THE CENTRALIZER OF THE LAGUERRE POLYNOMIAL SET

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1. Introduction. By a polynomial set (p.s.) we mean a sequence \( P = \{P_0(x), P_1(x), P_2(x), \ldots \} \) of polynomials in which \( P_0(x) \neq 0 \) and \( P_n(x) \) is of exact degree \( n \). In this work we shall be interested in sets (or classes) whose elements are themselves polynomial sets. This point of view is not new. Appell [2] considered the class \( \mathcal{A} \) of Appell polynomials \( A = \{A_n(x)\} \) whose generating function is

\[
A(t) = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!}.
\]

The Sheffer class \( \mathcal{S} \) [6] is the class of all p.s. \( S = \{S_n(x)\} \) for which

\[
A(t) e^{xH(t)} = \sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!}.
\]

Similarly the Boas-Buck class \( \mathcal{B} \) consists of all p.s. \( B \) for which [3]

\[
A(t) \Phi(xH(t)) = \sum_{n=0}^{\infty} \phi_n B_n(x) t^n,
\]

where in these formulas \( A(t), H(t) \) and \( \Phi(t) \) are formal power series such that \( A(0) \neq 0, H(0) = 0 \) but \( H'(0) \neq 0 \), and \( \Phi(t) = \phi_0 + \phi_1 t + \phi_2 t^2 + \cdots \) with \( \phi_k \neq 0 \) for all \( k \geq 0 \). (1.1) is obtained when \( H(t) = t \) and \( \Phi(t) = e^t \).

Many of the well known p.s. are included in one or more of the above classes. For example, the Hermite p.s. is in \( \mathcal{A} \) as well as in \( \mathcal{S} \). The Laguerre p.s. \( L(\alpha) \) is in \( \mathcal{S} \). Other examples are the Abel, the Meixner, the Bernoulli, and the Boole polynomial sets.

Appell [2], Sheffer [6] as well as Rota, Kahaner and Odlysko [5] (see also [4]) gave sets of polynomials (\( \mathcal{A} \) in [2], \( \mathcal{S} \) in [4], [5], [6]) an algebraic structure by defining multiplication in the following manner.

Let \( P = \{P_n(x)\} \) and \( Q = \{Q_n(x)\} \) be two elements of the set under consideration. Let, furthermore, \( P_n(x) = \sum_{k=0}^{n} p_{nk} x^k \) and \( Q_n(x) = \sum_{k=0}^{n} q_{nk} x^k \) where \( p_{nk}, q_{nk} \) are functions of \( n \) and \( k \) respectively. The product of \( P \) and \( Q \) is defined by

\[
P \ast Q = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} p_{nk} q_{nk} \right) x^n.
\]

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\[ \sum_{k=0}^{n} q_{nk}x^k \] for all \( n \). Then the (umbral) product \( R \equiv PQ \) is defined as the p.s. for which
\[ R_n(x) = P_n(Q) = \sum_{k=0}^{n} p_{nk}Q_k(x) \quad n = 0, 1, 2, \ldots . \]

It is clear that \( \pi \), the set of all p.s., with this multiplication is a group (non-commutative) in which the identity is \( I = \{ x^n, n = 0, 1, 2, \ldots \} \).

In [1] the present authors characterized the centralizer, \( C_{\pi}(L(\alpha)) \), of the Laguerre p.s. in the Boas-Buck group \( B \).

If we recall that \( B \subset \pi \) it becomes natural to characterize elements of \( C_{\pi}(L(\alpha)) \) the centralizer of \( L(\alpha) \) in \( \pi \).

As we shall see that, perhaps due to the fact that \( \pi \) lacks the nice structure that \( B \) has, this problem is somewhat more difficult than the problem considered in [1]. To our surprise the Euler numbers and polynomials played a prominent role in the solution (which did not arise in [1]).

2. Preliminaries. Let us recall the Euler polynomials

\[ \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \]

and the related tangent numbers \( C_0 = 1 \) and

\[ \tanh t = -\sum_{n=1}^{\infty} C_n \frac{t^n}{n!} \]

so that \( C_{2n} = 0 \) if \( n > 0 \) and \( C_{2n+1} = 2^{2n+1} E_{2n+1}(0) \). We shall abbreviate \( C_{2n+1}/2^{2n+1} \) by \((-1)^{n-1} \alpha_n \) \( (n = 0, 1, 2, \ldots) \).

It follows that in term of the Bernoulli numbers we have \( C_n = 1 + 2^n(1 - 2^n)B_n/n \quad (n \geq 1) \), and that

\[ (2.3) \quad C_n + (2 + C)^n = \begin{cases} 0 & (n > 0) \\ 2 & (n = 0), \end{cases} \]

and

\[ (2.4) \quad x^n = \frac{1}{2} \{ E_n(x + 1) + E_n(x) \}. \]

In (2.3) \((2 + C)^n\) is to be expanded by the binomial theorem and \( C^k \) be replaced by \( C_k \).

In this work we shall need the following lemmas:

**Lemma 1.** We have for \( N = 1, 2, 3, \ldots \)

\[ \alpha_N = -\frac{1}{2} \left[ 1 + \sum_{j=0}^{N-1} (-1)^{j-1} \alpha_j \binom{2N + 1}{2j + 1} \right], \]

\[ (2.5) \quad -1 = \sum_{j=0}^{N-1} (-1)^{j-1} \alpha_j \binom{2N}{2j + 1}. \]
These formulas follow from (2.3) with \( n = 2N + 1 \) and \( n = 2N \) respectively.

**Lemma 2.** For \( 0 \leq r \leq 2m + 1 \) we have

\[
-\frac{1}{2} \sum_{k \geq 0} \left\{ (-1)^k \binom{2m + 1 - r}{k} + \binom{r}{k} \right\} \frac{C_{2m+1-k}}{2^{2m+1-k}} = \delta_{0r}.
\]

**Lemma 3.** For \( 0 \leq r \leq 2m \) we have

\[
\sum_{k \geq 0} \left\{ \binom{r}{2k + 1} - \binom{2m - r}{2k + 1} \right\} \frac{C_{2m-2k-1}}{2^{2m-2k-1}} = \delta_{0r}.
\]

To prove (2.6) and (2.7) let \( f(x) \) be the polynomial defined by

\[ f(x) = x^r(x - 1)^{m-r} \quad (0 \leq r \leq m). \]

Then

\[ f(x) + f(x + 1) = \sum_{k \geq 0} \left\{ (-1)^k \binom{m - r}{k} + \binom{r}{k} \right\} x^{m-k}. \]

This, using (2.4), gives

\[ f(x) + f(x + 1) = \frac{1}{2} \sum_{k \geq 0} \left\{ (-1)^k \binom{m - r}{k} + \binom{r}{k} \right\} \left[ E_{m-k}(x) + E_{m-k}(x + 1) \right]. \]

But if \( g(x) \) is a polynomial such that \( g(x) + g(x + 1) \equiv 0 \) then \( g(x) \equiv 0 \). Hence we get

\[ f(x) \equiv \frac{1}{2} \sum_{k \geq 0} \left\{ (-1)^k \binom{m - r}{k} + \binom{r}{k} \right\} E_{m-k}(x). \]

Now putting \( m \) even or odd and \( x = 0 \) we get either (2.7) or (2.6).

3. The Centralizer \( C_\pi(L^{(\alpha)}) \). Let \( P = \{P_n(x)\} \) be an arbitrary p.s. in \( \pi \) and write for \( n = 0, 1, 2, \ldots \)

\[
P_n(x) = \sum_{k=0}^n \binom{n}{k} \frac{(1 + \alpha)_n}{(1 + \alpha)_k} p_{n,k} x^k \quad (p_{n,n} = \beta_n \neq 0).
\]

Let \( L = \{L_n^{(\alpha)}(x)\} \) be the Laguerre p.s. defined by

\[
L_n^{(\alpha)}(x) = \sum_{k=0}^n \binom{n}{k} \frac{(1 + \alpha)_n}{(1 + \alpha)_k} (-x)^k, \quad n = 0, 1, 2, \ldots .
\]

Our problem is, therefore, to determine \( p_{n,k} \) in (3.1) so that \( PL = LP \).

In this section we prove our main theorem, shown here.

**Theorem.** A p.s. \( P \in C_\pi(L^{(\alpha)}) \) if and only if

(a) \[ p_{n,n-2m-1} = \sum_{f=0}^m \frac{C_{2f+1}}{2^{2f+1}} \binom{2m + 1}{2f + 1} \nabla^{2f+1} p_{n,n-2m+2f} \]
and  
(b) \( p_{n,n-2k} \) are arbitrary with \( p_{n,n} \neq 0 \).

Here \( \nabla \) is the backward difference operator acting on \( n \): \( \nabla f(n) = f(n) - f(n - 1) \).

**Proof.** We first note that \( PL = LP \) is equivalent to requiring that for \( j = 0, 1, 2, \ldots, n \) and \( n \geq 0 \) we have

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} p_{k,n} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} p_{n,k}.
\]

(3.3)

We next see that by putting \( j = n, n - 1, \) etc. in (3.3) we get that \( p_{n,n} \) is arbitrary, that \( p_{n,n-1} = -(1/2)(\beta_n - \beta_{n-1}) \) for \( n = 1, 2, 3, \ldots \) so that (3.3) determines uniquely \( p_{n,n-2m-1} \), and that \( p_{n,n-2m} \) remains arbitrary.

To find the general solution of (3.3) we note that (3.3) can be rewritten in the form

\[
\sum_{k=0}^{s} (-1)^k \binom{s}{k} p_{n+k-s,n-s} = \sum_{k=0}^{s} \binom{s}{k} p_{n,n-k} \quad (0 \leq s \leq n)
\]

which implies, for \( s = 2m \) \( (m = 1, 2, \cdots) \),

\[
\sum_{k=0}^{m} \binom{2m}{2k} \{p_{n-2m+2k,n-2m} - p_{n,n-2k}\} = \sum_{k=0}^{m} \binom{2m}{2k+1} \{p_{n-2m+2k+1,n-2m} + p_{n,n-2k-1}\}.
\]

(3.4)

and for \( s = 2m + 1 \),

\[
2p_{n,n-2m-1} = \sum_{k=0}^{m} \binom{2m+1}{2k} \{p_{n-2m+2k-1,n-2m-1} - p_{n,n-2k}\}
\]

(3.5)

We now show that (3.4) and (3.5) are satisfied if \( p_{n,n-2m} \) are arbitrary and

\[
p_{n,n-2m-1} = \sum_{j=0}^{m} (-1)^{j-1} \alpha_j \binom{2m+1}{2j+1} \nabla^{2j+1} p_{n,n-2m+2j}.
\]

(3.6)

Indeed if we substitute (3.6) in the right hand side (RHS) of (3.4) we get

\[
\text{RHS} = \sum_{k=0}^{m} \binom{2m}{2k+1} \sum_{j=0}^{k} (-1)^{j-1} \alpha_j \binom{2k+1}{2j+1}
\]

\[
\{\nabla^{2j+1} p_{n-2m+2k+1,n-2m+2j+1} + \nabla^{2j+1} p_{n,n-2k+2j}\}.
\]

(3.7)

Since \( \nabla^{2j+1} f(n) = \sum_{r=0}^{j+1} (-1)^{r} \binom{2j+1}{r} f(n-r) \) then the above expression (3.7) is a sum of terms of the form \( p_{n-\mu,n-\mu-2k} \). To show that (3.6) satisfies
(3.4) we must show that the coefficient of $P_{n-\mu,n-\mu-2k}$ is $-\binom{2m}{\mu}$ if $\mu = 0$, is $\binom{2m}{\mu}$ if $\mu = 2m - 2k$ and is zero if $\mu \neq 0$ or $\mu \neq 2m - 2k$.

For example in the latter case, the coefficient of $P_{n-\mu,n-\mu-2k}$ in (3.7) is a multiple of

$$\sum_{j=0}^{m-k-1} (-1)^{j-1} \alpha_f \left( \binom{2m-2k}{\mu} - \binom{2m-2k-2j}{\mu} \right)$$

which is zero by Lemma 3.

Similarly $\mu = 0$ and $\mu = 2m - 2k$ follows from Lemma 1.

Formula (3.5) can be seen to be satisfied by (3.6). This time we need to use Lemmas 1 and 2 and also we must show that that coefficient of $P_{n-\mu,n-\mu-2k}$ is

$$\sum_{j=0}^{m-k-1} \left[ \binom{2m-2k+1-\mu}{2m-2k-2j} + \binom{2m-2k-2j}{\mu} \right] \alpha_f (-1)^{j-1} = 0.$$ 

This formula is a consequence of Lemma 2. This finishes the proof of the main theorem.

Formula 3.6 can be written operationally using the Euler polynomials $E_n(x)$. To do this let $\eta f(n, m) = f(n - 1, m)$ and $\mu f(n, m) = f(n, m - 1)$ so that $\nabla f(n, n) = (1 - \eta \mu) f(n, n).$ We get

$$P_{n,n-2m-1} = (1 - \eta \mu)^{2m+1} E_{2m+1} \left( \frac{\mu}{1 - \eta \mu} \right) \beta_n.$$ 

where we have again written $\beta_n = p_{n,n}$.

4. Special Cases. (a) $L^{(\alpha)}$ commutes with itself. This case follows when $P_{n,n-k} = (-1)^{n-k}$. Formula (3.6) can be seen to be satisfied since it implies that

$$P_{n,n-2m-1} = (-1)^n \{(1 + C)^{2m+1} - 1\} = (-1)^{n-1}.$$ 

This is easily seen because $(1 + C)^{2m+1} = 0$ for $m = 0, 1, 2, \cdots$.

(b) Let $\beta_n = p_{n,n} = n + \alpha$ and let $p_{n,n-2k} = 0$ for $k > 0$. Then easy calculations show that

$$P_n(x) = (n + \alpha)x^n - \frac{1}{2} n(n + \alpha)x_{n-1}.$$ 

The commutativity implies the known recurrence formula for the Laguerre polynomials $L_n^{(\alpha)}(x) - n L_{n-1}^{(\alpha)}(x) = L_n^{(\alpha - 1)}(x)$. The polynomial set $\{P_n(x)\}$ is not of the Boas-Buck type.

(c) The “symmetric subgroup” $\Sigma$. A p.s. $P$ is said to be symmetric if $P_n(-x) = (-1)^n P_n(x)$. It is easy to argue that the class of all symmetric p.s. $\Sigma$ with umbral composition forms a subgroup of $\pi$. We ask the question, what are the elements of $C_{\Sigma}(L^{(\alpha)})$?
To answer this question we note first that \( P \in \sum \Rightarrow p_{n,n-2m-1} = 0 \) for \( m = 0, 1, \ldots, \lceil n-1/2 \rceil \).

Putting \( m = 0 \) in (3.6) shows that \( p_{n,n} \) is independent of \( n \). It now follows by induction on \( m \) that \( p_{n,n-2m} = \gamma_{2m} \) is independent of \( n \). Thus such polynomial sets are given by

\[
P_{n}^{(r)}(x) = \sum_{k \geq 0} \binom{n}{2k} \frac{(1 + \alpha)_n}{(1 + \alpha)_{n-2k}} \gamma_{2k} x^{n-2k}.
\]

Furthermore one can easily show that \( P_{n}^{(r)}(P^{(u)}) = P_{n}^{(u)}(P^{(r)}) = P_{n}^{(\omega)}(x) \) where \( \delta_{2n}/(2n)! = \sum_k \binom{2k}{k} \mu_{2k} \gamma_{2n-2k} \) so that we have the following result.

**Theorem.** \( C_\pi(L^{(\alpha)}) \) is a commutative subgroup of \( C_\pi(L^{(\alpha)}) \).

We also remark that elements of \( C_\pi(L^{(\alpha)}) \) are related to Brenke polynomials since we can show that

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!(1 + \alpha)_n} P_{n}^{(r)}(x) = \left( \sum_{n=0}^{\infty} \gamma_{2n} \frac{t^n}{(2n)!} \right) \cdot 0F_1(-; 1 + \alpha; xt)
\]

where \( 0F_1(-; 1 + \alpha; u) = \sum_{n=0}^{\infty} u^n/(1 + \alpha)_n \).

The case \( r = \{1\} \) gives \( P_{n}^{(1)}(x) = 1/2\{(-1)^n L_n^{(\alpha)}(x) + L_n^{(\alpha)}(-x)\} \).

(d) As remarked earlier Appell showed that \( \mathcal{A} \) is a subgroup of \( \pi \). To determine \( C_{\mathcal{A}}(L) \) we see that if \( P \in \mathcal{A} \) then \( P_n(x) = \sum (\xi) a_n x^k \). Hence \( p_{n,n-k} = (1 + \alpha)_{n-k}/(1 + \alpha) a_k \) where \( a_k \) is independent of \( n \). Since \( n, n-2k \) are arbitrary so are \( a_{2k} \). Using (3.6) we can show that

\[
a_{2m+1} = - (2m)! \sum_{k=0}^{m} \binom{2m + 1}{2k + 1} \frac{C_{2k+1}}{2^{2k+1}} \frac{(2m - k)}{(2m - k)!} a_{2m-2k}.
\]

We can also show that such p.s. are generated by

\[
e^{E(\log(1-t)+xt)} = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!}
\]

where \( E(t) \) is an arbitrary even function of \( t \).

**References**


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