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A CHARACTERIZATION OF ORIENTED GRASSMANN MANIFOLDS

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Introduction. Let $G_{p,q}$ denote the oriented Grassmann manifold of p-planes in $\mathbf{R}^{p+q}$. Our purpose is to give a characterization of $G_{p,q}$ and its non-compact dual $G^*_{p,q}$ in terms of a parallel tensor field T satisfying certain algebraic conditions and its behaviour on geodesic spheres. When $q = 1$ our result contains that of L. Vanhecke and T. J. Willmore on spaces of constant curvature ([5]. [2]). For $q = 2$, a different characterization has been obtained by B. J. Papantoniou using the Hermitian structure which exists for that case [4].

In the course of our work we give (Proposition 3) an algebraic characterization of the tensor T on a vector space V^{pq}. Although every Riemannian manifold trivially carries a parallel tensor field satisfying the given conditions, namely $T(X, Y, Z) = g(Y, Z)X$, for $p, q \geqq 2$, T plays a significant role in the geometry of the Grassmann manifolds, somewhat analogous to the underlying almost complex structure on a Kähler manifold. In [5] Vanhecke and Willmore have also characterized the complex space forms in terms of their Kähler structures and the shape of their geodesic spheres. They have similarly characterized the remaining rank 1 symmetric spaces.

Some Properties of $G_{p,q}$. We consider $G_{p,q}$ as the Riemannian symmetric space $SO(p + q)/(SO(p) \times SO(q))$. Then following Kobayashi and Nomizu [3 pp. 271–273], for example, we may identify the tangent space at a point $m \in G_{p,q}$ with the vector space of real $p \times q$ matrices. Moreover the inner product

$$g(X, X) = \operatorname{tr} XX^t \tag{1}$$

at m gives rise to an invariant metric g on $G_{p,q}$ with curvature tensor R at m given by

$$R(X, Y)Z = XY^tZ + ZY^tX - ZX^tY - YX^tZ. \tag{2}$$

Similarly for the non-compact dual $G^*_{p,q}$ of $G_{p,q}$ the curvature at a point is given by the negative of this expression. Any other invariant metric

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g_c on these spaces is obtained by choosing $c > 0$ and defining g_c at m by $g_c = cg$, i.e. each g_c is homothetic to g.

The tensor T of type (1, 3) at m defined by

$$T(X, Y, Z) = XY^tZ \tag{3}$$

is invariant by $SO(p) \times SO(q)$ and hence extends to a parallel tensor field on $G_{p,q}$, also denoted by T. As a matter of notation we write

$$T_{XY}Z = T(X, Y, Z),\ T^{XY}Z = T(Z, X, Y),\ T^X_XY = T(X, Y, X).$$

and it is easy to check that the linear operators T_{XX}, T^{XX} and T^X_X are self-adjoint (see property P_1 below).

T has the following properties at m, and hence on $G_{p,q}$, which are immediate from (1) and (3):

P_1: $g(T(X, Y, Z), W) = g(T(Z, W, X), Y) = g(T(Y, X, W), Z)$,

P_2: $T(T(X, Y, Z), U, V) = T(X, T(U, Z, Y), V) = T(X, Y, T(Z, U, V))$,

P_3: $g(T^{XX^r}X, X) = 1/p\,\mathrm{tr}(T^{XX^{r+1}})$, $g(T^r_{XX}X, X) = 1/q\,\mathrm{tr}(T^{r+1}_{XX})$ for all integers $r \geqq 0$,

P_4: $\mathrm{tr}\,T^X_X = g(X, X)$.

One proves by elementary matrix operations that if V is a unit vector at m then $T(V, V, V) = V$, if and only if, as a matrix V has rank 1 and moreover in this case there exist orthogonal matrices P and Q such that

$$PVQ = \begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & & & & & \\ \vdots & & & \bigcirc & & \\ 0 & & & & & \end{pmatrix} \tag{4}$$

The map $X \to PXQ$ just corresponds to an orthonormal change of basis. For such a V it follows, either by direct computation or by using the canonical form (4), that the linear map of the tangent space at m

$$X \to R(V, X)V \tag{5}$$

has the following (possibly zero) eigenvectors:

(i) $T(V, X, V)$ and $X - T(V, V, X) - T(X, V, V) + T(V, X, V)$ in the zero eigenspace

(ii) $T(V, V, X) - T(V, X, V)$ and $T(X, V, V) - T(V, X, V)$ in the -1 eigenspace.

We now obtain a property of geodesic spheres in Riemannian locally symmetric spaces. For any Riemannian locally symmetric space M of dimension $\geqq 3$ let S_s denote the geodesic sphere with centre $m \in M$ and radius s contained in a normal neighborhood U of m. Let γ be a geodesic from m contained in U and let V be a parallel vector field along γ such that for some $c \in \mathbf{R}$, $R(\dot\gamma, V_m)\,\dot\gamma = cV_m$. Then let X be the Jacobi field along γ with initial conditions $X_m = 0$, $(\nabla_{\dot\gamma}X)_m = V_m$. Since $R(\dot\gamma, \cdot)\dot\gamma$ is parallel along γ we have $R(\dot\gamma, V)\dot\gamma = cV$ and, since $\nabla^2_{\dot\gamma}X = R(\dot\gamma, X)\dot\gamma$, we see that $X = fV$ where

$$f(s) = \begin{cases} |c|^{-1/2}\sin(|c|^{1/2}s) & \text{if } c < 0 \\ c^{-1/2}\sinh(c^{1/2}s) & \text{if } c > 0 \\ s & \text{if } c = 0. \end{cases}$$

Since the Riemannian curvature at m is bounded, the eigenvalues c are bounded, say $|c| < k^2$, $k > 0$. Thus if we take U to be a geodesic ball of radius $<\pi/k$. then $f \neq 0$ on U except at m. Now let N denote the unit vector field on $U\backslash\{m\}$ of tangent vectors to geodesics from m. We know from [1] that $\nabla_X N = \nabla_N X$ for the Jacobi field X as above. Hence the Weingarten map A_N of the geodesic spheres S_s satisfies $A_N X = -\nabla_N X$. This has two consequences. Firstly,

$$\begin{aligned} R(N, X)N &= [\nabla_N, \nabla_X]N - \nabla_{[N,X]}N \\ &= -\nabla_N AX \\ &= A^2X - (\nabla_N A)X. \end{aligned} \tag{6}$$

Since this equation is linear it is satisfied by all vector fields X along γ orthogonal to N. Secondly, we have $X = fV$, so that

$$A_N V = -\frac{f'}{f} V. \tag{7}$$

Thus we have proved the following consequence of (7).

PROPOSITION 1. *Let m be a point in a Riemannian locally symmetric space of dimension* $\geqq 3$. *then m has a normal neighborhood U such that for any geodesic* γ *from m, the parallel translate of an eigenspace of the linear map* $X \to R(N, X)N$ *at m is contained in an eigenspace of the Weingarten map* A_N *for each geodesic sphere in U about m.*

We next apply this result to $G_{p,q}$. Let $m \in G_{p,q}$ and U a normal neighborhood of m as in Proposition 1. Let γ be a geodesic in U from m and X a parallel vector field along γ. Finally, let N be the unit vector field tangent to γ. We then have the identity $X = X_1 + X_2$ where

$$\begin{aligned} X_1 &= X - T_{NN}X - T^{NN}X + 2T^N_N X, \\ X_2 &= T_{NN}X + T^{NN}X - 2T^N_N X. \end{aligned}$$

Now suppose that $T(N, N, N) = N$ and that X is orthogonal to N. Then from (i), (ii) following (5), X is the sum of two parallel eigenvector fields X_1 and X_2 of $R(N, \cdot)N$ along γ. Hence, as a consequence of Proposition 1, we see that at any point, other than m, on γ $A_N X = aX_1 + bX_2$ for some $a, b \in \mathbf{R}$. Equivalently

$$A_N X = aX + (a - b)(2T_N^N X - T_{NN}X - T^{NN}X).$$

Since this property holds at all points of $U \backslash \{m\}$ we have immediately the following result.

PROPOSITION 2. *Let $m \in G_{p,q}$ and choose a normal neighborhood U of m as in Proposition* 1. *Then for any geodesic sphere S_s in U with centre m and for any unit normal N to S_s such that $T(N, N, N) = N$, the Weingarten map of S_s satisfies*

$$A_N X = f(N)X + g(N)(2T_N^N X - T_{NN}X - T^{NN}X) \tag{8}$$

for some $f(N), g(N) \in \mathbf{R}$.

We remark that $f(N)$ and $g(N)$ could be determined for $G_{p,q}$ by the methods outlined earlier. However the above general form for $A_N X$ will be sufficient for our purposes.

A characterization of $\mathbf{G}_{p,q}$. We now state our main result.

THEOREM. *Let M be a complete, simply connected Riemannian manifold of dimension $pq \geqq 3$ with metric g. Let T be a parallel tensor field of type* (1, 3) *on M satisfying P_1 through P_4. Suppose that for each $m \in M$ there exists a normal neighborhood U of m such that for each geodesic sphere S_s in U with centre m and each unit normal N to S_s with $T(N, N, N) = N$, the Weingarten map satisfies* (8). *Then M is homothetic to either the Euclidean space E^{pq}, $G_{p,q}$* or *$G^*_{p,q}$.*

Before proving the theorem we first consider the tensor field T and show how it can be described as in (3) at any point.

PROPOSITION 3. *Let V be a real vector space of dimension pq with inner product $\langle\, ,\rangle$ and T a tensor of type* (1, 3) *on V satisfying P_1 through P_4 with $\langle\, ,\rangle$ replacing g. Then V is isomorphic to the vector space of all real $p \times q$ matrices and under the identification $T(X, Y, Z) = XY^tZ$ and $\langle X, X\rangle = \operatorname{tr} XX^t$.*

The proof of this proposition requires several lemmas. The first lemma is immediate from $P_1, \ldots, P_4$ and provides a useful duality between T_{XY} and T^{XY}.

LEMMA 1. *Define a tensor S on V by $S(X, Y, Z) = T(Z, Y, X)$ and write $S_{XY} = T^{YX}$, $S^{XY} = T_{YX}$, $S_X^X = T_X^X$. Then P_1, P_2 and P_4 are satisfied when T is replaced by S, and P_3 is satisfied when T^{XX} and T_{XX} are replaced by S_{XX} and S^{XX} respectively. In particular any property of T_{XX} is also satisfied by T^{XX} provided p and q are interchanged.*

LEMMA 2. *For any $X \in V$ and non-negative integer r, $T_{XX}^r X = T^{XX^r} X$. Moreover if $X \neq 0$ the T_{XX}, T^{XX} and T_X^X are nonzero self-adjoint endo-*

morphisms of V with T_{XX} and T^{XX} positive semi-definite. In particular $T(X, X, X) \neq 0$.

PROOF. The first statement follows from P_2 by induction on r. Also the self-adjoint properties are clear from P_1, and P_4 shows that T_X^X is non-zero. Now if $T_{XX} = 0$, then from P_2 we have for all $Y \in V$

$$T_X^{X^2} Y = T(X, T(X, Y, X), X) = T(T(X, X, Y), X, X) = 0$$

which is impossible since T_X^X is non-zero and self-adjoint. We now prove the positive semi-definiteness of T_{XX}. Let $\mu_1, \ldots, \mu_\ell$ be the distinct eigenvalues of T_{XX} with multiplicity $m_1, \ldots, m_\ell$ respectively, and let $X = X_1 + \cdots + X_\ell$ where $X_1, \ldots, X_\ell$ are the projections of X onto the corresponding eigenspaces. Then by P_3

$$\sum_{\alpha=1}^{\ell} \mu_\alpha^r (m_\alpha \mu_\alpha - q\langle X_\alpha, X_\alpha\rangle) = 0$$

for $r = 0, 1, 2, \ldots$. It follows that for each α, with $m_\alpha \neq 0$, $m_\alpha \mu_\alpha - q\langle X_\alpha, X_\alpha\rangle = 0$. Thus each $\mu_\alpha \geqq 0$ as required. Lemma 1 gives the result for T^{XX}. Finally, by choosing $r = 1$ in P_3, it is now clear that $T(X, X, X) \neq 0$.

LEMMA 3. *For any $X \in V$ and $r = 0, 1, 2, \ldots$*

$$T(T_{XX}^r X, T_{XX}^r X, T_{XX}^r X) = T_{XX}^{3r+1} X.$$

PROOF. We note that from P_3

$$T(Y, Z, T_{XX} U) = T(Y, T_{XX} Z, U)$$

and

$$T(T_{XX} U, Y, Z) = T_{XX} T(U, Y, Z).$$

The result follows by induction on r.

LEMMA 4. *Suppose X is a unit vector in V such that $T(X, X, X) = \lambda X$. Then λ is the only non-zero eigenvalue of T_{XX} (resp. T^{XX}) and $\lambda = q/m$ (resp. p/n) where m (resp. n) is the multiplicity of λ as an eigenvalue of T_{XX} (resp. T^{XX}).*

PROOF. Again we prove this only for T_{XX}, the result for T^{XX} following by Lemma 1. Suppose $T_{XX} Y = \mu Y$ where $\|Y\| = 1$. Then by P_2

$$\begin{aligned} \lambda\mu &= \lambda\langle Y, T(X, X, Y)\rangle \\ &= \langle Y, T(X, T(X, X, X), Y)\rangle \\ &= \langle Y, T(X, X, T(X, X, Y))\rangle \\ &= \mu^2. \end{aligned}$$

Hence either $\mu = 0$ or λ. Now $\lambda = q/m$ by virtue of P_3 with $r = 0$.

Let X be a unit vector in V; by virtue of Lemma 2, $\|T^r_{XX}X\| \neq 0$ and we set

$$Y_r = \frac{T^r_{XX}X}{\|T^r_{XX}X\|}$$

LEMMA 5. *The sequence* $\{Y_r\}$ *converges to a unit vector* Y *and* $T(Y, Y, Y) = \lambda Y$ *for some* $\lambda \in \mathbf{R}$. *Moreover* $rkT_{YY} \leqq rkT_{XX}$ *and* $rkT^{YY} \leqq rkT^{XX}$.

PROOF. First note from P_2 that $\langle T_{XX}Y, Z\rangle = \langle T_{XX}Z, Y\rangle$ and hence, as a consequence of P_3,

$$\|T^r_{XX}X\|^2 = \frac{1}{q}\mathrm{tr}(T^{2r+1}_{XX}).$$

Now with the same notation as in the proof of Lemma 2, let μ_θ be the greatest eigenvalue of T_{XX}.
Then

$$\frac{T^r_{XX}X_\theta}{\|T^r_{XX}X\|} = \mu^r_\theta\left(\frac{1}{q}\sum_{\alpha=1}^{\prime} m_\alpha\mu_\alpha^{2r+1}\right)^{-1/2} X_\theta = \left(\frac{q}{m_\theta\mu_\theta}\right)^{1/2} X_\theta$$

as $r \to \infty$. Also as T_{XX} has no negative eigenvalues, we have for any eigenvalue $\mu_\beta \neq \mu_\theta$

$$\begin{aligned}\frac{\|T^r_{XX}X_\beta\|}{\|T^r_{XX}X\|} &= \mu^r_\beta\left(\frac{1}{q}\sum_{\alpha=1}^{\prime} m_\alpha\mu_\alpha^{2r+1}\right)^{-1/2} \|X_\beta\| \\ &\leqq \mu^r_\beta\left(\frac{m_\theta}{q}\,\mu_\theta^{2r+1}\right)^{-1/2} \|X_\beta\| \\ &\to 0\end{aligned}$$

as $r \to \infty$. Thus $Y = \lim_{r\to\infty} Y_r = (q/(m_\theta\mu_\theta))^{1/2} X_\theta$. Since each Y_r is a unit vector then so is Y.

Next we use Lemma 3 to obtain, after a similar calculation

$$T(Y_r, Y_r, Y_r) = \frac{T^{3r+1}_{XX}X}{\|T^r_{XX}X\|^3} \to \frac{q}{m_\theta} Y$$

and hence $T(Y, Y, Y) = (q/m_\theta)Y$ as required. By Lemma 4, $rkT_{YY} = m_\theta$ so that $rkT_{YY} \leqq rkT_{XX}$. Finally the corresponding result for T^{XX} follows from Lemma 1.

Note that Lemma 5 proves the existence of a unit vector X satisfying $T(X, X, X) = \lambda X$. As an easy consequence of Lemmas 4 and 5 we have the following.

LEMMA 6. $k = \max\{\lambda \mid T(X, X, X) = \lambda X, \|X\| = 1\}$ *exists and is attained; moreover* rkT_{UU} *is the minimum over all unit vectors in* V *if and only if* $T(U, U, U) = kU$.

Now choose a unit vector U as in Lemma 6 and set $V_1 = \mathrm{im} T_{UU}$; V_1 is just the k-eigenspace of T_{UU} and $U \in V_1$.

LEMMA 7. *If $X \in V_1$ and $Y, Z \in V$, then $T(X, Y, Z) \in V_1$.*

PROOF. $X = T(U, U, W)$ for some $W \in V$. Hence from P_2, $T(X, Y, Z) = T(T(U, U, W), Y, Z) = T(U, U, T(W, Y, Z)) \in V_1$.

LEMMA 8. *If $X \in V_1$ and $Y \in V_1^\perp$, then $T(X, X, Y) = 0$.*

PROOF. By Lemma 7, $T(X, X, Y) \in V_1$, but for $Z \in V_1$, P_1 gives $\langle T(X, X, Y), Z\rangle = \langle T(X, X, Z), Y\rangle = 0$ and hence $T(X, X, Y) = 0$.

LEMMA 9. *For each unit vector $X \in V_1$, $rkT_{XX} = rkT_{UU}$, $T_{XX} = kI$ on V_1, $T_{XX} = 0$ on $V_1^\perp$ and* $\dim V_1 = q/k$.

PROOF. Write $X = T(U, U, W)$. Now $\ker T_{UU} \subset \ker T_{XX}$ for if $T(U, U, Y) = 0$, then $T(X, X, Y) = T(X, T(U, U, W), Y) = T(X, W, T(U, U, Y)) = 0$. Thus $rkT_{UU} \geqq rkT_{XX}$, but by Lemma 6 rkT_{UU} is the minimum over unit vectors in V_1 giving the equality. Furthermore by Lemma 8, $V_1 = \mathrm{im} T_{UU} = \mathrm{im} T_{XX}$. By Lemma 6, $T(X, X, X) = kX$ and so by Lemma 4, $T_{XX} = kI$ on V_1. Finally using P_3 we have $\dim V_1 = q/k$.

Next define $W_1 = \mathrm{im} T^{UU}$ with U as in Lemma 6. Then by Lemma 1, Lemma 9 holds for W_1 with T_{XX} replaced by T^{XX} and q by p.

Now for any $X \in V_1 \cap W_1$ we have $T_{XX} = T^{XX} = k\langle X, X\rangle I$ on $V_1 \cap W_1$. Hence for all $X, Y \in V_1 \cap W_1$

$$T(X, Y, X) + T(Y, X, X) = 2k\langle X, Y\rangle X$$

and

$$T(Y, X, X) = k\langle X, X\rangle Y.$$

These two equations give $T_X^X Y$ on $V_1 \cap W_1$ as we now state.

LEMMA 10. *For all $X, Y \in V_1 \cap W_1$ $T_X^X Y = 2k\langle X, Y\rangle X - k\langle X, X\rangle Y$.*

On the other hand we have the following for $Y \in (V_1 \cap W_1)^\perp$.

LEMMA 11. *If $X \in V_1 \cap W_1$ and $Y \in (V_1 \cap W_1)^\perp$ then $T_X^X Y = 0$.*

PROOF. From P_2 we have $T_X^{X^2} Y = T_{XX} T^{XX} Y = T^{XX} T_{XX} Y$ and hence $T_X^{X^2} Y \in V_1 \cap W_1$. Now for $Z \in V_1 \cap W_1$,

$$\begin{aligned}
\langle T_X^{X^2} Y, Z\rangle &= \langle T(X, X, T(Y, X, X)), Z\rangle \\
&= \langle T(X, X, Z), T(Y, X, X)\rangle \\
&= k\langle Z, T(Y, X, X)\rangle \\
&= k\langle Y, T(Z, X, X)\rangle \\
&= k^2\langle Y, Z\rangle \\
&= 0.
\end{aligned}$$

Thus $T_X^{X^2} Y = 0$, but T_X^X is self-adjoint, hence $T_X^X Y = 0$.

LEMMA 12. $k = 1 = \dim V_1 \cap W_1$.

PROOF. Let $d = \dim V_1 \cap W_1$. Now by Lemmas 10 and 11, if X is a unit vector in $V_1 \cap W_1$, then T_X^X has eigenvalues k with multiplicity 1, $-k$ with multiplicity $d - 1$, and 0. But by P_4 we have $1 = \operatorname{tr} T_X^X = (2 - d)k$. Now since d is a positive integer, $k = 1 = d$.

PROOF OF PROPOSITION 3. Choose a unit vector e_{11} in V such that $T(e_{11}, e_{11}, e_{11}) = e_{11}$ and define $V_1 = \operatorname{im} T_{e_{11}e_{11}}$, $W_1 = \operatorname{im} T^{e_{11}e_{11}}$ as before. Since $k = 1$ we know from Lemma 9 that dim $V_1 = q$ and dim $W_1 = p$. Now choose orthonormal bases $\{e_{11}, e_{12}, \ldots, e_{1q}\}$ for V_1 and $\{e_{11}, e_{21}, \ldots, e_{p1}\}$ for W_1. Then define $e_{i\alpha} = T(e_{i1}, e_{11}, e_{1\alpha})$ for $i = 2, \ldots, p$, $\alpha = 2, \ldots, q$; note that in fact $e_{1\alpha}$ and e_{i1} also satisfy this relation. We wish to prove that $\{e_{i\alpha}\}$ is an orthonormal basis for V. First note that by Lemma 9, $T(e_{1\alpha}, e_{1\alpha}, e_{1\beta}) = e_{1\beta}$. On the other hand taking $e_{1\alpha}$ in the role of e_{11}, as we may do since $e_{1\alpha} \in V_1$ and so $T(e_{1\alpha}, e_{1\alpha}, e_{1\alpha}) = e_{1\alpha}$, the dual of Lemma 8 or 9 together with Lemma 11 gives $T(e_{i\beta}, e_{i\alpha}, e_{i\alpha}) = 0$ for $\beta \neq \alpha$. Thus we have

$$\begin{aligned} T(e_{1\alpha}, e_{1\beta}, e_{1\gamma}) &= T(e_{1\alpha}, T(e_{1\beta}, e_{1\beta}, e_{1\beta}), e_{1\gamma}) \\ &= T(T(e_{1\alpha}, e_{1\beta}, e_{1\beta}), e_{1\beta}, e_{1\gamma}) \\ &= \delta_{\alpha\beta} T(e_{1\beta}, e_{1\beta}, e_{1\gamma}) \\ &= \delta_{\alpha\beta} e_{1\gamma} \end{aligned}$$

Similarly $T(e_{i1}, e_{j1}, e_{k1}) = \delta_{jk} e_{i1}$. From these results

$$\begin{aligned} \langle e_{i\alpha}, e_{j\beta} \rangle &= \langle T(e_{i1}, e_{11}, e_{1\alpha}), T(e_{j1}, e_{11}, e_{1\beta}) \rangle \\ &= \langle T(e_{11}, e_{i1}, T(e_{j1}, e_{11}, e_{1\beta})), e_{1\alpha} \rangle \\ &= \langle T(e_{11}, T(e_{11}, e_{j1}, e_{i1}), e_{1\beta}), e_{1\alpha} \rangle \\ &= \delta_{ij} \langle T(e_{11}, e_{11}, e_{1\beta}), e_{1\alpha} \rangle \\ &= \delta_{ij} \delta_{\alpha\beta}. \end{aligned}$$

Thus $\{e_{i\alpha}\}$ is orthonormal and by dimension a basis for V.

Now for any $X \in V$ write $X = x_{i\alpha} e_{i\alpha}$ where we have used the usual summation convention. Then for $Y = y_{i\alpha} e_{i\alpha}$, $Z = z_{i\alpha} e_{i\alpha}$ we have

$$T(X, Y, Z) = x_{i\alpha} y_{j\beta} z_{k\gamma} T(e_{i\alpha}, e_{j\beta}, e_{k\gamma}).$$

But we have

$$\begin{aligned} T(e_{i\alpha}, e_{j\beta}, e_{k\gamma}) &= T(T(e_{i1}, e_{11}, e_{1\alpha}), T(e_{j1}, e_{11}, e_{1\beta}), T(e_{k1}, e_{11}, e_{1\gamma})) \\ &= T(T(e_{i1}, e_{11}, e_{1\alpha}), T(e_{11}, e_{k1}, T(e_{j1}, e_{11}, e_{1\beta})), e_{1\gamma}) \\ &= T(T(e_{i1}, e_{11}, e_{1\alpha}), T(T(e_{11}, e_{k1}, e_{j1}), e_{11}, e_{1\beta}), e_{1\gamma}) \\ &= \delta_{jk} T(T(e_{i1}, e_{11}, e_{1\alpha}), e_{1\beta}, e_{1\gamma}) \\ &= \delta_{jk} T(e_{i1}, e_{11}, T(e_{1\alpha}, e_{1\beta}, e_{1\gamma})) \\ &= \delta_{jk} \delta_{\alpha\beta} e_{i\gamma}. \end{aligned}$$

Therefore $T(X, Y, Z) = x_{i\alpha}y_{j\alpha}z_{j\gamma}e_{i\gamma}$ Now identifying X with its $p \times q$ matrix of components $(x_{i\alpha})$ we have the desired formula $T(X, Y, Z) = XY^tZ$. Clearly $\langle X, X\rangle = \mathrm{tr}XX^t$ and the proposition is proved.

Before giving the proof of the theorem we prove one more Lemma.

LEMMA 13. *Let S be a tensor of type* (1, 3) *on the vector space of all $p \times q$ matrices with inner product $\langle , \rangle$ as before satisfying the symmetries of the curvature tensor including the Bianchi identity. Suppose that $S(N, X)N = 0$ for every N of rank* 1 *and $S(X, Y)T = 0$. Then $S = 0$.*

PROOF. First if M and N have rank 1, linearization of $S(N, X)N = 0$ with $rk(M + N) = 1$ gives $S(N, X)M + S(M, X)N = 0$. Thus setting $S(X, Y, Z, W) = \langle S(X, Y)Z, W\rangle$ we have $S(N, X, M, X) + S(M, X, N, X) = 0$ from which $S(N, X, M, X) = 0$. Linearizing this last equation then gives

$$S(N, X, M, Y) + S(N, Y, M, X) = 0. \tag{9}$$

We will now show that $S(X, Y)N = 0$ which implies that $S = 0$ since any basis vector $e_{i\alpha}$ may be regarded as a rank 1 matrix. Refering to (4) we take N as e_{11}. Suppose that $S(X, Y)N$ is given by the matrix $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ where A is 1 by 1 and D is $(p - 1)$ by $(q - 1)$. Then

$$\begin{aligned}\begin{pmatrix} A & B \\ C & D\end{pmatrix} &= S(X, Y)\,(T(N, N, N)) \\ &= T(S(X, Y)N, N, N) + T(N, S(X, Y)N, N) + T(N, N, S(X, Y)N) \\ &= \begin{pmatrix} A & 0 \\ C & 0\end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & 0\end{pmatrix} + \begin{pmatrix} A & B \\ 0 & 0\end{pmatrix}\end{aligned}$$

from which we see that $A = 0$ and $D = 0$. Thus we need only consider the components of $S(X, Y)e_{11}$ where Y is a basis vector in the firsr row or column and we compute here only $S(X, Y, e_{11}, e_{1\alpha})$.

$$\begin{aligned} S(X, Y, e_{11}, e_{1\alpha}) &= S(e_{11}, e_{1\alpha}, X, Y) \\ &= -S(e_{11}, X, Y, e_{1\alpha}) - S(e_{11}, Y, e_{1\alpha}, X) \\ &= 2S(e_{11}, X, e_{1\alpha}, Y)\end{aligned} \tag{10}$$

by (9), but just as $S(X, Y)e_{11}$ has no (1,1) component as a matrix, $S(e_{11}, X)e_{1\alpha}$ has no $(1, \alpha)$ component and hence $S(X, Y, e_{11}, e_{1\alpha})$ vanishes for $Y = e_{k\gamma}$ with $k \neq 1$, $\gamma \neq \alpha$. If Y is $e_{1\gamma}$ or $e_{k\alpha}$ we may use (10) twice giving $S(X, Y, e_{11}, e_{1\alpha}) = 2S(e_{11}, X, e_{1\alpha}, Y) = 4S(e_{1\alpha}, e_{11}, Y, X)$ and hence $S(X, Y, e_{11}, e_{1\alpha}) = 0$.

PROOF OF THE THEOREM. We first prove the theorem for the case when p and $q \geqq 2$. Suppose N is a unit tangent vector at a point $m \in M$ satisfying $T(N, N, N) = N$. As a consequence of our work above there exists a vector X at m normal to N such that X and $2T_N^N X - T_{NN}X - T^{NN}X$ are linearly independent. Let N also denote the unit tangent field to the geodesic $\gamma = \exp sN$. Then along γ, $T(N, N, N) = N$. By extending X to a parallel vector field along γ we see that the functions f and g in (8) are smooth along γ. Next it follows from equation (6) that along $\gamma\backslash\{m\}$, $R(N, X)N$ has the form

$$R(N, X)N = F(N)X + G(N)(2T_N^N X - T_{NN}X - T^{NN}X)$$

for any parallel vector field X orthogonal to N along γ. This is easily verified from the matrix representation which applies to all points of γ when parallel fields are used. In fact it can be seen that $F = f^2 - f'$ and $G = 2fg - g^2 - g'$ where the dash denotes differentiation along γ. It follows by continuity that at m for any unit vector N with $T(N, N, N) = N$ and any vector X

$$R(N, X)N = F(N)(X - g(N, X)N) + G(N)(2T_N^N X - T_{NN}X - T^{NN}X) \tag{11}$$

$F(N)$ and $G(N)$ being the limits as $s \to 0$.

We now show that for all vectors N at m satisfying $T(N, N, N) = N$, $F(N) = 0$ and $G(N)$ is independent of N. Taking N as e_{11} and $X = \left(\begin{smallmatrix} 0 & 0 \\ 0 & D \end{smallmatrix}\right)$ where D is $(p - 1)$ by $(q - 1)$ we have $R(N, X)N = F(N)X$. But T is parallel and so

$$\begin{aligned} F(N)X = R(N, X)N &= R(N, X)(T(N, N, N)) \\ &= T(R(N, X)N, N, N) + T(N, R(N, X)N, N) + T(N, N, R(N, X)N) \\ &= F(N)(T^{NN}X + T_N^{\ N}X + T_{NN}X) \\ &= 0. \end{aligned}$$

Again with N as above and X any other unit vector given by a rank 1 matrix we may write $X = \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$ where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Let $Z = \left(\begin{smallmatrix} B & 0 \\ 0 & 0 \end{smallmatrix}\right)$ where $B = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Then from (11)

$$\begin{aligned} -G(N)(b^2 + c^2) &= g(R(N, X)N, X) = -G(X)(b^2 + c^2) \\ -G(Z) &= g(R(N, Z)N, Z) = -G(N) \\ -G(Z)(a^2 + d^2) &= g(R(X, Z)X, Z) = -G(X)(a^2 + d^2). \end{aligned}$$

Since X is a unit vector $a^2 + b^2 + c^2 + d^2 = 1$ and hence these equations imply that $G(X) = G(N)$. Thus G is some constant k on this set of vectors.

Now set $S(X, Y)Z = R(X, Y)Z - k(T(X, Y, Z) + T(Z, Y, X) - T(Z, X, Y) - T(Y, X, Z,))$ and apply Lemma 13. Then

$$R(X, Y)Z = k(T(X, Y, Z) + T(Z, Y, X) - T(Z, X, Y) - T(Y, X, Z)).$$

We can also now compute the Ricci operator giving

$$\sum R(X, e_{i\alpha})e_{i\alpha} = k(p + q - 2)X.$$

Thus M is an Einstein manifold and k is a constant on M. In particular we see that M is locally symmetric.

If $k = 0$, then M is locally flat. Conversely on any locally flat manifold we can define T by $T(X, Y, Z) = g(Y, Z)X$. Then P_1 throuugh P_4 are satisfied and (8) becomes $A_N X = -(1/s)X$. With M complete and simply connected, as in the statement of the theorem, M is globally isometric to Euclidean space E^{pq}.

We remark that T may not be unique; for example for any factorization $n = pq$ we can regard E^n as the real p by q matrices and define $T(X, Y, Z) = XY^tZ$ so that P_1 through P_4 and (8) are satisfied.

Now suppose $k \neq 0$. It only remains to obtain equation (2) for a metric $\bar{g}$ on M homothetic to g. Define $\bar{g} = |k|g$ and $\bar{T}(X, Y, Z) = |k|T(X, Y, Z)$ on M. Then P_1 through P_4 are satisfied for $\bar{g}$ and $\bar{T}$, as is (8) with f and g divided by $|k|^{1/2}$ and N replaced by $\bar{N} = |k|^{-1/2}N$. Thus the conditions of the theorem still apply and since the curvature tensor is unchanged we have

$$R(X, Y)Z = \frac{k}{|k|}(\bar{T}(X, Y, Z) + \bar{T}(Z, Y, X) - \bar{T}(Z, X, Y) - \bar{T}(Y, X, Z))$$

for all vector fields X, Y, Z on M. Now assume $k > 0$. We know that M is a Riemannian locally symmetric space and it follows immediately from Proposition 3 and equation (2) that if m_1 and m_2 are points in $G_{p,q}$ and M respectively, then there is an isomorphism between their tangent spaces which preserves inner products and curvature tensors at m_1, m_2. Hence $G_{p,q}$ and M are locally isometric. Again with M complete and simply connected, M is globally isometric to $G_{p,q}$. When $k < 0$ we have the corresponding result for the non-compact dual $G^*_{p,q}$ and the proof is complete for $p, q \geqq 2$.

When p or q is equal to 1 we have for a unit vector N and any vector X orthogonal to N at a point $m \in M$ that $2T^N_N X - T_{NN}X - T^{NN}X = -X$. Thus (8) takes the form $A_N X = f(N)X$. Proceeding as before we have that (11) has the form $R(N, X)N = F(N)(X - g(N, X)N)$ where N is a unit vector and X is an arbitrary vector. Taking X as a unit vector we have

$$F(N)(1 - g(N, X)^2) = g(R(N, X)N, X) = F(X)(1 - g(X, N)^2$$

from which F is constant on unit vectors and hence M has constant curvature. Now $G_{p,1}$ (resp. $G^*_{p,1}$) has arbitrary positive (resp. negative) constant curvature depending on its chosen metric. Thus we obtain theorems 1 and 2 of [5] as a special case. We remark that again the tensor T is given by $T(X, Y, Z) = g(Y, Z)X$, cf. (3) with $q = 1$.

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