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HERMITEAN QUADRICS AS CONTACT MANIFOLDS

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ABSTRACT. The notions of real and complex contact manifolds are classical and it is evident that the projective cotangent bundle of real projective space, a real contact manifold, is a real form of the projective contangent bundle of complex projective space, a complex contact manifold. Hermitean quadrics, which are real manifolds in complex projective space, are real contact manifolds and are also real forms of the projective cotangent bundle of complex projective space. The latter is not evident and it is the purpose of this paper to establish these assertions, exhibit their connection with the anti-polarities of classical projective geometry, and to show that these two types of real contact manifolds constitute all of the real forms of the projective cotangent bundle of complex projective space, as homogeneous contact manifolds. The development of these facts leads to the observation that Hermitean quadrics are principal circle bundles over products of complex projective spaces and generalize the Hopf bundle as real contact manifolds.

1. Introduction. A real contact manifold is a $(2n-1)$-dimensional manifold with a contact structure given by a global maximal rank Pfaffian form [2]. The standard examples are odd-dimensional spheres and projective cotangent bundles of real n-dimensional manifolds. Boothby and Wang have shown how real contact manifolds arise as principal circle bundles over Kähler manifolds, the Pfaffian form being obtained from the Kähler form [2]. A well-known example is the projective cotangent bundle $M^{(2n-1)}$ of real projective space $P^{(n)}$; it is a principal circle bundle over a complex quadric Q^{n-1} in complex projective space P^n.

More interesting, and less known, are the Hermitean quadrics $\Phi_s^{(2n-1)}$, of signature s, in P^n. These are real contact manifolds, and are principal circle bundles over the products $P^s \times P^{n-s-1}$ of complex projective spaces. The contact manifolds $M^{(2n-1)}$ and $\Phi_s^{(2n-1)}$ are distinct since, for example, Q^{n-1} and $P^s \times P^{n-s-1}$ have different second Betti numbers.

However: The distinct real contact manifolds $M^{(2n-1)}$ and $\Phi_s^{(2n-1)}$ are, in fact, real forms of the same complex contact manifold, namely the projective contangent bundle M^{2n-1} of complex projective space P^n.

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A complex contact structure on an odd-dimensional complex manifold is given by, essentially, maximal rank local Pfaffian forms [1]. A real form is obtained at the fixed points of a complex conjugation and its global Pfaffian form as the real parts of the local Pfaffian forms [4]. It is evident how to obtain the real form $M^{(2n-1)}$ of M^{2n-1}. The real forms $\Phi_s^{(2n-1)}$ of M^{2n-1} are obtained from complex conjugations on M^{2n-1} which, when described in terms of P^n, are the classical "anti-polarities" associated with the Hermitean quadrics [3].

Now, the complex contact structure on M^{2n-1} is homogeneous under the group $PGL(n+1; \mathbf{C})$ [1, 7]. The real forms of M^{2n-1} correspond to certain real forms of the group and this allows one to classify all of the real forms of M^{2n-1} [4]. It turns out that there are none other than $M^{(2n-1)}$ and the $\Phi_s^{(2n-1)}$.

This, then, is the significance of the Hermitean quadrics as contact manifolds: They are the non-evident real forms of the projective cotangent bundle of complex projective space; their associated anti-polarities exhibit the complex conjugations.

Necessary preliminaries for this paper are summarized in §2 and §3. In §2, complex contact manifolds, the projective cotangent bundle M^{2n-1} of complex projective space and its evident real form, the projective cotangent bundle $M^{(2n-1)}$ of real projective space, are described. In §3, the construction by Boothby and Wang of real contact manifolds is described, and it is shown how it yields $M^{(2n-1)}$ in another manner.

The main results of this paper appear in Sections 4 and 5. In §4, a description of how Hermitean quadrics $\Phi_s^{(2n-1)}$ arise from classical anti-polarities in complex projective geometry is given, and it is shown that these anti-polarities give complex conjugations on M^{2n-1} which make these quadrics $\Phi_s^{(2n-1)}$ also real forms of M^{2n-1}. In §5, the additional fact that M^{2n-1} is homogeneous under the projective general linear group is used to show that the projective cotangent bundle M^{2n-1} of complex projective space has no real forms other than those exhibited: the evident one $M^{(2n-1)}$ and the Hermitean quadrics $\Phi_s^{(2n-1)}$.

An interesting sidelight on the real forms of M^{2n-1} is the role played by the Hopf bundle. An odd-dimensional sphere is a contact manifold and arises as a principal circle bundle, the Hopf bundle, over complex projective space. The Hermitean quadric $\Phi_s^{(2n-1)}$ as principal circle bundle over $P^s \times P^{n-s-1}$ generalizes the Hopf bundle, which is $s = 0$, and all the real forms of M^{2n-1} may be obtained from the Hopf bundle: $M^{(2n-1)}$ has a two-fold cover which is a subbundle of a Hopf bundle, and the $\Phi_s^{(2n-1)}$ are products of two Hopf bundles modulo a diagonal action.

Throughout, the complex dimension of a complex manifold is indicated by a superscript, and the real dimension of a real manifold by a superscript enclosed in parentheses.

2. Complex contact manifolds.

2.1. A complex contact structure on a complex manifold M^{2n-1} is given by a holomorphic principal $\mathbf{C}^*$-bundle $B^{2n} \to M^{2n-1}$ together with a Pfaffian form β on B^{2n} satisfying: (a) $(d\beta)^n$ vanishes nowhere, (b) β vanishes on vectors tangent to fibers, and (c) $R_a^*\beta = a\beta$, a in $\mathbf{C}^*$, where R_a is the right action of $\mathbf{C}^*$ on B^{2n} [1, 7]. If $\{U_i\}$ is an open cover of M^{2n-1} over which B^{2n} trivializes by sections s_i over U_i, then the holomorphic Pfaffian forms $\omega_i = s_i^*\beta$ give a contact structure in the classical sense: $\omega_i \wedge (d\omega_i)^{n-1}$ does not vanish on U_i, and ω_j is a non-vanishing multiple of ω_i on $U_i \cap U_j$. One may construct the bundle B^{2n} from the forms ω_i, so the above definition of a contact structure is equivalent to the classical one [5]. In case the bundle $B^{2n} \to M^{2n-1}$ is trivial with global section s, the Pfaffian form $\omega = s^*\beta$ on M^{2n-1} gives the contact structure.

The classical standard example is the projective cotangent bundle M^{2n-1} of a complex manifold V^n; B^{2n} is then the cotangent bundle of V^n, less its zero-section. If $x^1, x^2, \ldots, x^n$ are coordinates on an open set U of V^n, then $x^1, x^2, \ldots, x^n, u_1, u_2, \ldots, u_n$ are coordinates on B^{2n} over U, where $u_i(\xi)$ are the functions giving the components of $\xi = \sum_1^n u_i(\xi)dx^i$ in B^{2n}. $\beta = \sum_1^n u_i dx^i$ is the Pfaffian form on B^{2n}.

2.2. Let $x = {}^t(x^0, x^1, \ldots, x^n)$ be homogeneous coordinates for points of complex projective space P^n, and let $u = (u_0, u_1, \ldots, u_n)$ be homogeneous coordinates for hyperplanes of P^n. u represents the hyperplane $ux' = \sum_0^n u_i x'^i = 0$. The cotangent and projective cotangent bundles of P^n can be expressed $B^{2n} = I^{2n+1}/R^1$ and $M^{2n-1} = I^{2n+1}/R^2$, where

$$I^{2n+1} = \{(x, u) | ux = 0\} \subset \mathbf{C}^{n+1} \times \mathbf{C}^{n+1},$$

and R^1 and R^2 are the equivalence relations

$$R^1\colon (\lambda x, u\lambda^{-1}) \sim (x, u), \qquad \lambda \text{ in } \mathbf{C}^*,$$

$$R^2\colon (\lambda x, u\mu^{-1}) \sim (x, u), \qquad \lambda, \mu \text{ in } \mathbf{C}^*.$$

The form $u\, dx = \sum_0^n u_i dx^i$ on $\mathbf{C}^{n+1} \times \mathbf{C}^{n+1}$, when restricted to I^{2n+1}, is invariant under R^1. For, if $x' = \lambda x$ and $u' = u\lambda^{-1}$, then $u'dx' = u\lambda^{-1}d(\lambda x) = (ux)\lambda^{-1}d\lambda + u\, dx$. Since $ux = 0$ on I^{2n+1}, $u'dx' = u\, dx$. Hence this form pulls down to the Pfaffian form $\beta = u\, dx$ on B^{2n}. The right action of $\mathbf{C}^*$ on B^{2n} is $R_a((x, u) \bmod R^1) = (x, ua) \bmod R^1$, and β satisfies (a) to (c) of 2.1.

Throughout, B^{2n} and M^{2n-1} will denote the cotangent and projective contangent bundles of complex projective space P^n.

2.3. If, in 2.1, we replace complex by real, we have the definition of a real contact structure on a real manifold in terms of a principal $\mathbf{R}^*$-bundle.

If, in 2.2, we replace complex numbers by real numbers everywhere, we obtain the cotangent and projective cotangent bundles of real projective space $P^{(n)}$. Throughout, these will be denoted by $B^{(2n)}$ and $M^{(2n-1)}$. $B^{(2n)} \to M^{(2n-1)}$ is a principal $\mathbf{R}^*$-bundle.

A real contact structure is a real form of a complex contact structure when the real manifold, bundle, and structure group, consist of the fixed points of compatible complex conjugations on a complex manifold, bundle, and structure group $\mathbf{C}^*$, which are algebraic, and the real Pfaffian form is the restriction of a complex form which is invariant, up to a nonvanishing scalar multiple, under the conjugation. It suffices to give the conjugation on the bundle.

The projective cotangent bundle $M^{(2n-1)}$ of real projective space $P^{(n)}$ is a real form of the projective cotangent bundle M^{2n-1} of complex projective space P^n, the conjugation on B^{2n} being $(x, u) \bmod R^1 \rightsquigarrow (\bar{x}, \bar{u}) \bmod R^1$. Note: If $(x, u) \bmod R^1$ is fixed, $\bar{x} = \lambda x$ and $\bar{u} = u\lambda^{-1}$; necessarily $|\lambda| = 1$. Set $\lambda = \lambda_1^2$; then $(x, u) \sim (\lambda_1 x, u\lambda_1^{-1}) \pmod{R^1}$ and $\lambda_1 x$ and $u\lambda_1^{-1}$ are real.

We can express $B^{(2n)}$ and $M^{(2n-1)}$ as $B^{(2n)} = I^{(2n+1)}/R^{(1)}$ and $M^{(2n-1)} = I^{(2n+1)}/R^{(2)}$, where $I^{(2n+1)}$, $R^{(1)}$, $R^{(2)}$ are the real analogs of I^{2n+1}, R^1, R^2 of 2.2.

3. Real contact manifolds.

3.1. If $P \to M$ is a principal G-bundle over a manifold M, then a connection form ω on P is a Pfaffian form ω on P with values in the Lie algebra of G and is characterized by $\sigma^*\omega = g^{-1}\omega g + g^{-1}dg$ on $P \times G$, where $\sigma(u, g) = R_g u = ug$ is the right action of G on P, and $g^{-1}\,dg$ is the matrix of Maurer-Cartan forms of G. $d\omega + \omega \wedge \omega$ is the curvature of the connection. If G is Abelian: $R_g^*\omega = \omega$, so ω is obtained by pulling a form on M up to P, and $d\omega$ is the curvature, which also is obtained from a form on M.

3.2. Let Ω be a closed 2-form of integral cohomology class on a real manifold $W^{(2n-2)}$. By theorems of Kobayashi **[6]** there is a principal circle bundle $X^{(2n-1)} \xrightarrow{p} W^{(2n-2)}$ over $W^{(2n-2)}$ and a connection form ω on $X^{(2n-1)}$ such that $d\omega = p^*\Omega$. If Ω is a symplectic form, so that Ω^{n-1} does not vanish, then $\omega \wedge (d\omega)^{n-1}$ does not vanish and $X^{(2n-1)}$ has a contact structure given by the global form ω. By a theorem of Boothby and Wang, every regular contact structure on a compact real manifold is obtained from a symplectic manifold in this manner **[2]**. A frequent choice for the symplectic manifold is a compact algebraic manifold W^{n-1} with Ω the Kähler form, or a constant multiple of it, associated with the Hermitean metric.

3.3. The standard example of the Boothby-Wang construction is the odd-dimensional sphere $S^{(2n-1)}$ obtained from complex projective space P^{n-1}. Let

$$S^{(2n-1)} = \{z \mid {}^t\bar{z}z = \sum_1^n |z^i|^2 = 1\} \subset \mathbf{C}^n,$$

where $z = {}^t(z^1, z^2, \ldots, z^n)$. $S^{(2n-1)}$ is a principle circle bundle over P^{n-1}, a Hopf bundle, by sending z in $S^{(2n-1)}$ to the point in P^{n-1} with homogeneous coordinates z. The action of the structure group $U(1) = \{e^{2\pi\sqrt{-1}t} \mid 0 \leqq t < 1\}$ is $R_a z = za$. The Pfaffian form $\omega = {}^t\bar{z}\, dz = \sum_1^n \bar{z}^i\, dz^i$, when restricted to $S^{(2n-1)}$, satisfies $\sigma^*\omega = \omega + 2\pi\sqrt{-1}\, dt$ and hence is a connection form. For,

$$\sigma^*\omega = {}^t(\overline{za})d(za) = \bar{a}a\, {}^t\bar{z}\, dz + ({}^t\bar{z}z)\bar{a}\, da,$$

and $\bar{a}a = 1$, ${}^t\bar{z}z = 1$, $\bar{a}\, da = 2\pi\sqrt{-1}\, dt$. The 2-form $\Omega = \sum_1^n d\bar{z}^i \wedge dz^i$ on P^{n-1} is $1/\sqrt{-1}$ times the Kälher form of the metric

$$ds^2(z) = \sum_1^n d\bar{z}^i\, dz^i - \Big(\sum_1^n z^i d\bar{z}^i\Big)\Big(\sum_1^n \bar{z}^i dz^i\Big)$$

on P^{n-1}, and $\Omega/2\pi\sqrt{-1}$ has integer periods. The curvature $d\omega$ is the form Ω pulled up to $S^{(2n-1)}$.

Real projctive space $P^{(2n-1)}$ is obtained from the sphere $S^{(2n-1)}$ modulo the equivalence relation $z \sim \pm z$, hence we have a principal circle bundle $P^{(2n-1)} \to P^{n-1}$ with structure group $U(1)/\{\pm 1\} = \{e^{\pi\sqrt{-1}t} \mid 0 \leqq t < 1\}$. The form $\omega = {}^t\bar{z}dz$ on $S^{(2n-1)}$ is invariant under this equivalence relation, so yields a connection form on $P^{(2n-1)}$ given by the same formula. The contact structure on $P^{(2n-1)}$ is obtained by the Boothby-Wang construction from P^{n-1}.

3.4. Let $V^{(2n-1)}$ be the Stiefel manifold of two-frames in Euclidean $\mathbf{R}^{(n+1)}$:

$$V^{(2n-1)} = \left\{(x, y) \,\middle|\, \begin{array}{ll} x, y \text{ in } \mathbf{R}^{(n+1)}, & {}^t xx = 1 \\ {}^t xy = 0 & , {}^t yy = 1 \end{array}\right\}.$$

$V^{(2n-1)}$ is the unit tangent bundle of $S^{(n)} \subset \mathbf{R}^{(n+1)}$ and is simply-connected. For (x, y) in $V^{(2n-1)}$, $z = x + y\sqrt{-1}$ satisfies ${}^t\bar{z}z = 2$, so $(x, y) \rightsquigarrow x + y\sqrt{-1}$ includes $V^{(2n-1)}$ into the sphere $S^{(2n+1)} = \{z \mid {}^t\bar{z}z = 2\} \subset \mathbf{C}^{n+1}$. For (x, y) in $V^{(2n-1)}$, $z = x + y\sqrt{-1}$ satisfies ${}^t zz = 0$, so the map which sends z in $S^{(2n+1)}$ to the point of P^n having homogeneous coordinates z, sends $V^{(2n-1)}$ to the complex quadric $Q^{n-1} = \{z \mid {}^t zz = 0\}$ in P^n. This map is onto Q^{n-1}, and $V^{(2n-1)} \to Q^{n-1}$ is a principal circle bundle with structure group $U(1)$.

$$\begin{array}{ccc} V^{(2n+1)} & \xrightarrow{\text{incl.}} & S^{(2n-1)} \\ \downarrow & & \downarrow \\ Q^{n-1} & \xrightarrow{\text{incl.}} & P^n \end{array}$$

This bundle is the restriction of the Hopf bundle $S^{(2n+1)} \to P^n$ to $Q^{n-1} \subset P^n$.

The Pfaffian form $\omega = (1/2)\,{}^t\bar{z}\, dz$ on $S^{(2n+1)}$ restricts to $V^{(2n-1)}$ giving it a contact structure. $V^{(2n-1)}$ is obtained by the Boothby-Wang construction from Q^{n-1} and its Kähler form, which is the restriction of that of P^n. In terms of (x, y) in $V^{(2n-1)}$, ω is given by $\omega = (1/\sqrt{-1})\ {}^t y\ dx$ since ${}^t x\ dx = 0$, ${}^t y\ dy = 0$, ${}^t x\ dy = -{}^t y\ dx$.

The unit tangent bundle $N^{(2n-1)}$ of real projective space $P^{(n)}$ can be expressed $N^{(2n-1)} = V^{(2n-1)}/A$, where A is the equivalence relation A: $(x, y) \sim (x, y)$ or $(-x, -y)$. (x, y) mod $A \rightsquigarrow \pm z$, $z = x + y\sqrt{-1}$, includes $N^{(2n-1)}$ into $P^{(2n+1)}$ and its image, under the map from $P^{(2n+1)}$ to P^n, maps to Q^{n-1}. This map is onto, and $N^{(2n-1)} \to Q^{n-1}$ is a principal circle bundle with structure group $U(1)/\{\pm 1\}$. This bundle is the restriction of $P^{(2n+1)} \to P^n$ to $Q^{n-1} \subset P^n$.

$$\begin{array}{ccc} N^{(2n-1)} & \xrightarrow{\text{incl.}} & P^{(2n+1)} \\ \downarrow & & \downarrow \\ Q^{n-1} & \xrightarrow{\text{incl.}} & P^n \end{array}$$

The contact structure on $N^{(2n-1)}$ is obtained from that of $V^{(2n-1)}$ just as the contact structure on $P^{(2n+1)}$ is obtained from that of $S^{(2n+1)}$. The Pfaffian form $\omega = (1/2)\,{}^t\bar{z}\ dz$ on $P^{(2n+1)}$ restricts to $N^{(2n-1)}$ as $\omega = (1/\sqrt{-1})\,{}^t y\ dx$.

3.5. The unit tangent bundle $N^{(2n-1)}$ of real projective space $P^{(n)}$, when identified with the unit cotangent bundle, is a double cover of the projective cotangent bundle $M^{(2n-1)}$ of $P^{(n)}$. Express these bundles as $N^{(2n-1)} = V^{(2n-1)}/A$ and $M^{(2n-1)} = I^{(2n+1)}/R^{(2)}$, as in 3.4 and 2.2; the covering map is (x, y) mod $A \rightsquigarrow (x, {}^t y)$ mod $R^{(2)}$. The map from $\mathbf{R}^* \times N^{(2n-1)} = \mathbf{R}^* \times (V^{(2n-1)}/A)$ to $B^{(2n)} = I^{(2n+1)}/R^{(1)}$ given by $(r, (x, y)$ mod A$) \rightsquigarrow (x, {}^t yr)$ mod $R^{(1)}$ is 2-to-1 and onto. Thus, the $\mathbf{R}^*$-bundle $B^{(2n)} \to M^{(2n-1)}$, when

$$\begin{array}{ccc} \mathbf{R}^* \times N^{(2n-1)} & \xrightarrow{\text{2 to 1}} & B^{(2n)} \\ \downarrow \mathbf{R}^* & & \downarrow \mathbf{R}^* \\ N^{(2n-1)} & \xrightarrow{\text{2 to 1}} & M^{(2n-1)} \end{array}$$

pulled up to $N^{(2n-1)}$ by the covering map, becomes trivial. This triviality corresponds to the global section

$$s: N^{(2n-1)} \longrightarrow \mathbf{R}^* \times N^{(2n-1)}$$
$$s((x, y) \bmod A) = (1, (x, y) \bmod A).$$

The Pfaffian form $\beta = u\ dx$ on $B^{(2n)}$ giving the contact structure on $M^{(2n-1)}$, cf. 2.2, when pulled up to $\mathbf{R}^* \times N^{(2n-1)}$ by the map above, is $\beta = r\,{}^t y\ dx$, since $u = {}^t yr$. β pulled down to $N^{(2n-1)}$ by means of the section s above, is $s^*\beta = {}^t y\ dx$. Hence, $s^*\beta = \sqrt{-1}\,\omega$, where $\omega = (1/\sqrt{-1})\,{}^t y\ dx$ is the form on $N^{(2n-1)}$ giving the contact structure from the Boothby-Wang construction, cf. 3.4.

3.6. Consequently: The real form $M^{(2n-1)}$ of the complex contact manifold M^{2n-1} has $N^{(2n-1)}$ as double cover; the latter arises from the Boothby-Wang construction $N^{(2n-1)} \to Q^{n-1}$ and is the restriction of $P^{(2n+1)} \to P^n$.

$$\begin{array}{ccccccc}
 & & \mathbf{R}^* \times N^{(2n-1)} & \xrightarrow{\text{2 to 1}} & B^{(2n)} & \xrightarrow{\text{incl.}} & B^{2n} \\
 & & \downarrow{\scriptstyle \mathbf{R}^*} & & \downarrow{\scriptstyle \mathbf{R}^*} & & \downarrow{\scriptstyle \mathbf{C}^*} \\
P^{(2n+1)} & \xleftarrow{\text{incl.}} & N^{(2n-1)} & \xrightarrow{\text{2 to 1}} & M^{(2n-1)} & \xrightarrow{\text{inel.}} & M^{2n-1} \\
\downarrow{\scriptstyle U(1)/\{\pm 1\}} & & \downarrow{\scriptstyle U(1)/\{\pm 1\}} & & & & \\
P^n & \xleftarrow{\text{incl.}} & Q^{n-1} & & & &
\end{array}$$

4. Hermitean quadrics.

4.1. If H is a non-singular Hermitean matrix of order $n + 1$, the real connected manifold

$$\Phi^{(2n-1)} = \{x \mid {}^t\bar{x}Hx = 0\} \subset P^n$$

in complex projective space P^n is a Hermitean quadric. Associated with $\Phi^{(2n-1)}$ is the anti-polarity $u' = {}^t\bar{x}H$ and $x' = H^{-1}{}^t\bar{u}$ of P^n, cf. 2.2. This transformation interchanges points and hyperplanes, preserves incidence, and is involutive (since $H^{-1}\,{}^t(\overline{{}^t\bar{x}H}) = x$ and ${}^t(\overline{H^{-1}\,{}^t\bar{u}})H = u$). $\Phi^{(2n-1)}$ is obtained as the points of P^n which lie on the corresponding hyperplane under the anti-polarity: $u'x = 0$ with $u' = {}^t\bar{x}H$ [3].

By a suitable choice of coordinates z on P^n, H can be put in the form

$$H = \begin{pmatrix} -1_{s+1} & 0 \\ 0 & 1_{n-s} \end{pmatrix}.$$

Denote the Hermitean quadric by

$$\Phi_s^{(2n-1)} = \{z \mid {}^t\bar{z}Hz = -\sum_0^s |z^i|^2 + \sum_{s+1}^n |z^i|^2 = 0\}.$$

When $s = 0$, $\Phi_0^{(2n-1)}$ is a real sphere.

4.2. Separate the coordinates on P^n as $z = \binom{x}{y}$, where $x = {}^t(x^0, x^1, \ldots, x^s)$, $y = {}^t(y^{s+1}, y^{s+2}, \ldots, y^n)$. Then

$$\Phi_s^{(2n-1)} = \left\{\begin{pmatrix} x \\ y \end{pmatrix} \middle| -{}^t\bar{x}x + {}^t\bar{y}y = 0\right\} \subset P^n.$$

Let $S^{(2s+1)}$ and $S^{(2n-2s-1)}$ be the spheres

$$S^{(2s+1)} = \{x \mid {}^t\bar{x}x = 1\} \subset \mathbf{C}^{s+1},$$
$$S^{(2n-2s-1)} = \{y \mid {}^t\bar{y}y = 1\} \subset \mathbf{C}^{n-s},$$

and P^s and P^{n-s-1} the complex projective spaces with homogeneous coordinates x and y. Then

$$S^{(2s+1)} \times S^{(2n-2s-1)} \to \Phi_s^{(2n-1)} \text{ and } \Phi_s^{(2n-1)} \to P^s \times P^{n-s-1}$$

$$(x, y) \rightsquigarrow \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} \rightsquigarrow (x, y)$$

are principal circle bundles with structure groups

$$U(1) \text{ and } U(1)/\{\pm 1\} = \{e^{\pi\sqrt{-1}t} \mid 0 \leqq t < 1\},$$

and right actions

$$(x, y) \rightsquigarrow (ax, ay) \text{ and } \begin{pmatrix} x \\ y \end{pmatrix} \rightsquigarrow \begin{pmatrix} a^{-1}x \\ ay \end{pmatrix},$$

respectively. From the exact homotopy sequence of these bundles, one can conclude that $\Phi_s^{(2n-1)}$ is simply connected except when it is a circle ($n = 1$, $s = 0$). For $s = 0$, $\Phi_0^{(2n-1)} \to P^0 \times P^{n-1}$ is the Hopf bundle $S^{(2n-1)} \to P^{n-1}$.

The first bundle above allows one to express

$$\Phi_s^{(2n-1)} = S^{(2s+1)} \times S^{(2n-2s-1)}/U^{(1)},$$

where $U^{(1)}$ is the equivalence relation $U^{(1)}$: $\lambda z \sim z$, $|\lambda| = 1$, where z is in $S^{(2s+1)} \times S^{(2n-2s-1)} \subset \mathbf{C}^{s+1} \times \mathbf{C}^{n-s}$

The Pfaffian form $-{}^t\bar{x}\,dx + {}^t\bar{y}\,dy = -\sum_0^s \bar{x}^i dx^i + \sum_{s+1}^n \bar{y}^i\,dy^i$, when restricted to $S^{(2s+1)} \times S^{(2n-2s-1)} \subset \mathbf{C}^{s+1} \times \mathbf{C}^{n-s}$, is invariant under the action of $U(1)$, cf. 3.3, so defines a form $\omega = {}^t\bar{z}Hdz = -{}^t\bar{x}\,dx + {}^t\bar{y}\,dy$ on $\Phi_s^{(2n-1)}$. This form satisfies $\sigma^*\omega = \omega + 2\pi\sqrt{-1}\,dt$ for the action of $U(1)/\{\pm 1\}$ on $\Phi_s^{(2n-1)}$, and hence is a connection form. For, cf. 3.3,

$$^t(\overline{a^{-1}x})\,d(a^{-1}x) = {}^t\bar{x}\,dx - \pi\sqrt{-1}\,dt,$$
$$^t(\overline{ay})\,d(ay) = {}^t\bar{y}dy + \pi\sqrt{-1}\,dt,$$

where $a = e^{\pi\sqrt{-1}t}$. The 2-form $\Omega = -\sum_0^s d\bar{x}^i \wedge dx^i + \sum_{s+1}^n d\bar{y}^i \wedge dy^i$ on $P^s \times P^{n-s-1}$ is $1/\sqrt{-1}$ times the Kähler form of the indefinite metric $-ds^2(x) + ds^2(y)$ on $P^s \times P^{n-s-1}$, cf. 3.3. The curvature $d\omega$ is the form Ω pulled up to $\Phi_s^{(2n-1)}$. Hence ω on $\Phi_s^{(2n-1)}$ gives the contact structure obtained from Ω on $P^s \times P^{n-s-1}$ by the Boothby-Wang construction.

Viewed otherwise: If $X \to W$ is a principal circle bundle with connection ω obtained from the symplectic form Ω and similarly $X' \to W'$ with ω' obtained from Ω', then $-\omega + \omega'$ is invariant under the diagonal action of the circle group on $X \times X'$ and hence defines a form on the principal circle bundle $X \times X'/(\text{circle}) \to W \times W'$ which is the connection form obtained from the symplectic form $-\Omega + \Omega'$. From the Hopf bundles $S^{(2s+1)} \to P^s$ and $S^{(2n-2s-1)} \to P^{n-s-1}$, the principal circle bundle $\Phi_s^{(2n-1)} \to P^s \times P^{n-s-1}$, with connection as described above, is obtained in this manner.

4.3. The anti-polarity $u' = {}^t\bar{x}H$ and $x' = H^{-1}\,{}^t\bar{u}$ associated with $\Phi_s^{(2n-1)}$

gives rise to a complex conjugation on the projective cotangent bundle $M^{2n-1} = I^{2n+1}/R^2$ of complex projective space by

$$(x, u) \bmod R^2 \rightsquigarrow (H^{-1}\,{}^t\bar{u},\ {}^t\bar{x}H) \bmod R^2.$$

If $(x, u) \bmod R^2$ is fixed under the conjugation, then $H^{-1}\,{}^t\bar{u} = \lambda x$ and ${}^t\bar{x}H = u\mu^{-1}$, λ, μ in $\mathbf{C}^*$, so $(x, u) = (x, {}^t\bar{x}H\mu) \sim (x, {}^t\bar{x}H)$, and $(x, u) \bmod R^2$ is an image point of the map

$$\Phi_s^{(2n-1)} \longrightarrow M^{2n-1}$$
$$z \text{ (homogeneous)} \rightsquigarrow (z, {}^t\bar{z}H) \bmod R^2.$$

Every image point of this map is fixed under the conjugation, so this exhibits $\Phi_s^{(2n-1)}$ as the set of fixed points of this complex conjugation on M^{2n-1}.

From 4.2 and 2.2, express

$$\Phi_s^{(2n-1)} = S^{(2s+1)} \times S^{(2n-2s-1)}/U^{(1)} \text{ and } B^{2n} = I^{2n+1}/R^1.$$

The map

$$\mathbf{R}^* \times \Phi_s^{(2n-1)} \xrightarrow{\ \pi\ } B^{2n}$$
$$(r, z \bmod U^{(1)}) \rightsquigarrow (z, {}^t\bar{z}Hr) \bmod R^1$$

is well-defined and injective. For, if $\pi(r', z' \bmod U^{(1)}) = \pi(r, z \bmod U^{(1)})$, then $z' = \lambda z$ and ${}^t\bar{z}'Hr' = {}^t\bar{z}Hr\lambda^{-1}$. The first equation implies $|\lambda| = 1$ since z lies in $S^{(2s+1)} \times S^{(2n-2s-1)}$, and the second equation implies $z'r' = zr\bar{\lambda}^{-1}$. Hence $r\bar{\lambda}^{-1}/r' = \lambda$ and $r' = r$. Every image point of π is fixed under the complex conjugation

$$(x, u) \bmod R^1 \rightsquigarrow (H^{-1}\,{}^t\bar{u},\ {}^t\bar{x}H) \bmod R^1$$

on $B^{2n} = I^{2n+1}/R^1$, and conversely, every fixed point of this conjugation is an image point of π. For, if $(x, u) \bmod R^1$ is fixed, then $H^{-1}\,{}^t\bar{u} = \lambda x$ and ${}^t\bar{x}H = u\lambda^{-1}$ with λ in $\mathbf{C}^*$. Then $H^{-1}\,{}^t(\overline{\lambda {}^t\bar{x}H}) = \lambda x$ gives $\lambda = \bar{\lambda}$ is real. Since ${}^t\bar{x}Hx = (u\lambda^{-1})x = \lambda^{-1}ux = 0$, for suitable real $s \neq 0$, x/s lies in $S^{(2s+1)} \times S^{(2n-2s-1)}$. Set $r = \lambda$, $z = x/s$; then $(x, u) = (x, {}^t\bar{x}H\lambda) = (sz, {}^t\bar{z}Hsr) \sim (z, {}^t\bar{z}Hr)$ and $(x, u) \bmod R^1 = \pi(r, z \bmod U^{(1)})$. Hence π exhibits $\mathbf{R}^* \times \Phi_s^{(2n-1)}$ as the set of fixed points of this complex conjugation on B^{2n}. The triviality of the bundle $\mathbf{R}^* \times \Phi_s^{(2n-1)} \to \Phi_s^{(2n-1)}$ corresponds to the global section s.

$$\begin{array}{ccc} \mathbf{R}^* \times \Phi_s^{(2n-1)} & \xrightarrow{\ \pi\ } & B^{2n} \\ \downarrow{\scriptstyle \mathbf{R}^*} & & \downarrow{\scriptstyle \mathbf{C}^*} \\ \Phi_s^{(2n-1)} & \longrightarrow & M^{2n-1} \end{array}$$

$$s: \Phi_s^{(2n-1)} \longrightarrow \mathbf{R}^* \times \Phi_s^{(2n-1)}$$
$$s(z \bmod U^{(1)}) = (1, z \bmod U^{(1)}).$$

The Pfaffian form $\beta = u\,dx$ on B^{2n} giving the complex contact structure on M^{2n-1}, cf. 2.2, is essentially real with respect to the complex conjugation on B^{2n}; for under the conjugation, $\beta = u\,dx$ is sent to

$$({}^t\bar{x}H)d(H^{-1}\,{}^t\bar{u}) = {}^t\bar{x}\,{}^td\bar{u} = \overline{du.x} = -\overline{u\,dx} = -\bar{\beta},$$

since $du.x + u\,dx = 0$ by $ux = 0$. The Pfaffian form β pulled back to $\mathbf{R}^* \times \Phi_s^{(2n-1)}$ by π is $\pi^*\beta = r\,{}^t\bar{z}H\,dz$, since $u = {}^t\bar{z}Hr$ and $x = z$. $\pi^*\beta$ pulled down to $\Phi_s^{(2n-1)}$ by means of the section s above is $(\pi\circ s)^*\beta = {}^t\bar{z}Hdz$. Hence, $(\pi\circ s)^*\beta = \omega$, where $\omega = {}^t\bar{z}Hdz$ is the form on $\Phi_s^{(2n-1)}$ giving the contact structure from the Boothby-Wang construction, cf. 4.2.

4.4. Consequently: The real form $\Phi_s^{(2n-1)}$ of the complex contact manifold M^{2n-1} arises from the Boothby-Wang construction $\Phi_s^{(2n-1)} \to P^s \times P^{n-s-1}$.

$$\begin{array}{ccccc}
 & & \mathbf{R}^* \times \Phi_s^{(2n-1)} & \xrightarrow{\ \pi\ } & B^{2n} \\
 & & \big\downarrow \mathbf{R}^* & & \big\downarrow \mathbf{C}^* \\
S^{(2s+1)} \times S^{(2n-2s-1)} & \xrightarrow{U(1)} & \Phi_s^{(2n-1)} & \xrightarrow[\text{polarity}]{\text{anti-}} & M^{2n-1} \\
\big\downarrow U(1)\times U(1) & & \big\downarrow U(1)/\{\pm 1\} & & \\
P^s \times P^{n-s-1} & = & P^s \times P^{n-s-1} & &
\end{array}$$

5. Homogeneous contact manifolds.

5.1. Every connected homogeneous compact complex contact manifold which is algebraic, that is, the manifold, principal $\mathbf{C}^*$-bundle, and Pfaffian form, cf. 2.1, are algebraic, is obtained as a quotient G/P of a connected centerless simple complex Lie group G and a parabolic subgroup P. G is determined by its simple complex Lie algebra, and P, together with the principal $\mathbf{C}^*$-bundle and the Pfaffian form, is determined by the maximal root in the Lie algebra [**1, 7**]. G is the identity component of the group of contact automorphisms.

A real form of the complex contact manifold G/P is obtained as $G_0/G_0 \cap P$, where G_0 is a real form of G consisting of the fixed points of a complex conjugation on G which sends P to itself, or equivalently, for which the maximal root is real [**4**].

5.2. Throughout the remainder of this section: Let $G = GL(n+1; \mathbf{C})$ and let $P \subset G$ consist of matrices

$$\begin{pmatrix} * & * & \cdots & * \\ 0 & \vdots & & \vdots \\ \vdots & * & \cdots & * \\ 0 & \cdots & 0 & * \end{pmatrix},$$

where the starred entries are arbitrary. Denote by $G' = PGL(n+1; \mathbf{C})$ and P', respectively, G and P modulo nonzero scalar matrices $\mathbf{C}^* \cdot 1_{n+1}$. Then $M^{2n-1} = G/P = G'/P'$ is the projective cotangent bundle of complex projective space P^n, cf. 2.2. G acts transitively on $M^{2n-1} = I^{2n+1}/R^2$ by

$$g \cdot (x, u) \bmod R^2 = (gx, ug^{-1}) \bmod R^2,$$

and P is the isotropy subgroup of the point $(x_0, u_0) \bmod R^2$ in M^{2n-1}, $x_0 = {}^t(1\ 0\ \cdots\ 0)$ and $u_0 = (0\ \cdots\ 0\ 1)$. G'/P' expresses M^{2n-1} as the quotient of the centerless simple complex Lie group of type A_n and a parabolic subgroup.

5.3. Let c be a complex conjugation on M^{2n-1} giving rise to a real form, cf. 2.3. If c has $(x, u) \bmod R^2$ as a real point and a' in G' sends $(x, u) \bmod R^2$ to $(x_0, u_0) \bmod R^2$, then $a' \circ c \circ a'^{-1}$ is an equivalent conjugation on M^{2n-1} giving rise to an equivalent real form, and having $(x_0, u_0) \bmod R^2$ as a real point. Hence assume that the conjugation c has $(x_0, u_0) \bmod R^2$ as a real point. Then $cg' = c \circ g' \circ c^{-1}$ is a complex conjugation on G' which sends P' to itself: $cP' = P'$.

The conjugation $g \bmod \mathbf{C}^* \cdot 1_{n+1} \rightsquigarrow \bar{g} \bmod \mathbf{C}^* \cdot 1_{n+1}$ followed by c is an automorphism $g' \rightsquigarrow c\bar{g}'$ of G'. An automorphism of G' is induced by one of G since it is either inner or reverses the diagram of G'. Hence, complex conjugations on G' are induced by complex conjugations $g \rightsquigarrow Jg\bar{J}^{-1}$ and $g \rightsquigarrow H^{-1}\,{}^t\bar{g}^{-1}H$ on G, where $J\bar{J}$ and $H^{-1}\,{}^t\bar{H}$ are scalar matrices since c^2 is the identity on G'. The scalars are necessarily real and unimodular, respectively, so J and H may be replaced by scalar multiples so as to satisfy $J\bar{J} = \pm 1_{n+1}$ and ${}^t\bar{H} = H$.

Every fixed point of c in G' is the image of a fixed point of the conjugation in G which induces c.

5.4. The complex conjugation $g \rightsquigarrow J\bar{g}J^{-1}$ on G sends P to itself, and so does $g \rightsquigarrow \bar{g}$, so $JPJ^{-1} = P$. Since P' is a parabolic subgroup of G', P' is its own normalizer in G', and P is its own normalizer in G. Hence, J lies in P.

The conjugation $g \rightsquigarrow J\bar{g}J^{-1}$ on G corresponds to the conjugation $(x, u) \bmod R^2 \rightsquigarrow (J\bar{x}, \bar{u}J^{-1}) \bmod R^2$ on M^{2n-1}. For: If $gx_0 = x$ and $u_0g^{-1} = u$, then, since J lies in P, $J\bar{g}J^{-1}x_0 = J\bar{g}x_0 = J\bar{x}$ and $u_0(J\bar{g}J^{-1})^{-1} = u_0\bar{g}^{-1}J^{-1} = \bar{u}J^{-1}$. Cf. 5.2.

The conjugation $g \rightsquigarrow J\bar{g}J^{-1}$ on G, with $J\bar{J} = +1_{n+1}$, is induced by the complex conjugation $x \rightsquigarrow J\bar{x}$ on the vector space $\mathbf{C}^{n+1}$. This conjugation sends the subspace spanned by $x_0 = {}^t(1\ 0 \ldots 0)$ to itself, and likewise the subspace $u_0x = 0$, $u_0 = (0 \ldots 0\ 1)$. Choose a basis of $\mathbf{C}^{n+1}$, real with respect to $x \rightsquigarrow J\bar{x}$, the first element of which is x_0, and the first n elements of which are a basis of $u_0x = 0$. The matrix T expressing this basis in terms of the standard basis of $\mathbf{C}^{n+1}$ lies in P and $T^{-1}J\bar{T} = 1_{n+1}$. The conjugations $g \rightsquigarrow J\bar{g}J^{-1}$ and $g \rightsquigarrow \bar{g}$ on G are equivalent by an inner automorphism of G which sends P into itself, since $T^{-1}J(\overline{TgT^{-1}})J^{-1}T = (T^{-1}J\bar{T})\bar{g}\ (T^{-1}J\bar{T})^{-1}$. The corresponding conjugations on M^{2n-1} and their real forms are equivalent, so we obtain in this case the real form $M^{(2n-1)}$, the projective cotangent bundle of real projective space $P^{(n)}$, of M^{2n-1}, and the corresponding real form of $PGL(n+1; R)$ of G'.

The conjugation $g \rightsquigarrow J\bar{g}J^{-1}$ on G, with $J\bar{J} = -1_{n+1}$, is induced by the quaternion structure $x \rightsquigarrow J\bar{x}$ on $\mathbf{C}^{n+1}$, $n + 1 = 2m$ necessarily even. This conjugation does not correspond to one on M^{2n-1}, since for the latter, J lies in P and the first and last diagonal entries of $J\bar{J}$ are positive real. M^{2n-1} has no real form corresponding to the real form $PSU^*(2m)$ of G'.

5.5. The complex conjugation $g \rightsquigarrow H^{-1}\,{}^t\bar{g}^{-1}H$ on G sends P to itself, so $H^{-1}\,{}^t\bar{P}^{-1}H = P$ or $H^{-1}\,{}^tPH = P$, where ${}^tP$ is the group of transposes of elements of P. Since

$$ {}^tP = KPK^{-1}, \text{ where } K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}, $$

$(H^{-1}K)P(H^{-1}K)^{-1} = P$, and $H^{-1}K$ lies in P. The Hermitean matrix H and its inverse lie in KP and PK, respectively, and have the form

$$ H = \begin{pmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & * \end{pmatrix} \quad \text{and } H^{-1} = \begin{pmatrix} * & * & * \\ * & * & 0 \\ * & 0 & 0 \end{pmatrix}, $$

where the center starred entries are of order $n - 1$.

The conjugation $g \rightsquigarrow H^{-1}\,{}^t\bar{g}^{-1}H$ on G corresponds to the conjugation $(x, u) \bmod R^2 \rightsquigarrow (H^{-1}\,{}^t\bar{u}, {}^t\bar{x}H) \bmod R^2$ on M^{2n-1}. For: If $gx_0 = x$ and $u_0g^{-1} = u$, then $H^{-1}\,{}^t\bar{g}^{-1}\,Hx_0 = H^{-1}\,{}^t\bar{g}^{-1}\,{}^tu_0 = H^{-1}\,{}^t\bar{u}$ and $u_0(H^{-1}\,{}^t\bar{g}^{-1}H)^{-1} = u_0H^{-1}\,{}^t\bar{g}H = {}^tx_0\,{}^t\bar{g}H = {}^t\bar{x}H$, up to scalar factors. Cf. 5.4.

Using the Hermitean form ${}^t\bar{x}Hy$ on $\mathbf{C}^{n+1}$, choose a basis $e_0, e_1, \ldots, e_{n-1}, e_n$ of $\mathbf{C}^{n+1}$ so that e_0 is a multiple of x_0, ${}^t\bar{e}_nHe_n = 0$, ${}^t\bar{e}_0He_n = 2$, and $e_1, \ldots, e_{n-1}$ is an orthonormal basis of the space orthogonal to e_0 and e_n. Then $e_0, e_1, \ldots, e_{n-1}$ spans the subspace $u_0x = 0$. The matrix T expressing this basis in terms of the standard basis of $\mathbf{C}^{n+1}$ lies in P and ${}^t\bar{T}HT = H'$, where

$$ H' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1_s & 0 & 0 \\ 0 & 0 & 1_{n-s-1} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. $$

The conjugations $g \rightsquigarrow H^{-1}\ {}^t\bar{g}^{-1}H$ and $g \rightsquigarrow H'^{-1}\,{}^t\bar{g}^{-1}H'$ on G are equivalent by an inner automorphism of G which sends P into itself, since $T^{-1}H^{-1}\,{}^t(\overline{TgT^{-1}})^{-1}\,HT = ({}^t\bar{T}HT)^{-1}\,{}^t\bar{g}^{-1}({}^t\bar{T}HT)$. The corresponding conjugations on M^{n-1} and their real forms are equivalent. The latter conjugation is further equivalent to the one having

$$H' = \begin{pmatrix} -1_{s+1} & 0 \\ 0 & 1_{n-s} \end{pmatrix},$$

but in this case it no longer has (x_0, u_0) mod R^2 as a real point. From this we identify the real forms $\Phi_s^{(2n-1)}$, the Hermitean quadrics, of M^{2n-1}, and the corresponding real forms $PU(s+1, n-s)$ of G'.

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