algebraic extensions of F which are Henselian with real closed residue class field and which have Witt rings naturally isomorphic to the Witt ring of F modulo the preorder associated with a place from F into the real numbers.*

Our last result is motivated by a construction of Artin's of subfields of an algebraically closed field which are maximal with respect to the exclusion of an element of the algebraically closed field. Such fields clearly have exactly one minimal proper extension in the algebraically closed field.

THEOREM. *If F has a finite number, say n, of orderings, then F is a Rolle*

field if and only if F has exactly $2n$-1 *minimal proper extensions in some algebraic closure of F and F admits exactly one place into the real numbers.*

The fields with n orderings, $2n$-1 minimal proper algebraic extensions, and more than one place into the reals are also characterized.

A detailed discussion of these topics will appear in the Illinois Journal of Mathematics in a paper with the same title as this one.

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