ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 3, Summer 1984

THE CENTRALIZER OF THE LAGUERRE POLYNOMIAL SET

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1. Introduction. By a polynomial set (p.s.) we mean a sequence $P = \{P_0(x), P_1(x), P_2(x), \cdots\}$ of polynomials in which $P_0(x) \neq 0$ and $P_n(x)$ is of exact degree n. In this work we shall be interested in sets (or classes) whose elements are themselves polynomial sets. This point of view is not new. Appell [2] considered the class $\mathscr{A}$ of Appell polynomials $A = \{A_n(x)\}$ whose generating function is

$$A(t)e^{xt} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}. \tag{1.1}$$

The Sheffer class $\mathscr{S}$ [6] is the class of all p.s. $S = \{S_n(x)\}$ for which

$$A(t)e^{xH(t)} = \sum_{n=0}^{\infty} S_n(x)\frac{t^n}{n!}. \tag{1.2}$$

Similarly the Boas-Buck class $\mathscr{B}$ consists of all p.s. B for which [3]

$$A(t)\Phi(xH(t)) = \sum_{n=0}^{\infty} \phi_n B_n(x)t^n, \tag{1.3}$$

where in these formulas $A(t)$, $H(t)$ and $\Phi(t)$ are formal power series such that $A(0) \neq 0$, $H(0) = 0$ but $H'(0) \neq 0$, and $\Phi(t) = \phi_0 + \phi_1 t + \phi_2 t^2 + \cdots$ with $\phi_k \neq 0$ for all $k \geqq 0$. (1.1) is obtained when $H(t) = t$ and $\Phi(t) = e^t$.

Many of the well known p.s. are included in one or more of the above classes. For example, the Hermite p.s. is in $\mathscr{A}$ as well as in $\mathscr{S}$. The Laguerre p.s. $L^{(\alpha)}$ is in $\mathscr{S}$. Other examples are the Abel, the Meixner, the Bernoulli, and the Boole polynomial sets.

Appell [2], Sheffer [6] as well as Rota, Kahaner and Odlysko [5] (see also [4]) gave sets of polynomials ($\mathscr{A}$ in [2], $\mathscr{S}$ in [4], [5], [6]) an algebraic structure by defining multiplication in the following manner.

Let $P = \{P_n(x)\}$ and $Q = \{Q_n(x)\}$ be two elements of the set under consideration. Let, furthermore, $P_n(x) = \sum_{k=0}^{n} p_{nk}x^k$ and $Q_n(x) =$

Received by the editors on June 15, 1983.

$\sum_{k=0}^{n} q_{nk}x^k$ for all n. Then the (umbral) product $R \equiv PQ$ is defined as the p.s. for which $R_n(x) = P_n(Q) = \sum_{k=0}^{n} p_{nk}Q_k(x)$ $n = 0, 1, 2, \ldots$.

It is clear that π, the set of all p.s., with this multiplication is a group (non-commutative) in which the identity is $I = \{x^n, n = 0, 1, 2, \ldots\}$.

In [1] the present authors characterized the centralizer, $C_{\mathscr{B}}(L^{(\alpha)})$, of the Laguerre p.s. in the Boas-Buck group $\mathscr{B}$.

If we recall that $\mathscr{B} \subset \pi$ it becomes natural to characterize elements of $C_\pi(L^{(\alpha)})$ the centralizer of $L^{(\alpha)}$ in π.

As we shall see that, perhaps due to the fact that π lacks the nice structure that $\mathscr{B}$ has, this problem is somewhat more difficult than the problem considered in [1]. To our surprise the Euler numbers and polynomials played a prominent role in the solution (which did not arise in [1]).

2. Preliminaries. Let us recall the Euler polynomials

$$\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \tag{2.1}$$

and the related tangent numbers $C_0 = 1$ and

$$\tanh t = - \sum_{n=1}^{\infty} C_n \frac{t^n}{n!} \tag{2.2}$$

so that $C_{2n} = 0$ if $n > 0$ and $C_{2n+1} = 2^{2n+1}E_{2n+1}(0)$. We shall abbreviate $C_{2n+1}/2^{2n+1}$ by $(-1)^{n-1}\alpha_n$ $(n = 0, 1, 2, \ldots)$.

It follows that in term of the Bernoulli numbers we have $C_n = 1 + 2^n(1 - 2^n)B_n/n$ $(n \geqq 1)$, and that

$$C_n + (2 + C)^n = \begin{cases} 0 & (n > 0) \\ 2 & (n = 0), \end{cases} \tag{2.3}$$

and

$$x^n = \frac{1}{2}\{E_n(x + 1) + E_n(x)\}. \tag{2.4}$$

In (2.3) $(2 + C)^n$ is to be expanded by the binomial theorem and C^k be replaced by C_k.

In this work we shall need the following lemmas:

LEMMA 1. *We have for* $N = 1, 2, 3, \ldots$

$$\begin{aligned} \alpha_N &= -\frac{1}{2}\left\{1 + \sum_{j=0}^{N-1} (-1)^{j-1}\alpha_j \binom{2N+1}{2j+1}\right\}, \\ -1 &= \sum_{j=0}^{N-1} (-1)^{j-1}\alpha_j \binom{2N}{2j+1}. \end{aligned} \tag{2.5}$$

These formulas follow from (2.3) with $n = 2N + 1$ and $n = 2N$ respectively.

LEMMA 2. *For* $0 \leqq r \leqq 2m + 1$ *we have*

$$-\frac{1}{2}\sum_{k\geqq 0}\left\{(-1)^k\binom{2m+1-r}{k} + \binom{r}{k}\right\}\frac{C_{2m+1-k}}{2^{2m+1-k}} = \delta_{0r}. \tag{2.6}$$

LEMMA 3. *For* $0 \leqq r \leqq 2m$ *we have*

$$\sum_{k\geqq 0}\left\{\binom{r}{2k+1} - \binom{2m-r}{2k+1}\right\}\frac{C_{2m-2k-1}}{2^{2m-2k-1}} = \delta_{0r}. \tag{2.7}$$

To prove (2.6) and (2.7) let $f(x)$ be the polynomial defined by

$$f(x) = x^r(x-1)^{m-r} \qquad (0 \leqq r \leqq m).$$

Then

$$f(x) + f(x+1) = \sum_{k\geqq 0}\left\{(-1)^k\binom{m-r}{k} + \binom{r}{k}\right\}x^{m-k}.$$

This, using (2.4), gives

$$f(x)+f(x+1) = \frac{1}{2}\sum_{k\geqq 0}\left\{(-1)^k\binom{m-r}{k}+\binom{r}{k}\right\}\{E_{m-k}(x)+E_{m-k}(x+1)\}.$$

But if $g(x)$ is a polynomial such that $g(x) + g(x+1) \equiv 0$ then $g(x) \equiv 0$. Hence we get

$$f(x) \equiv \frac{1}{2}\sum_{k\geqq 0}\left\{(-1)^k\binom{m-r}{k} + \binom{r}{k}\right\}E_{m-k}(x).$$

Now putting m even or odd and $x = 0$ we get either (2.7) or (2.6).

3. The Centralizer $\mathbf{C}_\pi(\mathbf{L}^{(\alpha)})$. Let $P = \{P_n(x)\}$ be an arbitrary p.s. in π and write for $n = 0, 1, 2, \ldots$

$$P_n(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{(1+\alpha)_n}{(1+\alpha)_k}p_{n,k}x^k \qquad (p_{n,n} = \beta_n \neq 0). \tag{3.1}$$

Let $L = \{L_n^{(\alpha)}(x)\}$ be the Laguerre p.s. defined by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{(1+\alpha)_n}{(1+\alpha)_k}(-x)^k, \qquad n = 0, 1, 2, \ldots. \tag{3.2}$$

Our problem is, therefore, to determine $p_{n,k}$ in (3.1) so that $PL = LP$.
In this section we prove our main theorem, shown here.

THEOREM. *A.p.s.* $P \in C_\pi(L^{(\alpha)})$ *if and only if*

$$p_{n,n-2m-1} = \sum_{j=0}^{m}\frac{C_{2j+1}}{2^{2j+1}}\binom{2m+1}{2j+1}\nabla^{2j+1}p_{n,n-2m+2j} \tag{a}$$

and

(b) $p_{n,n-2k}$ *are arbitrary with* $p_{n,n} \neq 0$.

Here ∇ is the backward difference operator acting on n: $\nabla f(n) = f(n) - f(n-1)$.

PROOF. We first note that $PL = LP$ is equivalent to requiring that for $j = 0, 1, 2, \ldots, n$ and $n \geqq 0$ we have

$$\sum_{k=j}^{n} (-1)^k \binom{n-j}{k-j} p_{k,j} = \sum_{k=j}^{n} (-1)^j \binom{n-j}{k-j} p_{n,k}. \tag{3.3}$$

We next see that by putting $j = n, n-1$, etc. in (3.3) we get that $p_{n,n}$ is arbitrary, that $p_{n,n-1} = -(1/2)(\beta_n - \beta_{n-1})$ for $n = 1, 2, 3, \cdots$ so that (3.3) determines uniquely $p_{n,n-2m-1}$, and that $p_{n,n-2m}$ remains arbitrary.

To find the general solution of (3.3) we note that (3.3) can be rewritten in the form

$$\sum_{k=0}^{s} (-1)^k \binom{s}{k} p_{n+k-s,n-s} = \sum_{k=0}^{s} \binom{s}{k} p_{n,n-k} \qquad (0 \leqq s \leqq n)$$

which implies, for $s = 2m$ $(m = 1, 2, \cdots)$,

$$\begin{aligned} &\sum_{k=0}^{m-1} \binom{2m}{2k} \{p_{n-2m+2k,n-2m} - p_{n,n-2k}\} \\ &= \sum_{k=0}^{m-1} \binom{2m}{2k+1} \{p_{n-2m+2k+1,n-2m} + p_{n,n-2k-1}\}, \end{aligned} \tag{3.4}$$

and for s odd, $s = 2m + 1$,

$$\begin{aligned} 2p_{n,n-2m-1} &= \sum_{k=0}^{m} \binom{2m+1}{2k} \{p_{n-2m+2k-1,\, n-2m-1} - p_{n,n-2k}\} \\ &\quad - \sum_{k=0}^{m-1} \binom{2m+1}{2k+1} \{p_{n-2m+2k,n-2m-1} + p_{n,n-2k-1}\}. \end{aligned} \tag{3.5}$$

We now show that (3.4) and (3.5) are satisfied if $p_{\nu,\nu-2\mu}$ are arbitrary and

$$p_{n,n-2m-1} = \sum_{j=0}^{m} (-1)^{j-1} \alpha_j \binom{2m+1}{2j+1} \nabla^{2j+1} p_{n,n-2m+2j}. \tag{3.6}$$

Indeed if we substitute (3.6) in the right hand side (RHS) of (3.4) we get

$$\begin{aligned} \text{RHS} = &\sum_{k=0}^{m-1} \binom{2m}{2k+1} \sum_{j=0}^{k} (-1)^{j-1} \alpha_j \binom{2k+1}{2j+1} \\ &\{\nabla^{2j+1} p_{n-2m+2k+1,n-2m+2j+1} + \nabla^{2j+1} p_{n,n-2k+2j}\}. \end{aligned} \tag{3.7}$$

Since $\nabla^{2j+1} f(n) = \sum_{r=0}^{2j+1} (-1)^r \binom{2j+1}{r} f(n-r)$ then the above expression (3.7) is a sum of terms of the form $p_{n-\mu,n-\mu-2k}$. To show that (3.6) satisfies

(3.4) we must show that the coefficient of $p_{n-\mu,n-\mu-zk}$ is $-\binom{2m}{2k}$ if $\mu = 0$, is $\binom{2m}{2k}$ if $\mu = 2m - 2k$ and is zero if $\mu \neq 0$ or $\mu \neq 2m - 2k$.

For example in the latter case, the coefficient of $p_{n-\mu,n-\mu-2k}$ in (3.7) is a multiple of

$$\sum_{j=0}^{m-k-1} (-1)^{j-1}\alpha_j\left\{\binom{\mu}{2m-2k-2j-1} - \binom{2m-2k-\mu}{2m-2k-2j-1}\right\}$$

which is zero by Lemma 3.

Similarly $\mu = 0$ and $\mu = 2m - 2k$ follows from Lemma 1.

Formula (3.5) can be seen to be satisfied by (3.6). This time we need to use Lemmas 1 and 2 and also we must show that that coefficient of $p_{n-\mu,\,n-\mu-2k}$ is

$$\sum_{j\geq 0}\left[\binom{2m-2k+1-\mu}{2m-2k-2j} + \binom{\mu}{2m-2k-2j}\right]\alpha_j(-1)^{j-1} = 0. \tag{3.8}$$

This formula is a consequence of Lemma 2. This finishes the proof of the main theorem.

Formula 3.6 can be written operationally using the Euler polynomials $E_n(x)$. To do this let $\eta f(n, m) = f(n-1, m)$ and $\mu f(n, m) = f(n, m-1)$ so that $\nabla f(n, n) = (1 - \eta\mu)f(n, n)$. We get

$$p_{n,n-2m-1} = (1 - \eta\mu)^{2m+1} E_{2m+1}\left(\frac{\mu}{1-\eta\mu}\right)\cdot \beta_n.$$

where we have again written $\beta_n = p_{n,n}$.

4. Special Cases. (a) $L^{(\alpha)}$ commutes with itself. This case follows when $p_{n,n-k} = (-1)^{n-k}$. Formula (3.6) can be seen to be satisfied since it implies that

$$p_{n,n-2m-1} = (-1)^n\{(1 + C)^{2m+1} - 1\} = (-1)^{n-1}.$$

This is easily seen because $(1 + C)^{2m+1} = 0$ for $m = 0, 1, 2, \cdots$.

(b) Let $\beta_n = p_{n,n} = n + \alpha$ and let $p_{n,n-2k} = 0$ for $k > 0$. Then easy calculations show that

$$P_n(x) = (n + \alpha)x^n - \frac{1}{2}n(n + \alpha)x_{n-1}.$$

The commutativity implies the known recurrence formula for the Laguerre polynomials $L_n^{(\alpha)}(x) - n\,L_{n-1}^{(\alpha)}(x) = L_n^{(\alpha-1)})\,(x)$. The polynomial set $\{P_n(x)\}$ is not of the Boas-Buck type.

(c) The "symmetric subgroup" Σ. A p.s. P is said to be symmetric if $P_n(-x) = (-1)^n P_n(x)$. It is easy to argue that the class of all symmetric p.s. Σ with umbral composition forms a subgroup of π. We ask the question, what are the elements of $C_\Sigma(L^{(\alpha)})$?

To answer this question we note first that $P \in \sum \Rightarrow p_{n,n-2m-1} = 0$ for $m = 0, 1, \ldots, [n-1/2]$.

Putting $m = 0$ in (3.6) shows that $p_{n,n}$ is independent of n. It now follows by induction on m that $p_{n,n-2m} = \gamma_{2m}$ is independent of n. Thus such polynomial sets are given by

$$P_n^{(\gamma)}(x) = \sum \binom{n}{2k} \frac{(1+\alpha)_n}{(1+\alpha)_{n-2k}} \gamma_{2k} x^{n-2k}.$$

Furthermore one can easily show that $P_n^{(\gamma)}(P^{(\mu)}) = P_n^{(\mu)}(P^{(\gamma)}) = P_n^{(\delta)}(x)$ where $\delta_{2n}/(2n)! = \sum_k \binom{2n}{2k} \mu_{2k} \gamma_{2n-2k}$ so that we have the following result.

THEOREM. *$C_\Sigma(L^{(\alpha)})$ is a commutative subgroup of $C_\pi(L^{(\alpha)})$.*

We also remark that elements of $C_\Sigma(L^{(\alpha)})$ are related to Brenke polynomials since we can show that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!(1+\alpha)_n} P_n^{(\gamma)}(x) = \left(\sum_{n=0}^{\infty} \gamma_{2n} \frac{t^n}{(2n)!}\right) \cdot {}_0F_1(-;1+\alpha; xt)$$

where ${}_0F_1(-;1+\alpha;u) = \sum_{n=0}^{\infty} u^n/n!(1+\alpha)_n$.

The case $\gamma = \{1\}$ gives $P_n^{(1)}(x) = 1/2\{(-1)^n L_n^{(\alpha)}(x) + L_n^{(\alpha)}(-x)\}$.

(d) As remarked earlier Appell showed that $\mathscr{A}$ is a subgroup of π. To determine $C_{\mathscr{A}}(L)$ we see that if $P \in \mathscr{A}$ then $P_n(x) = \sum \binom{n}{k} a_{n-k} x^k$. Hence $p_{n,n-k} = (1+\alpha)_{n-k}/(1+\alpha)_n a_k$ where a_k is independent of n. Since ${}_{n,n-2k}$ are arbitrary so are a_{2k}. Using (3.6) we can show that

$$a_{2m+1} = -(2m)! \sum_{k=0}^{m} \binom{2m+1}{2k+1} \frac{C_{2k+1}}{2^{2k+1}} \frac{(2m-2k)}{(2m-2k)!} a_{2m-2k}.$$

We can also show that such p.s. are generated by

$$e^{E(\log(1-t))+xt} = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!}$$

where $E(t)$ is an arbitrary even function of t.

REFERENCES

1. N. A. Al-Salam and W. A. Al-Salam, *The Centralizer of the Laguerre Polynomial Set.*, Linear and Multilinear Algebra **7** (1979), 149–154.

2. P. Appell, *Sur une classe de polynomes*, Ann. Sci. Ecole Norm. Sup. (2) **9** (1880), 119–144.

3. R. P. Boas and R. C. Buck, *Polynomials defined by generating relations*, Amer. Math. Monthly **63** (1956), 626–632.

4. J. Goldberg and J. W. Brown, *Generalized Appell connection sequences*, J. Math. Analysis and Appl. **46** (1974), 242–248.

5. G. C. Rota, D. Kahaner and A. Odlysko, *On the foundation of combinatorial theory* VIII. Finite Operator Calculus, J. Math. Anal. and Appl. **42** (1973), 685–760.

6. I. M. Sheffer, *On sets of polynomials and associated linear functional operators and equations*, Amer. J. Math. **53** (1931) 15–38.

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