ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 3, Summer 1984

SADDLE POINT APPROXIMATIONS IN n-TYPE EPIDEMICS AND CONTACT BIRTH PROCESSES

J. RADCLIFFE AND L. RASS

ABSTRACT. An n-type epidemic is considered. This model encompasses the measles, host-vector, and carrier-borne epidemics, and in addition rabies involving several species of animal. The saddle point approximation indicates that the asymptotic velocity of propagation in the deterministic model is the same as the minimum velocity, c_0, for which wave solutions exist in the deterministic model. It also suggests an approximation to the asymptotic expectation velocity in the stochastic model.

An n-type contact birth process is also defined. The analogue of the McKean connection between the distribution function of the position of furthest spread of the infection and the equations for the n-type simple epidemic is established. This suggests the asymptotic speed of translation of the distribution function is c_0.

1. Introduction. The equations for the spatial spread of a deterministic epidemic have been shown to only have wave solutions travelling with velocity greater than or equal to a critical velocity c_0. This has been proved by Atkinson and Reuter [2] for the case of constant infectivity with removals; and by Diekmann [4] for a more general model in which infectivity is allowed to vary with the time since infection, instead of infected individuals being removed at a steady rate. Use of the saddle point approximation, (Daniels [3]), indicates that the asymptotic velocity of propagation of infection in this model is in fact c_0. This has been proved rigourously by Aronson [1], Diekmann [5] and Thieme [11]. The saddle point approximation gives the correct velocity of propagation using a comparatively simple method.

Let S_t be the position of the furthest spread of a one-type contact birth process. It is stated in Mollison [7] that $y(s, t) = P(S_t > s)$ satisfies the deterministic simple epidemic equations. It then follows that $y(s, t)$ propagates with speed c_0, where c_0 is the minimal possible speed at which

Received by the editors on September 29, 1983.

Key words and phrases: n-Type epidemic, host-vector and carrier-borne epidemics, rabies involving several species, contact birth process, velocity of propagation, saddle point approximation.

wave solutions exist to the deterministic equations for the simple epidemic.

A deterministic model has been formulated in Radcliffe and Rass [9] to describe the spatial spread of an epidemic involving n types of individual. This model encompasses the measles, host-vector and carrier-borne epidemics, and in addition rabies involving several species of animal. It also includes models where some of the population have been vaccinated against the disease; the vaccination not conferring complete immunity. It was shown that the main results for the measles epidemic, concerning the existence, uniqueness and non-existence of wave solutions for different speeds, are still valid for this more general model.

In the present paper we formulate a deterministic model with constant infectivity and removal, to describe the spatial spread of an n-type epidemic. The asymptotic velocity of propagation of infection for this model is investigated. Use of the saddle point approximation indicates this to be the minimum velocity, c_0, at which wave solutions exist to the deterministic model. An n-type contact birth process is also defined. The analogue of the McKean connection between the distribution function of the position of furthest spread of the infection and the equations for the n-type simple epidemic is established; this suggests that the asymptotic speed of translation of this distribution function is c_0. We hope to pursue the approach of Aronson [1] and Diekmann [5], for the more general model of the epidemic in which the infection rate varies with the time since infection, in a subsequent paper. This would give a rigorous proof of these results for the deterministic n-type epidemic and the n-type contact birth process.

In the final section, we look briefly at a model for the n-type stochastic epidemic; and show that c_0 is also an approximation, possibly rather crude, for the asymptotic expectation velocity of this model.

2. The deterministic epidemic. In our paper Radcliffe and Rass [9], we formulated an n-type deterministic epidemic based on the one-type model analysed by Diekmann [4]. Here we consider an n-type version of the two-type model 1 of Radcliffe, Rass and Stirling [10].

Consider n populations of uniform densities on the real line $\mathbf{R}$, each of which consists of susceptible, infectious and removed individuals. Denote the proportions of susceptible, infectious and removed individuals in the i-th population at position s and time t by $x_i(s, t)$, $y_i(s, t)$ and $z_i(s, t)$, $(i = 1, \ldots, n)$, respectively; so that $x_i(s, t) + y_i(s, t) + z_i(s, t) = 1$. The density of individuals in the i-th population is σ_i. Let λ_{ij} be the rate of infection of susceptible individuals in population i by infectious individuals in population j. The contact distribution representing the distance r over which infection occurs has density $p_{ij}(r)$. The removal rate for infectious individuals in population i is μ_i.

The epidemic occuring over all real t is described by the equations

$$(1)\quad \begin{cases} \dfrac{\partial x_i(s,t)}{\partial t} = -x_i(s,t)\displaystyle\sum_{j=1}^{n}\lambda_{ij}\sigma_j\int_{-\infty}^{\infty}p_{ij}(r)y_j(s-r)\,dr, \\ \dfrac{\partial y_i(s,t)}{\partial t} = x_i(s,t)\displaystyle\sum_{j=1}^{n}\lambda_{ij}\sigma_j\int_{-\infty}^{\infty}p_{ij}(r)y_j(s-r)\,dr-\mu_i y_i(s,t), \\ \dfrac{\partial z_i(s,t)}{\partial t} = \mu_i y_i(s,t),\ (i=1,\ldots,n). \end{cases}$$

In the tail of the epidemic, making the assumption that $x_i(s, t) = 1$ it is seen that the $y_i(s, t)$ approximately satisfy the equations

$$(2)\quad \frac{\partial y_i(s,t)}{\partial t} = \sum_{j=1}^{n}\lambda_{ij}\sigma_j\int_{-\infty}^{\infty}p_{ij}(r)y_j(s-r,t)\,dr-\mu_i y_i(s,t),\ \text{for } i=1,\ldots n.$$

3. The n-type contact birth process. Consider an n-type contact birth process, where the probability that a type i individual has a type j offspring in the time interval $(t, t + \delta t)$ is $\lambda_{ij}\delta t + o(\delta t)$. If S is the distance of the offspring of type j from a parent of type i, then S has density $p_{ij}(s)$. Let $X_m(t)$, $m = 1, 2, \ldots$, be the position of all individuals of a specific type k at time t. Let $U(t) = \max_m X_m(t)$ and

$$z_i(s,t) = P(U(t) \leqq s \mid \text{one type } i \text{ individual at time } t = 0).$$

In this section we obtain equations satisfied by $y_i(s, t) = 1 - z_i(s, t)$. These are shown to be identical to a special case of the equations for the model for the spatial spread of the deterministic n-type simple epidemic considered in §2. This result was noted for the one-type epidemic by Mollison [7].

Let

$$f(s) = \begin{cases} 1 & \text{if } s \geqq 0, \\ 0 & \text{if } s < 0, \end{cases}$$

then $z_i(s, t) = E[\prod_m f(s - X_m(t))]$.

Consider the birth of the first offspring. Let T be the time of birth of the first offspring and y be its position. Take V to be a random variable such that $V = j$ if the offspring is of type j. Then T has density $f_T(t) = \alpha_i e^{-\alpha_i t}$, $(0 < t < \infty)$, where $\alpha_i = \sum_{j=1}^n \lambda_{ij}$ and $P(V = j) = \lambda_{ij}/\alpha_i$, for $j = 1, \ldots, n$.

$$E[\prod_m f(s - X_m(t)) \mid T = t_1, Y = y, V = j] = z_i(t - t_1, s)\, z_j(t - t_1, s - y).$$

Then $E[\prod_m f(s - X_m(t)) \mid T = t_1] = z_i(t - t_1, s)\sum_{j=1}^n(\lambda_{ij}/\alpha_i)\int_{-\infty}^{\infty} z_j(t - t_1, s - y)p_{ij}(y)dy$. Now $z_i(s, t) = \delta_{ik}\, P(T > t)f(s) + \int_0^t \alpha_i\, e^{-\alpha_i t_1} E[\prod_m f(s - X_m(t)) \mid T = t_1]\, dt_1$, where

$$\delta_{ik} = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

Thus $e^{\alpha_i t} z_i(s, t) = \delta_{ik} f(s) + \int_0^t e^{\alpha_i \tau} z_i(s, \tau) \sum_{j=1}^n \lambda_{ij} \int_{-\infty}^{\infty} z_j(s-y, \tau) p_{ij}(y) dy\, d\tau$. Differentiating with respect to t we obtain the equations

$$\frac{\partial z_i(s,t)}{\partial t} = z_i(s,t) \sum_{j=1}^n \lambda_{ij} \int_{-\infty}^{\infty} z_j(s-y, t) p_{ij}(y)\, dy - \alpha_i z_i(s,t), \ (i=1, \ldots, n).$$

Let $y_i(s, t) = 1 - z_i(s, t)$. Then,

$$(3) \quad \frac{\partial y_i(s,t)}{\partial t} = (1 - y_i(s,t)) \sum_{j=1}^n \lambda_{ij} \int_{-\infty}^{\infty} y_j(s-r, t) p_{ij}(r)\, dr, (i=1, \ldots, n).$$

Note that these equations are identical to equations (1) with $\mu_i = 0$ and $\sigma_i = 1$ for all i. This establishes the McKean connection.

If instead of using $U(t)$, the furthest spread of a specific type of individual, we use the furthest spread of all individuals, we obtain exactly the same equations.

In the tail of the process, $y_i(s, t)$ is approximately zero. Hence in the tail, $y_i(s, t)$ should approximately satisfy the equations,

$$(4) \qquad \frac{\partial y_i(s,t)}{\partial t} = \sum_{j=1}^n \lambda_{ij} \int_{-\infty}^{\infty} y_j(s-r, t) p_{ij}(r)\, dr, \ (i = 1, \ldots, n).$$

4. Properties of a class of matrices and the roots of equations. In this section we first set up some notation and definitions, which are the same as those used in Radcliffe and Rass [**9**]. We then state two lemmas, the first of which is proved in Dieudonné [**6**, p. 248] and the second of which is proved in Radcliffe and Rass [**9**, lemma 1].

Let $\mathbf{B} = (b_{ij})$ denote a matrix with (i, j)-th element b_{ij}. The i-th element of a vector $\mathbf{a}$ is denoted by $\{\mathbf{a}\}_i$. We denote a vector with all zero elements by $\mathbf{0}$ and one with all its elements unity by $\mathbf{1}$. Inequalities between matrices imply the corresponding inequalities between the elements of the matrices.

A matrix is said to be non-negative if all its elements are real non-negative. It is said to be finite if all its elements are finite. A square matrix $\mathbf{B} = (b_{ij})$ is said to be non-reducible if for every $i \neq j$ there exists a distinct sequence $i_1, \ldots, i_r$ with $i_1 = i$ and $i_r = j$ such that $b_{i_s, i_{s+1}} \neq 0$ for $s = 1, \ldots, (r-1)$. Otherwise $\mathbf{B}$ is called reducible. When $\mathbf{B}$ is a non-negative, finite square matrix we define $\rho(\mathbf{B})$ to be the maximum of the moduli of the eigenvalues of $\mathbf{B}$. We define $\rho(\mathbf{B}) = \infty$ when $\mathbf{B}$ is a non-negative, non-reducible square matrix with at least one infinite element. This is merely a notational convenience.

LEMMA 1. (*Continuity of the roots of an equation as a function of parameters*). *Let A be an open set in* $\mathbf{C}$, *F a metric space, f a continuous complex valued function in $A \times F$, such that for each $\alpha \in F$, $z \to f(z, \alpha)$ is analytic* *in A. Let D be an open subset of A, whose closure $\bar{D}$ in C is compact and contained in A, and let $\alpha^* \in F$ be such that no zero of $f(z, \alpha^*)$ is on the frontier of D. Then there exists a neighborhood W of α^* in F such that:*

(i) *for any $\alpha \in W$, $f(z, \alpha)$ has no zeros on the frontier of D;*

(ii) *for any $\alpha \in W$, the sum of the orders of the zeros of $f(z, \alpha)$ belonging to D is independent of α.*

LEMMA 2. *The class $\mathscr{B}$ of non-negative, non-reducible, finite square matrices has the following properties:*

(i) *If $\mathbf{B} \in \mathscr{B}$ then $\rho(\mathbf{B})$ is a simple eigenvalue of $\mathbf{B}$. When $\mathbf{B}$ is not the zero matrix of order 1 then $\rho(\mathbf{B}) > 0$. Corresponding to $\rho(\mathbf{B})$ there exists a positive left eigenvector $\mathbf{u}' > \mathbf{0}'$, which is unique up to a multiple.*

(ii) *If $\mathbf{B} \in \mathscr{B}$, then $\rho(\mathbf{B})$ increases as any element of $\mathbf{B}$ increases.*

(iii) *For any matrix $\mathbf{B} = (b_{ij})$ of order n in $\mathscr{B}$, and any $s = 1, \ldots, n$,*

$$b_{ss} \leqq \rho(\mathbf{B}) \leqq \max_i \sum_{j=1}^{n} b_{ij}.$$

(iv) *Let $\mathbf{C} = (c_{ij})$ be a matrix of complex valued elements and $\mathbf{C}^+ = (|c_{ij}|)$. If $\mathbf{C}^+ \leqq \mathbf{B}$, where $\mathbf{B} \in \mathscr{B}$, with strict inequality for at least one element, then for any eigenvalue μ of $\mathbf{C}$, $|\mu| < \rho(\mathbf{B})$.*

(v) *When $\mathbf{B} \in \mathscr{B}$ is of order $n > 1$, and $\mathbf{B}^*$ is any k dimensional principal minor of $\mathbf{B}$, where $k < n$ then $\rho(\mathbf{B}^*) < \rho(\mathbf{B})$.*

(vi) *Define the adjoint of a square matrix of order 1 to be the identity matrix of order 1. Then for any $\mathbf{B} \in \mathscr{B}$,*

$$Adj(\lambda\mathbf{I} - \mathbf{B}) > \mathbf{0}, \textit{for } \lambda \geqq \rho(\mathbf{B});$$
$$|\lambda\mathbf{I} - \mathbf{B}| > 0, \textit{for } \lambda > \rho(\mathbf{B});$$
$$|\lambda\mathbf{I} - \mathbf{B}| = 0, \textit{for } \lambda = \rho(\mathbf{B}).$$

(vii) *If $\mathbf{B} \in \mathscr{B}$ is of order $n > 1$ and $\mathbf{B}^*$ is any $(n-1)$ dimensional principal minor of $\mathbf{B}$, then $|\lambda\mathbf{I} - \mathbf{B}| < 0$ for $\rho(\mathbf{B}^*) < \lambda < \rho(\mathbf{B})$.*

(viii) *When $\mathbf{B}(\theta) = (b_{ij}(\theta)) \in \mathscr{B}$ and the $b_{ij}(\theta)$ are continuous functions of θ for $\theta_1 < \theta < \theta_2$, then $\rho(\mathbf{B}(\theta))$ is a continuous function of θ within this range. If $\lim_{\theta \downarrow \theta_1} \mathbf{B}(\theta)$ exists and is non-reducible then $\lim_{\theta \downarrow \theta_1} \rho(\mathbf{B}(\theta)) = \rho(\lim_{\theta \downarrow \theta_1} \mathbf{B}(\theta))$. A similar result holds as $\theta \uparrow \theta_2$.*

(ix) *Let $\mathbf{B}(\theta) = (b_{ij}(\theta)) \in \mathscr{B}$ and its entries be continuous functions of θ for $\theta \in A$. There is a left eigenvector $\underset{\sim}{u}'(\theta)$ corresponding to $\rho(\mathbf{B}(\theta))$ with n-th entry unity, such that $\{\mathbf{u}(\theta)\}_i$ is a continuous function of θ for $i = 1, \ldots, n$ and $\theta \in A$.*

5. The critical velocity. Define $\mathbf{P}(\lambda) = (P_{ij}(\lambda))$ and $\boldsymbol{\Lambda}(\lambda) = (\Lambda_{ij}(\lambda))$ where $P_{ij}(\lambda) = \int_{-\infty}^{\infty} e^{\lambda r}\, p_{ij}(r)\, dr$ and $\Lambda_{ij}(\lambda) = \lambda_{ij} \int_0^{\infty} e^{-\lambda t} e^{-\mu_j t}\, dt = \lambda_{ij}(\lambda + \mu_j)^{-1}$. Let $V_{ij}(\lambda) = \sigma_j P_{ij}(\lambda) \Lambda_{ij}(c\lambda)$ and $\mathbf{V}(\lambda) = (V_{ij}(\lambda))$. We restrict our attention to the case when $\Lambda(0)$ is non-reducible, so that $\mathbf{V}(0)$ is non-reducible. Note that $\rho(\mathbf{V}(0))$ does not depend on c.

It was shown in Radcliffe and Rass [9] that wave solutions to equations (4) do not exist for any speed $c > 0$ if $\rho(\mathbf{V}(0)) \leqq 1$. If $\rho(\mathbf{V}(0)) > 1$ then wave solutions are only possible for some positive speed if $p_{ij}(r) = 0(e^{-\delta r})$ as $r \to \infty$ for some positive real δ, all i, j. In this case let $\Delta_{V_{ij}} = \sup\{\lambda \in \mathbf{R}\colon\ V_{ij}(\lambda) < \infty\} = \sup\{\lambda \in \mathbf{R}\colon\ P_{ij}(\lambda) < \infty\}$ and $\Delta_V = \min_{i,j} \Delta_{V_{ij}}$. Then wave solutions exist for some speed $c > 0$ and a critical speed of propagation c_0 was defined to be $c_0 = \inf\{c \in \mathbf{R}_+\colon\ \rho(\mathbf{V}(\lambda)) < 1$ for some $\lambda \in (0, \Delta_V)\}$. From the results in our paper [9], this is easily seen to be equivalent to $c_0 = \inf\{c \in \mathbf{R}_+\colon\ \rho(\mathbf{V}(\lambda)) = 1$ for some $\lambda \in (0, \Delta_V)\}$. If $c_0 = 0$, then it was shown that wave solutions exist for all speeds $c > 0$. If $c_0 > 0$, wave solutions exist for all speeds $c \geqq c_0$. No wave solutions exist for speeds c such that $0 < c < c_0$.

For the model of §2 an explicit expression for c_0 exists. Let $f(\lambda) = \{\rho(\mathbf{K}(\lambda)) - \mu\}/\lambda$, where $\mathbf{K}(\lambda) = (\mathbf{K}_{ij}(\lambda))$ and $K_{ij}(\lambda) = \sigma_j\lambda_{ij}P_{ij}(\lambda) + \delta_{ij}(\mu - \mu_i)$ and $\mu = \max(\mu_1, \ldots, \mu_n)$. Observe that $K_{ij}(\lambda)$ is finite for all i, j and $\lambda \in \mathbf{R}$ such that $0 \leqq \lambda < \Delta_V$.

LEMMA 3. *The critical speed of propagation for the deterministic epidemic is given by* $c_0 = \max(\inf_{\lambda \in (0, \Delta_V)} f(\lambda), 0)$.

PROOF. Let $\mathbf{Q}(\lambda) = (Q_{ij}(\lambda))$ where $Q_{ij}(\lambda) = \sigma_j\lambda_{ij}P_{ij}(\lambda)$. Let $\mathbf{F}(\lambda) = \operatorname{diag}(\mu_1 + c\lambda, \ldots, \mu_n + c\lambda)$. Then $\mathbf{V}(\lambda) = \mathbf{Q}(\lambda)\{\mathbf{F}(\lambda)\}^{-1}$.

Consider any $c > 0$ such that there exists a λ such that $\rho(\mathbf{V}(\lambda)) = 1$. Since $\mathbf{V}(\lambda)$ is non-negative, from lemma 2(i) there exists a $\mathbf{u} > \mathbf{0}$ such that $\mathbf{u}'(\mathbf{V}(\lambda) - \mathbf{I}) = \mathbf{0}'$. Hence $\mathbf{u}'(\mathbf{Q}(\lambda) - \mathbf{F}(\lambda)) = \mathbf{0}'$. Let $\mathbf{D} = \operatorname{diag}(\mu - \mu_1, \ldots, \mu - \mu_n)$. Then $\mathbf{K}(\lambda) = \mathbf{Q}(\lambda) + \mathbf{D}$, and $\mathbf{K}(\lambda)$ is non-negative. Also $\mathbf{u}'(\mathbf{K}(\lambda) - (c\lambda + \mu)\mathbf{I}) = \mathbf{0}'$.

From lemma 2(i), since $\mathbf{u}' > \mathbf{0}'$, $(c\lambda + \mu) = \rho(\mathbf{K}(\lambda))$. Hence $c = f(\lambda)$. We may reverse the steps so that for any λ such that $f(\lambda) > 0$, then $\rho(\mathbf{V}(\lambda)) = 1$ for $c = f(\lambda)$. Hence the critical velocity c_0 is given by $c_0 = \max(\inf_{\lambda \in (0, \Delta_V)} f(\lambda), 0)$.

6. The analyticity and convexity of $\rho(\mathbf{K}(\lambda))$. So far $\rho(\mathbf{K}(\lambda))$ has been defined for λ real such that $0 \leqq \lambda < \Delta_V$. In order to use the saddle point approximation in §8, it is necessary to extend the definition of $\rho(\mathbf{K}(\lambda))$ to a region about the interval $(0, \Delta_V)$ of the real axis. We then show that $\rho(\mathbf{K}(\lambda))$ and $\mathbf{E}(\lambda)$, the idempotent corresponding to $\rho(\mathbf{K}(\lambda))$ in the spectral expansion of $\mathbf{K}(\lambda)$, are analytic in a region of the complex plane about $(0, \Delta_V)$.

Consider any real λ_0 such that $0 < \lambda_0 < \Delta_V$. Let the eigenvalues of $\mathbf{K}(\lambda_0)$ be $\mu_1, \ldots, \mu_n$, where $\mu_1 = \rho(\mathbf{K}(\lambda_0))$ and $\operatorname{Re}(\mu_j) < \rho(\mathbf{K}(\lambda_0))$ for

$j > 1$. Take $\varepsilon = \min((1/2)\{\rho(\mathbf{K}(\lambda_0)) - \max(\mathrm{Re}\mu_j,\ \text{for}\ j \geqq 2)\},\ (1/2)\min\{|\mu_i - \mu_j|\ \text{for}\ \mu_i \neq \mu_j\})$.

From lemma 1 there exists a δ_0, with $0 < \delta_0 < \min(\lambda_0, \Delta_V - \lambda_0)$, such that, for $|\lambda - \lambda_0| < \delta_0$, exactly one eigenvalue of $\mathbf{K}(\lambda)$ lies in an ε-neighborhood of $\rho(\mathbf{K}(\lambda_0))$ and that is the eigenvalue of $\mathbf{K}(\lambda)$ with the largest real part. For $|\lambda - \lambda_0| < \delta_0$, define $\rho(\mathbf{K}(\lambda))$ to be equal to the eigenvalue of $\mathbf{K}(\lambda)$ with largest real part. It also follows from lemma 1 that $\lim_{\lambda\to\lambda_0}\rho(\mathbf{K}(\lambda)) = \rho(\mathbf{K}(\lambda_0))$.

LEMMA 4. *For each $\lambda \in \mathbf{R}$ such that $0 < \lambda < \Delta_V$, there exists a neighborhood of λ in which $\rho(\mathbf{K}(\lambda))$ and the elements of $\mathbf{E}(\lambda)$ are analytic.*

PROOF. We first show that for each real $\lambda_0 \in (0, \Delta_V)$ there exists a $\delta^* > 0$ such that $\rho(\mathbf{K}(\lambda))$ is differentiable for $\lambda \in \mathbf{C}$ and $|\lambda - \lambda_0| < \delta^*$. Define $\rho(\lambda) = \rho(\mathbf{K}(\lambda))$. Note that from lemma 1, $\rho(\lambda)$ is a continuous function of λ for $|\lambda - \lambda_0| < \delta_0$. Hence $\mathrm{Adj}(\rho(\lambda)\mathbf{I} - \mathbf{K}(\lambda))$ is a continuous function of λ for $|\lambda - \lambda_0| < \delta_0$ with $\mathrm{Adj}(\rho(\lambda_0)\mathbf{I} - \mathbf{K}(\lambda_0)) > \mathbf{0}$ by lemma 2(vi). Therefore there exists a $\delta^* < \delta_0$ such that $\mathrm{Re}\{\mathrm{Adj}(\rho(\lambda)\mathbf{I} - \mathbf{K}(\lambda))\} > \mathbf{0}$ for $|\lambda - \lambda_0| < \delta^*$. Now for λ such that $|\lambda - \lambda_0| < \delta^*$ and $d\lambda$ such that $|\lambda + d\lambda - \lambda_0| < \delta^*$, $|\{\rho(\lambda + d\lambda) - \rho(\lambda)\}\mathbf{I} + \{\rho(\lambda)\mathbf{I} - \mathbf{K}(\lambda)\}| = -\{|\rho(\lambda + d\lambda)\mathbf{I} - \mathbf{K}(\lambda + d\lambda)| - |\rho(\lambda + d\lambda)\mathbf{I} - \mathbf{K}(\lambda)|\}$.

Let $g(\theta) = |\rho(\lambda + d\lambda)\mathbf{I} - \mathbf{K}(\lambda + d\lambda - \theta)|$. Then $g(\theta)$ is an analytic function of θ. Hence using the Taylor expansion of $g(\theta)$,

$$\begin{aligned}
&|\{\rho(\lambda + d\lambda) - \rho(\lambda)\}\mathbf{I} + \{\rho(\lambda)\mathbf{I} - \mathbf{K}(\lambda)\}| \\
&\quad = d\lambda \sum_i \sum_j K'_{ij}(\lambda + d\lambda)\,\{\mathrm{Adj}(\rho(\lambda + d\lambda)\mathbf{I} - \mathbf{K}(\lambda + d\lambda)\}_{ij} + o(d\lambda) \\
&\quad = d\lambda \sum_i \sum_j K'_{ij}(\lambda)\,\{\mathrm{Adj}(\rho(\lambda)\mathbf{I} - \mathbf{K}(\lambda))\}_{ij} + o(d\lambda).
\end{aligned}$$

Now $|\{\rho(\lambda + d\lambda) - \rho(\lambda)\}\mathbf{I} - \{\rho(\lambda)\mathbf{I} - \mathbf{K}(\lambda)\}| = \sum_{j=1}^{n} a_j(\lambda)\{\rho(\lambda + d\lambda) - \rho(\lambda)\}^j$, where the $a_j(\lambda)$, for $j = 1, \ldots, n$, are continuous functions of λ and $a_n(\lambda) = 1$. Note that $a_1(\lambda) = \mathrm{trace}\ \mathrm{Adj}\{\rho(\lambda)\mathbf{I} - \mathbf{K}(\lambda)\}$. Hence $\mathrm{Re}(a, (\lambda)) > 0$ for $|\lambda - \lambda_0| < \delta^*$.

Let $A(\lambda, d\lambda) = \sum_{j=1}^{n} a_j(\lambda)(\rho(\lambda + d\lambda) - \rho(\lambda))^{j-1}$. Then $\lim_{d\lambda\to 0} A(\lambda, d\lambda) = a_1(\lambda)$. Hence, for $d\lambda$ sufficiently small, $\mathrm{Re}(A(\lambda, d\lambda)) > 0$. We can therefore write, for $d\lambda$ sufficiently small,

$$\frac{\rho(\lambda + d\lambda) - \rho(\lambda)}{d\lambda} = \frac{\sum\sum K'_{ij}(\lambda)\{\mathrm{Adj}(\rho(\lambda)\mathbf{I} - \mathbf{K}(\lambda)\}_{ij} + (o(d\lambda)/(d\lambda))}{A(\lambda, d\lambda)}.$$

Hence

$$\lim_{d\lambda\to 0} \frac{\rho(\lambda + d\lambda) - \rho(\lambda)}{d\lambda} = \frac{\sum\sum K'_{ij}(\lambda)\{\mathrm{Adj}(\rho(\lambda)\mathbf{I} - \mathbf{K}(\lambda)\}_{ij}}{\mathrm{trace}\ \mathrm{Adj}(\rho(\lambda)\mathbf{I} - \mathbf{K}(\lambda)}.$$

This establishes the existence of $\rho'(\lambda)$ for a region $|\lambda - \lambda_0| < \delta^*$. In this region

$$\rho'(\lambda) = \frac{\sum\sum K'_{ij}(\lambda)\{\mathrm{Adj}(\rho(\lambda)\mathbf{I} - \mathbf{K}(\lambda))\}_{ij}}{\mathrm{trace}\ \mathrm{Adj}(\rho(\lambda)\mathbf{I} - \mathbf{K}(\lambda))}.$$

Hence $\rho(\lambda)$ is analytic in a region $|\lambda - \lambda_0| < \delta^*$.

We now prove the analyticity of $\mathbf{E}(\lambda)$. Now $\rho(\lambda)$ is defined and is analytic in some region $|\lambda - \lambda_0| < \delta^*$. Let $\mathbf{v}'(\lambda)$ be a left eigenvector corresponding to $\rho(\lambda)$. Note that $\mathbf{v}'(\lambda)$ is unique up to a multiple. We partition $\mathbf{v}'(\lambda)$ and $\mathbf{K}(\lambda)$ so that

$$\mathbf{v}'(\lambda) = (\mathbf{v}^{*\prime}(\lambda), \mathrm{v}_n(\lambda)) \text{ and } \mathbf{K}(\lambda) = \begin{pmatrix} \mathbf{K}^*(\lambda) & \boldsymbol{\beta}(\lambda) \\ \boldsymbol{\alpha}'(\lambda) & K_{nn}(\lambda) \end{pmatrix}.$$

Then $\mathbf{v}^{*\prime}(\lambda)(\rho(\lambda)\mathbf{I} - \mathbf{K}^*(\lambda)) = \boldsymbol{\alpha}'(\lambda)\mathrm{v}_n(\lambda)$.

From lemma 2 parts (v) and (vi), $|\rho(\lambda_0)\mathbf{I} - \mathbf{K}^*(\lambda_0)| > 0$, and $\mathrm{Adj}(\rho(\lambda_0)\mathbf{I} - \mathbf{K}^*(\lambda_0)) > \mathbf{0}$. Hence there exists a δ, with $0 < \delta < \delta^*$, such that $|\rho(\lambda)\mathbf{I} - \mathbf{K}^*(\lambda)| \neq 0$ and $\mathrm{Re}(\mathrm{Adj}(\rho(\lambda)\mathbf{I} - \mathbf{K}^*(\lambda))) > \mathbf{0}$ for $|\lambda - \lambda_0| < \delta$.

It then follows immediately that, for $|\lambda - \lambda_0| < \delta$, $\mathrm{v}_n(\lambda) \neq 0$. We may then consider $\mathbf{v}'(\lambda)$ in this region to be that eigenvector with $\mathrm{v}_n(\lambda) \equiv 1$. Then

$$\mathbf{v}^{*\prime}(\lambda) = \frac{\boldsymbol{\alpha}'(\lambda)\,\mathrm{Adj}(\rho(\lambda)\mathbf{I} - \mathbf{K}^*(\lambda))}{|\rho(\lambda)\mathbf{I} - \mathbf{K}^*(\lambda)|}.$$

Then the elements of $\mathbf{v}^{*\prime}(\lambda)$ are the ratios of two analytic functions, with the denominators non-zero for $|\lambda - \lambda_0| < \delta$. Hence $\mathbf{v}'(\lambda)$, with n-th element unity, is analytic for $|\lambda - \lambda_0| < \delta$.

Let $\mathbf{w}(\lambda)$ be the right eigenvector corresponding to $\rho(\lambda)$. Again we may take $\mathbf{w}(\lambda)$ to have n-th element unity, and establish the analyticity of $\mathbf{w}(\lambda)$ in a region $|\lambda - \lambda_0| < \delta_1$ for some $\delta_1 > 0$. The proof is identical to the proof for $\mathbf{v}'(\lambda)$.

Now $\mathbf{E}(\lambda) = \mathbf{w}(\lambda)\mathbf{v}'(\lambda)/\{\mathbf{v}'(\lambda)\mathbf{w}(\lambda)\}$. By lemma 2(i), $\mathbf{v}'(\lambda_0)$ and $\mathbf{w}(\lambda_0)$ have positive elements. Hence from the analyticity, and therefore continuity, of the elements of $\mathbf{v}'(\lambda)$ and $\mathbf{w}(\lambda)$, there exists a δ_2 with $0 < \delta_2 < \min(\delta, \delta_1)$ such that $\mathbf{v}'(\lambda)\mathbf{w}(\lambda) \neq 0$ for $|\lambda - \lambda_0| < \delta_2$. The elements of $\mathbf{E}(\lambda)$ are the ratios of analytic functions with non-zero denominators, i.e., the elements of $\mathbf{E}(\lambda)$ are analytic.

LEMMA 5. *$\rho(\lambda)$ is a convex function of λ for $0 < \lambda < \Delta_V$. It is strictly convex if $p_{ij}(s)$ is symmetric about zero for all i, j.*

PROOF. We first show that for λ real such that $0 < \lambda < \Delta_V$, and $|y|$ sufficiently small such that $\rho(\lambda + iy)$ is defined and analytic, $|\rho(\lambda + iy)| < \rho(\lambda)$, for $y \neq 0$.

Now $|P_{ij}(\lambda)| > |P_{ij}(\lambda + iy)|$ for $y \neq 0$. Therefore $Q_{ij}(\lambda) \geqq |Q_{ij}(\lambda + iy)|$, with strict inequality if $\lambda_{ij} \neq 0$.

For $\lambda_{ij} = 0$, $K_{ij}(\lambda + iy) = K_{ij}(\lambda)$. For $\lambda_{ij} \neq 0$,

$$\begin{aligned} |K_{ij}(\lambda + iy)| &= |Q_{ij}(\lambda + iy) + (\mu - \mu_i)\,\delta_{ij}| \\ &\leqq |Q_{ij}(\lambda + iy)| + \delta_{ij}(\mu - \mu_i)| \\ &< Q_{ij}(\lambda) + \delta_{ij}(\mu - \mu_i) \\ &= K_{ij}(\lambda). \end{aligned}$$

Therefore for $y \neq 0$, if μ is an eigenvalue of $\mathbf{K}(\lambda + iy)$, from lemma 2 (iv), $|\mu| < \rho(\mathbf{K}(\lambda))$. Thus $|\rho(\lambda + iy)| < \rho(\lambda)$.

For λ real such that $0 \leqq \lambda < \Delta_V$ and y small with $y \neq 0$.

$$|\rho(\lambda + iy)| = \{(\rho(\lambda) - (1/2)y^2\,\rho''(\lambda))^2 + (y\rho'(\lambda))^2 + o(y^3)\}^{1/2}. \tag{5}$$

In order that $|\rho(\lambda + iy)| < |\rho(\lambda)|$ it is necessary that $\rho''(\lambda) \geqq 0$, i.e., $\rho(\lambda)$ is convex.

If the contact distributions are all symmetric about zero, then $K'_{ij}(0) = 0$ and $K'_{ij}(\lambda) > 0$ for $\lambda > 0$. Let λ be such that $0 \leqq \lambda < \Delta_V$. Now

$$\rho'(\lambda) = \frac{\sum\sum K'_{ij}(\lambda)\,\mathrm{Adj}(\rho(\lambda)\mathbf{I} - \mathbf{K}(\lambda))}{\mathrm{trace}\ \mathrm{Adj}(\rho(\lambda)\mathbf{I} - \mathbf{K}(\lambda))}.$$

Since $\mathrm{Adj}(\rho(\lambda)\mathbf{I} - \mathbf{K}(\lambda)) > \mathbf{0}$, it follows that $\rho'(0) = 0$ and $\rho'(\lambda) > 0$ for $\lambda > 0$. Hence from (5) $\rho''(\lambda) > 0$ for λ real with $0 < \lambda < \Delta_V$, and $\rho(\lambda)$ is strictly convex for $0 < \lambda < \Delta_V$.

The following corollary will be used in §8 to show the existence, for each positive s and t, of a unique saddle point of the function $\mathrm{Re}\{\rho(\lambda)t - \lambda s\}$ on the real axis.

COROLLARY. *If the $p_{ij}(r)$ are symmetric about zero for all i, j, and the limit of $P_{ij}(\lambda)$ as $\lambda \uparrow \Delta_{V_{ij}}$ is infinite for all i, j, then $\rho'(\lambda)$ is a strictly increasing continuous function of λ for λ real with $0 \leq \lambda < \Delta_V$, with $\rho'(0) = 0$ and* $\lim_{\lambda \uparrow \Delta_V} \rho'(\lambda) = \infty$.

PROOF. From lemma 5, $\rho'(0) = 0$ and $\rho'(\lambda)$ is strictly increasing. The continuity follows from lemma 4. It remains to show that $\lim_{\lambda \uparrow \Delta_V} \rho'(\lambda) = \infty$.

Suppose Δ_V is finite. From lemma 2 (viii) $\lim_{\lambda \uparrow \Delta_V} \rho(\lambda) = \infty$. It herefore follows that $\lim_{\lambda \uparrow \Delta_V} \rho'(\lambda) = \infty$.

Now consider the case when Δ_V is infinite. Since each $p_{ij}(r)$ is symmetric about zero, $\lim_{\lambda \to \infty} P(\lambda)$ is non-reducible. Thus by lemma 2 (viii), $\lim_{\lambda \to \infty} \rho(\lambda) = \infty$. Using l'hôpital's rule, $\lim_{\lambda \to \infty} \rho(\lambda)/\lambda = \lim_{\lambda \to \infty} \rho'(\lambda)$. Also using this rule,

$$\lim_{\lambda\to\infty} P_{ij}(\lambda)/\lambda = \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} r\, e^{\lambda r} p_{ij}(r)\, dr = \infty.$$

Then $\rho(\lambda)/\lambda = \rho(\mathbf{K}(\lambda)/\lambda) \geqq \rho(\mathbf{Q}(\lambda)/\lambda)$, which tends to infinity as λ tends to infinity. Hence $\lim_{\lambda\to\infty} \rho'(\lambda) = \infty$.

7. Properties of f(λ). The following lemma describes the behavior of $f(\lambda)$ for real λ such that $\lambda \in (0, \Delta_V)$, when $\rho(\mathbf{V}(0)) > 1$. We define $\mathbf{M} = \operatorname{diag}(\mu_1, \ldots, \mu_n)$.

LEMMA 6. *If $p_{ij}(r)$ is symmetric about zero with the limit of $P_{ij}(\lambda)$ as $\lambda \uparrow \Delta_{V_{ij}}$ infinite, for all i, j, and $\rho(\mathbf{V}(0)) > 1$, then*

(i) *$f(\lambda)$ is a continuous function of λ, with $f(\lambda) > 0$ for $0 < \lambda < \Delta_V$, $\lim_{\lambda\downarrow 0} f(\lambda) = \infty$ and $\lim_{\lambda\uparrow\Delta_V} f(\lambda) = \infty$,*

(ii) *there is a unique real value λ in the range $0 < \lambda < \Delta_V$ such that $f'(\lambda) = 0$.*

PROOF.

(i) The continuity of $f(\lambda)$ is an immediate consequence of lemma 4.

We now consider $\lim_{\lambda\downarrow 0} f(\lambda)$. Suppose that $\rho(\mathbf{V}(0))$ is finite, and let $\eta = \rho(\mathbf{V}(0))$. By lemma 2 (i) there exists a $\mathbf{u}' > \mathbf{0}'$ such that $\mathbf{u}'(\mathbf{V}(0) - \eta\mathbf{I}) = \mathbf{0}'$. Thus $\mathbf{u}'(\eta^{-1}\mathbf{Q}(0) - \mathbf{M}) = \mathbf{0}'$. Hence $\mathbf{u}'(\eta^{-1}\mathbf{Q}(0) + \mathbf{D} - \mu\mathbf{I}) = \mathbf{0}'$, and so $\rho(\eta^{-1}\mathbf{Q}(0) + \mathbf{D}) = \mu$. Since $\eta > 1$, from lemma 2 (ii) $\rho(\mathbf{K}(0)) = \rho(\mathbf{Q}(0) + \mathbf{D}) > \rho(\eta^{-1}\mathbf{Q}(0) + \mathbf{D}) = \mu$. Hence $(\rho(\mathbf{K}(0) - \mu) > 0$ and $\lim_{\lambda\downarrow 0} f(\lambda) = \infty$.

If $\lim_{\lambda\downarrow 0} \rho(\mathbf{V}(0))$ is infinite, then $\mu_j = 0$ for some j. Since $\mathbf{K}(\lambda) = \mathbf{Q}(\lambda) + \mathbf{D}$, and $\lim_{\lambda\downarrow 0} \mathbf{K}(\lambda) = \mathbf{K}(0)$ is finite and non-reducible, $K_{jj}(0) \geqq \mu$ for some j. If $n > 1$ then, by lemma 2 (iii), $\rho(\mathbf{K}(0)) > K_{jj}(0) \geqq \mu$. If $n = 1$, then $\rho(\mathbf{K}(0)) = K_{11}(0) > \mu = 0$. Hence $\lim_{\lambda\downarrow 0} f(\lambda) = \infty$.

We next show that $\lim_{\lambda\uparrow\Delta_V} f(\lambda) = \infty$. For Δ_V finite, from the corollary to lemma 5, $\lim_{\lambda\uparrow\Delta_V} \rho(\lambda) = \infty$. Hence $\lim_{\lambda\uparrow\Delta_V} f(\lambda) = \infty$. For Δ_V infinite, we use the result that $\lim_{\lambda\to\infty} \rho(\lambda)/\lambda = \infty$ which was obtained in the proof of the corollary to lemma 5. Therefore $\lim_{\lambda\to\infty} f(\lambda) = \infty$.

It only remains to show that $f(\lambda) > 0$ for $0 < \lambda < \Delta_V$. We have shown that $\rho(K(0)) > \mu$. From the corollary to lemma 5, $\rho(K(\lambda)) > \rho(\mathbf{K}(0)) > \mu$ for $0 < \lambda < \Delta_V$. The result then follows.

(ii) From part (i), using the differentiability of $\rho(\lambda)$, $f'(\lambda) = 0$ for some $0 < \lambda < \Delta_V$. Now $f'(\lambda) = (\rho'(\lambda) - f(\lambda))/\lambda$ and $f''(\lambda) = (\rho''(\lambda) - 2f'(\lambda))/\lambda$. Hence, for any λ such that $f'(\lambda) = 0$, $f''(\lambda) = \rho''(\lambda)/\lambda$, which is strictly positive for $0 < \lambda < \Delta_V$ from the corollary to lemma 5. Hence every solution to $f'(\lambda) = 0$ is a minimum. Since $f(\lambda)$ is a continuous function of λ for $0 < \lambda < \Delta_V$, the result follows.

8. The saddle point approximation. It was shown in §2 that, in the tail of the epidemic, the proportion of infectives in the deterministic model satisfy approximately the equations

$$\frac{\partial y_i(s, t)}{\partial t} = \sum_{j=1}^{n} \sigma_j \lambda_{ij} \int_{-\infty}^{\infty} y_j(s - r, t) p_{ij}(r)\, dr - \mu_i y_i(s, t). \tag{6}$$

Also equation (3) is a special case of this equation with $\sigma_j = 1$ and $\mu_j = 0$, $(j = 1, \ldots, n)$. Thus a study of these equations is pertinent to the models considered in this paper.

We take $p_{ij}(r)$ to be symmetric and exponential in the tail for all i and j, so that $P_{ij}(\lambda)$ exists for $-\Delta_{V_{ij}} < \operatorname{Re} \lambda < \Delta_{V_{ij}}$ for some positive $\Delta_{V_{ij}}$. Then $\mathbf{P}(\lambda) = (P_{ij}(\lambda))$ exists and has analytic entries for $-\Delta_V < \operatorname{Re} \lambda < \Delta_V$. We restrict $\mathbf{P}(\lambda)$ so that for all i, j, $\lim_{\lambda \uparrow \Delta_{V_{ij}}} P_{ij}(\lambda) = \infty$. In addition, for any real θ_1 and θ_2 with $[\theta_1, \theta_2] \subset (0, \Delta_V)$, there exists a $k_{ij}(y)$ with $|P_{ij}(\theta + iy)| \leqq k_{ij}(y)$ for all $\theta \in [\theta_1, \theta_2]$ and $\int_{-\infty}^{\infty} k_{ij}(y)\, dy < \infty$. These conditions hold for most common contact distributions.

Let $L_i(\lambda, t) = \int_{-\infty}^{\infty} e^{\lambda s} y_i(s, t) ds$ and $\{\mathbf{L}(\lambda, t)\}_i = L_i(\lambda, t)$. From equations (6) we obtain

$$\frac{\partial L_i(\lambda, t)}{\partial t} = \sum_{j=1}^{n} \sigma_j \lambda_{ij} P_{ij}(\lambda) L_i(\lambda, t) - \mu_i L_i(\lambda, t).$$

Hence

$$\frac{\partial \mathbf{L}(\lambda, t)}{\partial t} = (\mathbf{K}(\lambda) - \mu \mathbf{I}) \mathbf{L}(\lambda, t),$$

where $\mu = \max(\mu_1, \ldots, \mu_n)$ and $\{\mathbf{K}(\lambda)\}_{ij} = \sigma_j \lambda_{ij} P_{ij}(\lambda) + \delta_{ij}(\mu - \mu_i)$. Thus,

$$\mathbf{L}(\lambda, t) = e^{(\mathbf{K}(\lambda) - \mu \mathbf{I})t} \mathbf{u}(\lambda),$$

where $\mathbf{u}(\lambda) = \mathbf{L}(\lambda, 0)$. Now $u_i(\lambda) = \{\mathbf{u}(\lambda)\}_i$ is the transform of $y_i(s, 0)$, which is taken to have finite support. $u_i(\lambda)$ is also assumed to be analytic for $-\infty < \operatorname{Re} \lambda < \infty$. This is certainly reasonable for the models described in §2 and 3. Hence $\mathbf{u}(\lambda)$ is bounded for $\operatorname{Re} \lambda$ bounded.

Let $\mathbf{L}^*(\lambda, t) = \mathbf{L}(\lambda, t) - e^{-\mathbf{M}t} \mathbf{u}(\lambda)$, where $\mathbf{M} = \operatorname{diag}(\mu_1, \ldots, \mu_n)$. Then

$$\mathbf{L}(\lambda, t) = \mathbf{L}^*(\lambda, t) + \sum_{j=1}^{n} e^{-\mu_j t} \mathbf{E}_j \mathbf{u}(\lambda), \tag{7}$$

where $\mathbf{E}_j$ has (j, j)-th element 1 and all other elements zero.

We now prove a lemma to justify the use of the inversion formula for $\mathbf{L}^*(\lambda, t)$. Let $|\mathbf{A}|$ denote the matrix with $\{|\mathbf{A}|\}_{ij} = |\{\mathbf{A}\}_{ij}|$.

LEMMA 7. *For any real θ such that $-\Delta_V < \theta < \Delta_V$, $L_1^*(\lambda, t)$ is absolutely integrable over the line $\lambda = \theta + iy$ for $-\infty < y < \infty$.*

PROOF.

$$|\mathbf{L}^*(\lambda, t)| = |[\sum_{j=0}^{\infty} (\mathbf{Q}(\lambda) - \mathbf{M})^j t^j/j! - \sum_{j=0}^{\infty} (-\mathbf{M})^j t^j/j!]\mathbf{u}(\lambda)|$$
$$\leqq [\sum_{s=1}^{\infty} t^s/s! \sum_{j=1}^{s} {}^sC_j(|\mathbf{Q}(\lambda)|)^j \mu^{s-j}]\,|\mathbf{u}(\lambda)|$$
$$\leqq [\sum_{j=1}^{\infty} t^j(|\mathbf{Q}(\lambda)|)^j/j! \sum_{s=j}^{\infty} (\mu t)^{s-j}/(s-j)!]\,|\mathbf{u}(\lambda)|$$
$$= t\, e^{\mu t}|\mathbf{Q}(\lambda)| \sum_{j=0}^{\infty} (|\mathbf{Q}(\lambda)|)^j t^j |\mathbf{u}(\lambda)|/(j+1)!$$
$$\leqq t\, e^{\mu t}|\mathbf{Q}(\lambda)|\, e^{|\mathbf{Q}(\lambda)|t}\, |\mathbf{u}(\lambda)|.$$

Now $\lim_{y\to\pm\infty} |\mathbf{Q}(\theta + iy)| = 0$ and the elements of $|\mathbf{u}(\lambda)|$ are bounded. Thus, for a given t and a real θ with $0 \leqq \theta < \Delta_V$, $e^{|\mathbf{Q}(\theta+iy)|}\, |\mathbf{u}(\theta + iy)|$ has bounded elements for $-\infty < y < \infty$. The elements of $|\mathbf{Q}(\theta + iy)|$ are absolutely integrable over the range $-\infty < y < \infty$. It follows immediately that $L_i^*(\lambda, t)$ is absolutely integrable over the line $\lambda = \theta + iy$ for $-\infty < y < \infty$, for each $\theta \in (-\Delta_V, \Delta_V)$ and each $i = 1, \ldots, n$.

Since $\{\mathbf{u}(\lambda)\}_i$ is the transform of a function with finite support, it follows from (7) and lemma 7 that

$$y_i(s, t) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} e^{-s\lambda}\{\mathbf{L}^*(\lambda, t)\}_i\, d\lambda + e^{-\mu_i t}\, y_i(s, 0),$$

where $y_i(s, 0) = 0$ for $|s|$ sufficiently large. Hence, for any θ such that $-\Delta_V < \theta < \Delta_V$, if $|s|$ is sufficiently large

$$y_i(s, t) = \frac{1}{2\pi i} \int_{\theta-i\infty}^{\theta+i\infty} e^{-s\lambda}\{L^*(\lambda, t)\}_i\, d\lambda. \tag{8}$$

We wish to obtain the speed of propagation in the upper tail, i.e., $\lim_{t\to\infty} s/t$, where s is such that $\int_s^\infty y_i(x, t)dx = \eta$, for small η. It is assumed that $\lim_{t\to\infty} s/t = c$ exists and is positive.

The line of integration of $L_i(\lambda, t)$ is taken to pass through the saddle point of $\mathrm{Re}\{\rho(\lambda)t - \lambda s\}$ on the real axis. By the corollary to lemma 5, there is a unique value $\theta(t)$ such that $\rho'(\theta(t)) = s/t$; $\theta(t)$ is then the required saddle point. Note that $\theta(t)$ will tend to a limit θ_0 as s/t tends to c, where $\rho'(\theta_0) = c$. For any θ_1 and θ_2 with $0 < \theta_1 < \theta_0 < \theta_2 < \Delta_V$, we can choose T such that $\theta(t) \in [\theta_1, \theta_2]$ for $t > T$.

If t is sufficiently large, then

$$\eta = \int_s^\infty y_i(x, t)dx = \frac{1}{2\pi i} \int_s^\infty \int_{\theta(t)-i\infty}^{\theta(t)+i\infty} e^{-\lambda x}\, L_i^*(\lambda, t)\, d\lambda\, dx.$$

Since this is absolutely integrable, we may interchange the order of integration and obtain

$$\eta = \frac{1}{2\pi i} \int_{\theta(t)-i\infty}^{\theta(t)+i\infty} \lambda^{-1} e^{-\lambda s} L_i^*(\lambda, t)\, d\lambda.$$

We obtain an approximation to $\eta\, e^{\mu t - \gamma(t)}$, where $\gamma(t) = \rho(\theta(t))t - s\theta(t)$, for t large. t is taken to be sufficiently large so that $\theta(t) \in [\theta_1, \theta_2] \subset (0, \Delta_V)$. The proof is restricted to the case where $\mathbf{K}(\theta_0)$ has distinct eigenvalues. This holds for $n = 1$ and 2, and hence covers the important models for the measles, host-vector and carrier-borne epidemics, and measles with vaccinated and non-vaccinated individuals.

THEOREM 1. *Suppose that* $\mathbf{K}(\theta_0)$ *has distinct eigenvalues. Given any* $\varepsilon > 0$ *and* $\delta > 0$, *there exists a* $T > 0$ *such that*

$$|\eta e^{\mu t - \gamma(t)} - \frac{1}{2\pi i} e^{-\gamma(t)} \int_{\theta(t)-i\delta}^{\theta(t)+i\delta} \lambda^{-1}\{\mathbf{E}(\lambda)\mathbf{u}(\lambda)\}_i\, e^{\rho(\lambda)t - \lambda s}\, d\lambda\,| < \varepsilon$$

for all $t > T$.

PROOF. Now

$$\eta\, e^{\mu t - \gamma(t)} = \frac{1}{2\pi i} e^{-\gamma(t)} \int_{\theta(t)-i\infty}^{\theta(t)+i\infty} \lambda^{-1} e^{-\lambda s} \{(e^{\mathbf{K}(\lambda)t} - e^{\mathbf{D}t})\mathbf{u}(\lambda)\}_i\, d\lambda.$$

Let $\eta\, e^{\mu t - \gamma(t)} = \sum_{j=1}^{5} I_j$, where in the integrals $\lambda = \theta(t) + iy$ and

$$I_1 = \frac{1}{2\pi i} e^{-\gamma(t)} \int_{|y|>\alpha} \lambda^{-1} e^{-\lambda s}\{(e^{\mathbf{K}(\lambda)t} - e^{\mathbf{D}t})\mathbf{u}(\lambda)\}_i\, dy,$$

$$I_2 = -\frac{1}{2\pi i} e^{-\gamma(t)} \int_{|y|<\alpha} \lambda^{-1} e^{-\lambda s}\{e^{\mathbf{D}t}\,\mathbf{u}(\lambda)\}_i\, dy,$$

$$I_3 = \frac{1}{2\pi i} e^{-\gamma(t)} \int_{\delta<|y|<\alpha} \lambda^{-1} e^{-\lambda s}\{e^{\mathbf{K}(\lambda)t}\,\mathbf{u}(\lambda)\}_i\, dy,$$

$$I_4 = \frac{1}{2\pi i} e^{-\gamma(t)} \int_{|y|<\delta} \lambda^{-1} e^{-\lambda s}\{(e^{\mathbf{K}(\lambda)t} - \mathbf{E}(\lambda)\, e^{\rho(\lambda)t})\,\mathbf{u}(\lambda)\}_i\, dy, \text{ and}$$

$$I_5 = \frac{1}{2\pi i} e^{-\gamma(t)} \int_{|y|<\delta} \lambda^{-1} e^{-\lambda s}\{\mathbf{E}(\lambda)\,\mathbf{u}(\lambda)\}_i\, e^{\rho(\lambda)t}\, dy.$$

The proof consists of showing that, given any $\varepsilon > 0$ and $\delta > 0$, there exists an α with $\alpha > \delta$ and $T > 0$ such that $|I_i| < \varepsilon/4$, for $i = 1, \ldots, 4$, for all $t > T$. Parts (i)–(iv) consider the integrals $|I_1|$ to $|I_4|$ respectively.

(i) Choose T_1 so that $\theta(t)$ is contained in an interval $[\theta_1, \theta_2] \subset (0, \Delta_V)$ for all $t > T_1$. In the integrals let $\lambda = \theta(t) + iy$.

$$|I_1| \leqq \frac{1}{2\pi\theta_1} \int_{|y|>\alpha} e^{-\rho(\theta(t))t}\, \{|e^{(\mathbf{Q}(\lambda)+\mathbf{D})t} - e^{\mathbf{D}t}|\ |\mathbf{u}(\lambda)|\}_i\, dy.$$

It follows in a similar manner to the proof of lemma 7 that

$$|I_1| \leqq \frac{1}{2\pi\theta_1}\int_{|y|>\alpha} e^{-\rho(\theta(t))t}\, t\, e^{dt}\{|\mathbf{Q}(\lambda)|\, e^{|\mathbf{Q}(\lambda)|t}\, |\mathbf{u}(\lambda)|\}_i\, dy,$$

where $d = \max(d_1, \ldots, d_n)$, and $d_i = \{\mathbf{D}\}_{ii}$.

Let $\lambda = \theta(t) + iy$, with $|y| > \alpha$ and $t > T_1$. From the conditions on $\mathbf{u}(\lambda)$ and $\mathbf{P}(\lambda)$ there exist u^* and $k_{ij}(y)$ such that $|u_i(\lambda)| < u^*$ and $|P_{ij}(\lambda)| \leqq k_{ij}(y)$ with $\int_{-\infty}^{\infty} k_{ij}(y)dy < \infty$. If $\beta = \max_{i,j|y|>\alpha} \sigma_j\lambda_{ij}k_{ij}(y)$, then for α sufficiently large, β can be made arbitrarily small. If $\mathbf{H}$ is a matrix of unit elements then

$$e^{|\mathbf{Q}(\lambda)|t} = \sum_{k=0}^{\infty} |\mathbf{Q}(\lambda)|^k t^k/k! < \sum_{k=0}^{\infty} (\beta t\mathbf{H})^k/k! = e^{n\beta t}\,\mathbf{H}.$$

Hence

$$|I_1| \leqq [u^*t/(2\pi\theta_1)]\exp[(d + n\beta - \rho(\theta(t)))t] \sum_{j=1}^{n} \int_{|y|>\alpha} k_{ij}(y)\, dy.$$

Using the corollary to lemma 5 we obtain

$$|I_1| \leqq [u^*t/(2\pi\theta_1)]\exp[(d + n\beta - \rho(\theta_1))t] \sum_{j=1}^{n} \int_{|y|>\alpha} k_{ij}(y)\, dy.$$

From lemma 6, $\rho(\theta_1) > \mu \geqq d$, hence for α sufficiently large $\rho(\theta_1) > d + n\beta$. Then there exists an $\alpha > 0$ and a $T_2 > T_1$ such that $|I_1| < \varepsilon/4$ for $t > T_2$.

(ii) For $t > T_2$ and α as in (i), we have

$$|I_2| \leqq \frac{1}{2\pi\theta_1}\int_{|y|<\alpha} \exp[-t(\rho(\theta(t)) - \mu)]\, |u_i(\lambda)|\, dy$$
$$\leqq \frac{\alpha}{\pi\theta_1} \sup_{\theta(t)\in[\theta_1,\theta_2]} |u_i(\theta(t))| \exp[-t(\rho(\theta_1) - \mu)].$$

Since $\rho(\theta_1) > \mu$ by lemma 6, there exists a $T_3 > T_2$ such that $|I_2| < \varepsilon/4$ for $t > T_3$.

(iii) For $t > T_3$, any $\delta > 0$ and α as in (i);

$$|I_3| \leqq \frac{1}{2\pi\theta_1}\int_{\delta\leqq|y|\leqq\alpha} e^{-\rho(\theta(t))}\{e^{|\mathbf{K}(\lambda)|t}\, |\mathbf{u}(\lambda)|\}_i\, dy.$$

Let $K_{ij}^*(\theta(t)) = \sup_{\delta\leqq|y|\leqq\alpha} |K_{ij}(\theta(t) + iy)|$. Then $K_{ij}^*(\theta(t)) \leqq K_{ij}(\theta(t))$, with equality if and only if $K_{ij}(\theta(t)) = 0$, i.e., $\lambda_{ij} + \delta_{ij}(\mu - \mu_i) = 0$. Hence if $K^*(\theta(t)) = (K_{ij}^*(\theta(t)))$, then $\rho(\mathbf{K}^*(\theta(t))) < \rho(\mathbf{K}(\theta(t)))$ by lemma 1 (ii). Let $K_{ij}^* = \sup_{\theta(t)\in[\theta_1,\theta_2]} K_{ij}^*(\theta(t))$ and $\mathbf{K}^* = (K_{ij}^*)$. Then

$$|I_3| \leqq \frac{1}{2\pi\theta_1}\int_{\delta\leqq|y|\leqq\alpha} \exp[-t \inf_{\theta(t)\in[\theta_1,\theta_2]} \rho(\mathbf{K}(\theta(t)))]\, \{e^{\mathbf{K}^*t}\, |\mathbf{u}(\theta(t) + iy)|\}_i\, dy.$$

There exists an M such that $\{e^{\mathbf{K}^*t}\}_{ij} < M\, e^{\rho(\mathbf{K}^*)t}$, for $i, j = 1, \ldots, n$, and $t > T_3$. Hence

$$|I_3| \leqq \frac{nMu}{2\pi\theta_1}\int_{\delta\leqq|y|\leqq\alpha}\exp[t\{\rho(\mathbf{K}^*) - \inf_{\theta(t)\in[\theta_1,\theta_2]}\rho(\mathbf{K}(\theta(t)))\}]dy,$$

where $u = \max_i \sup u_i(\lambda)$, and the sup is over $\theta_1 \leqq \operatorname{Re}\lambda \leqq \theta_2$ and $\delta \leqq |\operatorname{Im}\lambda| \leqq \alpha$.

We now show that K_{ij}^* tends to $K_{ij}^*(\theta_0)$ as both $\theta_1\uparrow\theta_0$ and $\theta_2\downarrow\theta_0$. $|K_{ij}(\lambda)|$ is a continuous function of λ for $-\Delta_{V_{ij}} < \operatorname{Re}\lambda < \Delta_{V_{ij}}$, and is therefore uniformly continuous in a closed region within this strip. Thus for every $\zeta > 0$ there exists a δ_1 such that $|K_{ij}(\theta + iy) - K_{ij}(\theta_0 + iy)| < \zeta$, for $\theta \in [\theta_1, \theta_2]$ with $|\theta_1 - \theta_2| < \delta_1$ and $\delta \leqq |y| \leqq \alpha$. Therefore $K_{ij}(\theta + iy) < K_{ij}(\theta_0 + iy) + \zeta$ for all θ and y such that $\theta_1 \leqq \theta \leqq \theta_2$ and $\delta \leqq |y| \leqq \alpha$. Hence, for all θ with $\theta_1 \leqq \theta \leqq \theta_2$, $K_{ij}^*(\theta) \leqq K_{ij}^*(\theta_0) + \zeta$. Therefore it follows that $K_{ij}^* \leqq K_{ij}^*(\theta_0) + \zeta$. Clearly also $K_{ij}^*(\theta_0) \leqq K_{ij}^*$, and thus $|K_{ij}^* - K_{ij}^*(\theta_0)| < \zeta$ for $|\theta_1 - \theta_2| < \delta_1$. Hence for every $\phi > 0$, there exists a $\delta_2 > 0$ such that $|\rho(\mathbf{K}^*) - \rho(\mathbf{K}^*(\theta_0))| < (1/2)\phi$ for $|\theta_1 - \theta| < \delta_2$.

Let $\phi = \{\rho(\mathbf{K}(\theta_0)) - \rho(\mathbf{K}^*(\theta_0))\} > 0$. Using the continuity of $\rho(\mathbf{K}(\theta))$ for $\theta \in [\theta_1, \theta_2]$ we can choose a $T_4 > T_3$ such that

$$|\inf_{t>T_4}\rho(\mathbf{K}(\theta(t))) - \rho(\mathbf{K}(\theta_0))| < (1/2)\phi.$$

In addition for $t > T_4$, $\theta(t) \in [\theta_1, \theta_2]$ where $|\theta_2 - \theta_1| < \delta_2$. Hence $|\rho(\mathbf{K}^*) - \rho(\mathbf{K}^*(\theta_0))| < (1/2)\phi$. Therefore $\{\inf_{t>T_4}\rho(\mathbf{K}(\theta(t))) - \rho(\mathbf{K}^*)\} > 0$ and it immediately follows that $|I_3|$ tends to zero as t tends to infinity. Hence there exists a $T_5 > T_4$ such that $|I_3| < \varepsilon/4$ for $t > T_5$.

(iv) Let the eigenvalues of $\mathbf{K}(\theta_0)$ be $\rho(\theta_0), \mu_2, \ldots, \mu_n$. Note that $\mathbf{K}(\theta_0)$ has distinct eigenvalues. Using lemma 1, there exists a $\delta > 0$ and $\theta_1 < \theta_0 < \theta_2$ such that $\mathbf{K}(\lambda)$ has eigenvalues $\mu_1(\lambda) \equiv \rho(\lambda), \mu_2(\lambda), \ldots, \mu_n(\lambda)$ which are distinct for $\theta_1 < \operatorname{Re}\lambda < \theta_2$ and $|\operatorname{Im}\lambda| < \delta$. It also follows that $\rho(\lambda) \to \rho(\theta_0)$ and $\mu_j(\lambda) \to \mu_j$ as $\lambda \to \theta_0$.

We use this $\delta > 0$ in part (iii) to obtain T_5. Then we choose $T_6 > T_5$ so that $\theta(t) \in [\theta_1, \theta_2]$ for $t > T_6$.

Thus for the restricted range of λ,

$$e^{\mathbf{K}(\lambda)t} = \mathbf{E}(\lambda)\, e^{\rho(\lambda)t} + \sum_{j=2}^{n}\mathbf{E}_j(\lambda)\, e^{\mu_j(\lambda)t},$$

where $\mathbf{E}_j(\lambda) = \prod_{i\neq j}(\mathbf{K}(\lambda) - \mu_j(\lambda)\mathbf{I})/\prod_{i\neq j}(\mu_j(\lambda) - \mu_i(\lambda))$, which tends to $\mathbf{E}_j$ as $\lambda \to \theta_0$. In this closed neighbourhood of θ_0, $\mathbf{E}_j(\lambda)$ is continuous and hence $|\mathbf{E}_j(\lambda)| \leqq \mathbf{D}_j$, for some $\mathbf{D}_j$, $j = 2, \ldots, n$.

Now

$$\begin{aligned}|I_4| &\leqq \frac{1}{2\pi\theta_1}\sum_{j=2}^{n}\int_{|y|\leqq\delta}\exp[-t(\rho(\theta(t)) - \operatorname{Re}\mu_j(\lambda))]\,\{|\mathbf{E}_j(\lambda)|\,|\mathbf{u}(\lambda)|\}_i\,dy\\ &\leqq \frac{1}{2\pi\theta_1}\sum_{j=2}^{n}\{\mathbf{D}_j\mathbf{u}\}_i\exp[-t(\rho(\theta(t)) - \mu^*(\theta(t)))],\end{aligned}$$

where $\mu^*(\theta(t)) = \max_{j\geqq 2}[\sup_{|y|\leqq\delta}\{\mathrm{Re}(\mu_j(\theta(t) + iy)\}]$.

Noe $\mathrm{Re}(\mu_j(\lambda))$ is a continuous function of λ for $\lambda = \theta(t) + iy$ with $|y| < \delta$ and $t > T_5$. Hence for any $\varepsilon^* > 0$ there exists a $\delta^* > 0$ such that, for λ, $\lambda^* \in [\theta_1, \theta_2]$ and $|\lambda - \lambda^*| < \delta^*$, $|\mathrm{Re}(\mu_j(\lambda)) - \mathrm{Re}(\mu_j(\lambda^*))| < \varepsilon^*$. Hence, as in the proof of part (iii), the continuity of $\sup_{|y|\leqq\delta}\mathrm{Re}(\mu_j(\theta(t) + iy))$ is established. It then immediately follows that $\mu^*(\theta)$ is also continuous for θ in some small interval containing θ_0.

Since $\{\rho(\theta) - \mu^*(\theta)\}$ is a continuous function of θ for θ in a closed interval containing θ_0 it achieves its inf in this closed interval. This inf must then be positive. Hence there exists a $T_7 > T_6$ such that $\inf_{t>T_7}\{\rho(\theta(t)) - \mu^*(\theta(t))\} = \eta > 0$. Hence

$$|I_4| \leqq \frac{1}{2\pi\theta_1}\sum_{j=2}^{n}\{\mathbf{D}_j\mathbf{u}\}_i\, e^{-\eta t} \text{ for } t > T_7.$$

Therefore $|I_4|$ can be made arbitrarily small for t sufficiently large. Hence there exists a $T_8 > T_7$ such that $|I_4| < \varepsilon/4$ for $t > T_8$.

The theorem then follows taking $T > T_8$.

We now obtain the saddle point approximation. From lemma 4, $\rho(\lambda)$ and $\mathbf{E}(\lambda)$ are analytic in a neighborhood of $\lambda = \theta_0$. In theorem 1 we take δ sufficiently small and T sufficiently large so that $\lambda = \theta(t) + iy$ lies in the region of analyticity for $|y| < \delta$ and $t > T$.

Defining $\simeq$ to mean 'for t sufficiently large can be made arbitrarily close to', the standard saddle point method may be applied. From theorem 1, with $\lambda = \theta(t) + iy$,

$$\begin{aligned}\eta\, e^{\mu t-\gamma(t)} &\simeq \frac{1}{2\pi}e^{-\gamma(t)}\int_{-\delta}^{\delta}\lambda^{-1}\{\mathbf{E}(\lambda)\mathbf{u}(\lambda)\}_i\, e^{\rho(\lambda)t-\lambda s}\,dy\\ &\simeq \frac{1}{2\pi}\int_{-\delta}^{\delta}[\theta(t)]^{-1}\{\mathbf{E}(\theta(t))\mathbf{u}(\theta(t))\}_i\\ &\qquad \exp[iy\,\rho'(\theta(t))t-(1/2)\,\rho''(\theta(t))y^2\,t - iys]\,dy\end{aligned}$$

Since $\theta(t)$ is defined by $\rho'(\theta(t)) = s/t$, it follows that

$$\eta\, e^{\mu t-\gamma(t)} \simeq \frac{1}{2\pi}\int_{-\delta}^{\delta}[\theta(t)]^{-1}\{\mathbf{E}(\theta(t))\mathbf{u}(\theta(t))\}_i \exp[-(1/2)y^2\,\rho''(\theta(t))t]\,dy.$$

Hence for t sufficiently large,

$$\eta\, e^{\mu t-\gamma(t)} \simeq \{\mathbf{E}(\theta(t))\mathbf{u}(\theta(t))\}_i/[(2\pi)^{1/2}\,\theta(t)(\rho''(\theta(t))t)^{1/2}]. \tag{9}$$

9. The speed of propagation. We now obtain $\not{c} = \lim_{t\to\infty} s/t$. From equation (9) it follows that

$$\exp\{(\rho(\theta(t)) - \mu)t - \theta(t)s\} \simeq \eta(2\pi)^{1/2}\,\theta(t)(\rho''(\theta(t))t)^{1/2}/\{\mathbf{E}(\theta(t))\mathbf{u}(\theta(t))\}_i.$$

Thus

$$\frac{s}{t} \simeq \frac{\rho(\theta(t)) - \mu}{\theta(t)} - \frac{1}{\theta(t)t}\log\left[\frac{\eta(2\pi)^{1/2}\theta(t)(\rho''(\theta(t))t)^{1/2}}{\{\mathbf{E}(\theta(t))\mathbf{u}(\theta(t))\}_i}\right].$$

Therefore $\lim_{t\to\infty} s/t = \lim_{t\to\infty}[\{\rho(\theta(t)) - \mu\}/\theta(t)]$.

Note that $\theta(t)$ is such that $s/t = \rho'(\theta(t))$. Now $\rho'(x)$ is a continuous function of x since $\rho(\lambda)$ is analytic. Hence $\lim_{t\to\infty} \theta(t) = \theta_0$, where $\rho'(\theta_0) = \lim_{t\to\infty} s/t$. Therefore $\ell = \lim_{t\to\infty} s/t = \{\rho(\theta_0) - \mu\}/\theta_0 = \rho'(\theta_0)$. Now $f(\theta) = \{\rho(\theta) - \mu\}/\theta$, and $f'(\theta) = \{\rho'(\theta) - f(\theta)\}/\theta$. Therefore $\rho'(\theta) = f(\theta)$ if and only if $f'(\theta) = 0$. From lemma 6(ii) θ_0 is the unique value of θ such that $f'(\theta) = 0$, and hence $\ell = \min_{\theta\in(0,\Delta_V)} f(\theta)$. From §5 it therefore follows that $\ell = c_0$.

For equations (6), this result is exact and is set out in theorem 2. However, for the models considered in this paper the results are only approximate and are summarized after the theorem.

THEOREM 2. *Let s and θ_0 be defined as in* §8. *Suppose the following condition hold*:

(i) $\ell = \lim_{t\to\infty} s/t$ *exists and is positive.*

(ii) $\mathbf{K}(\theta_0)$ *has distinct eigenvalues.*

(iii) *The $p_{ij}(r)$ are symmetric, and exponential in the tail, for all i and j.*

(iv) *For any interval* $[\theta_1, \theta_2] \subset (0, \Delta_V)$, *and for all* $\theta \in [\theta_1, \theta_2]$, *there exist* $k_{ij}(y)$ *such that* $|P_{ij}(\theta + iy)| \leqq k_{ij}(y)$ *and* $\int_{-\infty}^{\infty} k_{ij}(y)dy < \infty$.

(v) *Each $y_i(s, 0)$ is a function of bounded support, for $i = 1, \ldots, n$.*

The speed of propagation in the tail of equations (6) *is* $c_0 = \min_{\theta\in(0,\Delta_V)} f(\theta)$. This is the minimum velocity for which wave solutions exist in the model for the deterministic epidemic.

Summary. Since equations (6) correspond in the tail to the epidemic model of §2 and the contact birth process of §3, this indicates the following results:

(i) The asymptotic velocity of propagation of the spread of infection in the deterministic model of §2 is c_0.

(ii) Let $U_i(t)$ be the position of the individual of type i with maximum position at time t in the n-type contact birth process of §3. Then $y_i(s, t) = P(U_i(t) > s)$ has asymptotic speed of translation c_0.

10. The stochastic n-type epidemic. In this section we consider the stochastic analogue of the model in §2; which is an n-type version of the two-type model of Radcliffe [**8**].

Consider n populations of uniform densities on the real line $\mathbf{R}$, each of which consists of susceptible and removed individuals. Let $X_i(s, t)\delta s$, $Y_i(s, t)\delta s$ and $Z_i(s, t)\delta s$ denote the numbers of susceptibles, infectious and removed individuals respectively in the interval $(s, s + \delta s)$. The density of individuals of type i is σ_i, i.e., $\sigma_i\delta s$ is the number of type i individuals in $(s, s + \delta s)$. Let λ_{ij} be the rate of infection of an individual in population i by an individual in population j. The contact distribution represents the distance r over which infection occurs and has density $p_{ij}(r)$. The removal rate for infected individuals in the i-th population is μ_i.

More specifically the assumptions regarding infection are taken to be: (i) The probability that a specific susceptible in the i-th population in $(s, s + \delta s)$ is infected by a specific infectious individual in the j-th population in $(r, r + \delta r)$ in the time interval $(t, t + \delta t)$ is $\lambda_{ij} p_{ij}(s - r)\delta t + o(\delta t)$. (ii) The probability that an infectious individual in population i dies in the time interval $(t, t + \delta t)$ is $\mu_i \delta t + o(\delta t)$.

We set up equations for the mean distributions which are approximately true in the tail of the epidemic. In the tail $X_i(s, t)$ is very close to σ_i, and we approximate by letting $X_i(s, t)$ remain equal to σ_i. This makes the process multiplicative in the tail, i.e., it is behaving like a position dependent Markov branching process.

The mean distributions $f_i(r, t)$ are such that $f_i(r, t)\delta r$ represents the expected number of infectious individuals in $(r, r + \delta r)$ at time t. These then satisfy the equations

$$\frac{\partial f_i(r, t)}{\partial t} = -\mu_i f_i(r, t) + \sum_{j=1}^{n} \sigma_i \int_{-\infty}^{\infty} \lambda_{ij} p_{ij}(r - s) f_j(s, t) ds, \; (i = 1, \ldots, n).$$

Let $y_i(r, t)$ be the expected proportion of infectives at position r and time t, i.e., $y_i(r, t) = f_i(r, t)/\sigma_i$, $(i = 1, \ldots, n)$. Then in the tail of the epidemic the $y_i(r, t)$ satisfy the equations

$$(9) \quad \frac{\partial y_i(r, t)}{\partial t} = -\mu_i y_i(r, t) + \sum_{j=1}^{n} \sigma_j \int_{-\infty}^{\infty} \lambda_{ij} p_{ij}(r - s) y_j(s, t) ds, \; (i = 1, \ldots, n).$$

These equations are identical to equations (1). This suggests c_0 as an approximation for the asymptotic expectation velocity for this model. It is assumed that the contact distributions are not too spread out so that only contributions near the tail, where the approximation is likely to be valid, are important. However, even under these conditions it is not clear to what extent the approximation is valid. Thus in general c_0 may only be a crude approximation for the asymptotic expectation velocity.

References

1. D.G. Aronson, *The asymptotic speed of propagation of a simple epidemic. Nonlinear Diffusion,* ed. W.E. Fitzgibbon and H.F. Walker, Research Notes in Mathematics, Pitman, London, 1977.

2. C. Atkinson, and G.H. Reuter, *Deterministic epidemic wave,* Math. Proc. Cambridge Philos. Soc. **80** (1976), 315–330.

3. H.E. Daniels, *The deterministic spread of a simple epidemic, Perspectives in Probability and Statistics,* Academic Press, London, 1975.

4. O. Diekmann, *Thresholds and travelling waves for the geographical spread of infection*, J. Math. Biol. **6** (1978), 109–130.

5. ———, *Run for your life. A note on the asymptotic speed of propagation of an epidemic*, J. of Differential Equations **33** (1978), No. 1, 58–73.

6. J. Dieudonné, *Foundations of Modern Analysis*, Chelsea, New York, 1969.

7. D. Mollison, *Markovian contact processes*, Advances in Appl. Prob. **10** (1978), 85–108.

8. J. Radcliffe, *The initial geographical spread of host-vector and carrier-borne epidemics*, J. Appl. Prob. **10** (1973), 703–717.

9. ———, and Rass L. *Wave solutions for the deterministic non-reduciable n-type epidemic*, J. Math. Biol. **17** (1983), 45–66.

10. J. Radcliffe, L. Rass, and W.D. Stirling, *Wave solutions for the deterministic host-vector epidemic*, Math. Proc. Cambridge Philos. Soc. **91** (1982), 131–152.

11. H.R. Thieme, *Asymptotic estimates of the solutions of nonlinear integral equations and asymptotic speeds for the spread of populations*. J. Reine. Angew. Math. **306** (1979), 94–121.

Queen Mary College, London