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# PRODUCTS OF TWO ABELIAN SUBGROUPS

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Every group $G = AB$, which is the product of two abelian subgroups $A$ and $B$, is metabelian by a well-known result of Itô [4]. In this short note some further statements on the structure of such groups are given. For instance, the center, the $FC$-center, the hypercenter and the $FC$-hypercenter of $G$ are 'factorized' as products of a subgroup of $A$ and a subgroup of $B$ (Theorem 2.2). The Fitting subgroup and the Hirsch-Plotkin radical of $G$ are in general not factorized in a corresponding way. However, some sufficient conditions are given, under which these important characteristic subgroups are factorized (Theorems 2.4 and 2.5). It is also shown that if $G$ is not cyclic of prime order and if $A \neq G$ or $B \neq G$, then there is at least one factorized normal subgroup $N$ of $G = AB$ with $1 \neq N \neq G$ (Theorem 3.1).

The notation is standard; see for instance [8] and [9].

**1. The factorizer.** The following result of Wielandt [12] is useful for the investigation of factorized groups.

LEMMA 1.1. *If the group $G = AB$ is the product of two subgroups $A$ and $B$, then the following conditions of the subgroup $S$ of $G$ are equivalent*:

(a) $S = (A \cap S)(B \cap S)$ *and* $A \cap B \subseteqq S$,

(b) *If* $ab \in S$ *with* $a \in A$ *and* $b \in B$, *then* $a \in S$.

A subgroup $S$ of the factorized group $G = AB$ which satisfies the equivalent conditions of Lemma 1.1 is called *factorized.*

Since intersections of arbitrary many factorized subgroups of $G = AB$ are factorized subgroups of $G$, every normal subgroup $N$ of $G$ is contained in a smallest factorized subgroups $X = X(N)$ of $G$, which we call the *factorizer* of $N$ in $G$. By [1], Theorem 1.7, p. 108, the following holds.

*Lemma* 1.2. *If the group $G = AB$ is the product of two subgroups $A$ and $B$ and if $N$ is a normal subgroup of $G$, then*

$$X = X(N) = AN \cap BN = N(A \cap BN) = N(B \cap AN) = (A \cap BN)(B \cap AN).$$

This implies the following result.

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LEMMA 1.3. *If the group $G = AB$ is the product of two abelian subgroups $A$ and $B$, then the factorizer $X = X(N)$ of every normal subgroup $N$ of $G$ is a normal subgroup of $G$.*

PROOF. By Lemma 1.2, $X/N = N(A \cap BN)/N = N(B \cap AN)/N$, and this group is normalized by $G/N = (AN/N)(BN/N)$. Hence $X/N$ is a normal subgroup of $G/N$, so that $X$ is a normal subgroup of $G$.

Thus, a normal subgroup of $G = AB$ is factorized if and only if it is equal to its factorizer in $G$. In general however, a normal subgroup of $G = AB$ is properly contained in its factorizer. This can be seen from the following simple example.

EXAMPLE 1.4. If $Y \neq 1$ is any group, the direct square $G = Y \times Y = AB$ is factorized by two subgroups $A$ and $B$ isomorphic to $Y$. The diagonal subgroup $D$ of $G$, which consists of all elements $(y, y)$ where $y$ is an element of $Y$, is also a subgroup of $G$ isomorphic to $Y$. If $Y$ is abelian, $G$ is abelian and $D$ is a normal subgroup of $G$. The factorizer of $D$ is $X(D) = AD \cap BD = G$. Note that if $Y$ is cyclic of prime order, then $D$ is a minimal and maximal subgroup of $G$.

**2. Some factorized subgroups.** Which characteristic subgroups of a product $G = AB$ of two abelian subgroups $A$ and $B$ are always factorized?

LEMMA 2.1. *If the group $G = AB$ is the product of two abelian subgroups $A$ and $B$, then the FC-center and the centralizer of every factorized subgroup of $G$ are factorized.*

PROOF. Let $c = ab^{-1}$ with $a \in A$, $b \in B$, be an element of the $FC$-center of $G$, so that $c$ has only finitely many conjugates in $G$. If $s = a^*b^*$ is an element of $G$ with $a^* \in A$ and $b^* \in B$, then

$$a^s = a^{a^*b^*} = a^{b^*} = (cb)^{b^*} = c^{b^*}b^{b^*} = c^{b^*}b.$$

Hence also $a$ has only finitely many conjugates in $G$ and belongs to the $FC$-center of $G$. By Lemma 1.1 the $FC$-center of $G$ is factorized.

Let $S$ be a subgroup of $G$ such that $S = (A \cap S)(B \cap S)$, and let $s = a_1b_1$ be an element of $S$ with $a_1 \in A \cap S$ and $b_1 \in B \cap S$. If $c = ab^{-1}$ is contained in the centralizer of $S$ with $a \in A$ and $b \in B$, then $[c, b_1] = 1$. We have

$$\begin{aligned}[a, s] &= [a, a_1b_1] = [a, b_1][a, a_1]^{b_1} = [a, b_1] \\ &= [cb, b_1] = [c, b_1]^b[b, b_1] = [c, b_1]^b = 1.\end{aligned}$$

Hence $a$ commutes with every $s$ in $F$, so that the centralizer of $S$ is factorized by Lemma 1.1.

Corresponding to the upper central series of $G$ the upper $FC$-central series of a group $G$ is defined by

$$F_0 = 1,$$
$$F_{\alpha+1}/F_\alpha = FC\text{-center of } G/F_\alpha \text{ for every ordinal } \alpha,$$
$$F_\gamma = \bigcup_{\beta<\gamma} F_\beta \text{ for limit ordinals } \gamma.$$

Corresponding to the hypercenter, the limit of the upper $FC$-central series of $G$ is called the $FC$-hypercenter; see **[8]**, p. 129.

THEOREM 2.2. *If the group $G = AB$ is the product of two abelian subgroups $A$ and $B$, then every term of the upper central series and every term of the upper FC-central series of $G$ are factorized; in particular, the center, the FC-center, the hypercenter and the FC-hypercenter of $G$ are factorized characteristic subgroups of $G$.*

PROOF. If $N$ is a normal subgroup of $G$, then $G/N = (AN/N)(BN/N)$, and by Lemma 2.1 the center $Z/N$ and the $FC$-center $F/N$ of $G/N$ are factorized subgroups of $G/N$. Hence, by **[1]**, Lemma 1.3(b), p. 107, the inverse images $Z$ and $F$ are factorized subgroups of $G = AB$. By **[1]**, Lemma 1.3(a), p. 107, the union of every chain of factorized subgroups of $G$ is a factorized subgroup of $G$. Therefore, every term of the upper central series and every term of the upper $FC$-central series of $G$ are factorized. In particular, the hypercenter and the $FC$-hypercenter of $G$ are factorized.

EXAMPLE 2.3. There exist finite $p$-groups $G = AB$ which are the product of two subgroups $A$ and $B$, but their centers are not factorized. Consider for instance the multiplicative group $G = U(4, F)$ of upper triangular matrices of degree 4 over the (finite) field $F$ with eigenvalues 1. Then $G$ is a nilpotent group which is factorized by the two subgroups

$$A = \left\{ \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & x & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{where } x, y, z \text{ are in } F \right\}$$

and

$$B = \left\{ \begin{pmatrix} 1 & u & v & 0 \\ 0 & 1 & w & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{where } u, v, w \text{ are in } F \right\}.$$

It is easy to see that $A \cap Z(G) = 1 = B \cap Z(G)$, so that $Z(G) \neq 1$ is not factorized; see also Gillam **[3]**. Hence Theorem 2.2 does not hold if the two subgroups $A$ and $B$ (or even $G$) are merely nilpotent.

In [13] an example of V. G. Vasil'ev is given of a group which is the product of two abelian subgroups and which is the factorizer of one of its abelian normal subgroups, but which is not locally nilpotent. This example also shows that the Fitting subgroup and the Hirsch-Plotkin radical of a product of two abelian subgroups need not be factorized. In the following some sufficient conditions for the factorization of these two characteristic subgroups are given. The first result is concerned with the Fitting subgroup.

THEOREM 2.4. *If the group $G = AB$ is the product of two abelian subgroups $A$ and $B$ and if $X = X(F)$ is the factorizer of the Fitting subgroup $F = F(G)$, then the following holds*:

(a) *If $A$ and $B$ have finite torsionfree rank, then $X$ is locally nilpotent*,

(b) *If $A$ and $B$ are minimax groups, then $F$ is factorized.*

PROOF. Let $Y = X(N)$ be the factorizer of a nilpotent normal subgroup $N \neq 1$ of $G$. The commutator subgroup $N'$ of $N$ is a normal subgroup of ($G$ and) $Y$ and $N/N'$ is an abelian normal subgroup of $Y/N'$.

By Lemma 1.2, $Y = A^*B^*$ with $A^* = A \cap BN$ and $B^* = B \cap AN$. The group $Y/N' = (A^*N'/N')(B^*N'/N')$ is factorized by two epimorphic images of $A^*$ and $B^*$. Apply Theorem 2 of Zaičev [13], p. 418. In case (a) it follows that $Y/N'$ is locally nilpotent. In case (b) it follows that $Y/N'$ is even nilpotent. Application of Robinson [7] shows that $Y$ is locally nilpotent and even nilpotent in case (b).

The Fitting subgroup $F = \prod_i N_i$ is the product of all nilpotent normal subgroups $N_i$ of $G$. By the first part each factorizer $X(N_i)$ is locally nilpotent and even nilpotent in case (b). By Lemma 1.3 each $X(N_i)$ is a normal subgroup of $G$, so that their product $\prod_i X(N_i)$ is also a factorized locally nilpotent normal subgroup of $G$; see [1], Corollary 1.4 (b), p. 108. Since this product contains $F$, it also contains $X(F)$. In particular $X(F)$ is locally nilpotent. In case (b) $X(F)$ is even contained in the Fitting subgroup of $G$, so that $F = X(F)$. This proves the theorem.

Of course, Theorem 2.4 also implies that the Fitting subgroup $F = F(G)$ of a product $G$ of two abelian subgroups of finite torsionfree rank is factorized if $X(F)$ is finitely generated or lies in a class of groups for which local nilpotency implies nilpotency (or Fitting) such as for instance linear groups without quasicyclic subgroups or linear groups over the rational field; see [9], p. 31.

The following theorem gives some criteria for the Hirsch-Plotkin radical to be factorized.

THEOREM 2.5. *If the group $G = AB$ is the product of two abelian subgroups $A$ and $B$, then the Hirsch-Plotkin radical $R = R(G)$ is factorized if the factorizer $X = X(R)$ satisfies at least one of the following conditions*:

(i) *$X$ is locally finite,*

(ii) *$X$ is finitely generated,*

(iii) *$X$ is a polyminimax group without quasicyclic groups of type $p^\infty$ in the center of its Fitting subgroups for every prime $p$,*

(iv) *$X$ is a linear group over a finitely generated integral domain.*

PROOF. If $X$ is locally finite, by Amberg [2], Satz, p. 234, $X$ is locally nilpotent. If $X$ is finitely generated, $X$ is even nilpotent by [1], Lemma 4.1, p. 112.

Let $X$ satisfy (iii) or (iv). If $H$ is a finite epimorphic image of $X$, then there exists a normal subgroup $N$ of $X$ such that $H = X/N$. By Lemma 1.2 $X = NA^* = RB^* = A^*B^*$ with $A^* = A \cap BR$ and $B^* = B \cap AR$. Therefore

$$H = X/N = (A^*N/N)(B^*N/N) = (A^*N/N)(RN/N) = (B^*N/N)(RN/N)$$

is factorized by three nilpotent subgroups $A^*N/N$, $B^*N/N$ and $RN/N$. Then by [1], Lemma 4.1, p. 112, $H$ is nilpotent. Thus every finite epimorphic image of $G$ is nilpotent.

If $X$ satisfies (iii), it is nilpotent by Robinson [6], Theorem A and Theorem 5.11, p. 496 and p. 517; see also [9], Theorem 9.37, p. 135. If $X$ satisfies (iv), it is nilpotent by Wehrfritz [11], Lemma 2, p. 620.

By Lemma 1.3, $X$ is normal in G. Hence $X = N$, and the theorem is proved.

COROLLARY 2.6. *If the group $G = AB$ is the product of two abelian subgroups $A$ and $B$, which are locally finite or finitely generated, then the Hirsch-Plotkin radical $R = R(G)$ is factorized.*

PROOF. The factorizer $X = X(R) = (A \cap X)(B \cap X)$ is the product of two subgroups $A \cap X$ and $B \cap X$ of $A$ resp. $B$. If $A$ and $B$ are finitely generated, then $X$ is finitely generated. If $A$ and $B$ are locally finite, so are $A \cap X$ and $B \cap X$, so that $X$ is locally finite by [1], Theorem 5.4, p. 118. The assertion follows now from Theorem 2.5.

QUESTION 2.7. Is the Hirsch-Plotkin radical of a product of two abelian subgroups of finite torsionfree rank always factorized?

**3. Factorized normal subgroups.** Except in trivial cases a product of two abelian subgroups always contains nontrivial factorized normal subgroups. This follows from the following theorem.

THEOREM 3.1. *If the group $G = AB$, which is not cyclic of prime order, is the product of two abelian subgroups $A$ and $B$ with $A \neq G$ or $B \neq G$, then there exists a factorized normal subgroup $N$ of $G$ with $1 \neq N \neq G$.*

PROOF. Assume that the theorem is false, and let $G \neq AB$ be a counterexample. If $G$ is abelian, then $A \neq G$ or $B \neq G$ is a proper factorized normal subgroup of $G$. If this is trivial, $A = 1$, say, then $B = G$ contains a factorized normal subgroup $N$ of $G$ with $1 \neq N \neq G$, since $G$ is not cyclic of prime order. Hence $G$ is not abelian, so that the center $Z(G)$ is properly contained in $G$. Since $Z(G)$ is factorized by Theorem 2.2, $Z(G) = 1$. Then also $A \cap B = 1$, since this subgroup is contained in $Z(G)$. If $N \neq 1$ is a normal subgroup of $G$, by Lemma 1.3 the factorizer $X = X(N)$ is a factorized normal subgroup of $G$. Hence by Lemma 1.2

$$(1) \qquad G = N(A \cap BN) = N(B \cap AN) = (A \cap BN)(B \cap AN)$$

for every normal subgroup $N \neq 1$ of $G$.

By Itô's theorem, $G$ is metabelian. Since $G$ is not abelian, $1 \neq G' \neq G$. There exists an element $a \neq 1$ in $A \cap BG'$ such that $E = [G', a] \neq 1$. Clearly $a = bm^{-1}$ with $b \in B$, $m \in G'$. Since $G'$ is abelian, $E$ is the set of all elements $[x, a]$ with $x \in G'$. Also $E = [G', a] = [G', b]$, so that $E$ is a normal subgroup of $AB = G$.

Since the factorizer $X(E) = AE \cap BE = G$, we have by Lemma 1.2

$$(2) \qquad G = AE = BE = AG' = BG'.$$

Assume that $A \cap C(G') \neq 1$. Then there is an element $a \neq 1$ in $A$ which is centralized by $A$ and $G'$. By (2), $a$ is a nontrivial element of the center of $G$. This contradiction shows that $A \cap G' = A \cap C(G') = 1$.

Assume that $E = G'$. Since $a^{-1}$ is contained in $A \cap BG'$, we have $a^{-1} = xb^{-1}$ where $x \in G'$ and $b \in B$. Therefore $a^{-1}b = x$ is contained in $G'$. Hence $a^{-1}b = [a, a^*b^*]$ with $a^*b^* \in G'$, $a^* \in A$ and $b^* \in B$. This implies $a^{-1}b = a^{-1}(a^*b^*)^{-1}a(a^*b^*)$ and $b^*bb^{*-1} = a^{*^{-1}}aa^*$. Since $A \cap B = 1$, $a = b = 1$, a contradiction. It follows that $E$ is properly contained in $G'$. This leads to the final contradiction $G = AE \subset AG' = G$. The theorem is proved.

COROLLARY 3.2. *If the group $G = AB$ is the product of two abelian subgroups $A$ and $B$ with $A \neq G \neq B$, then every maximal factorized normal subgroup of $G$ contains at least one of the subgroups $A$ and $B$.*

PROOF. If $M$ is a maximal factorized normal subgroup of $G$, $M$ equals its factorizer $X(M) = AM \cap BM$. Then $G/M = (AM/M)(BM/M)$, $AM/M \neq G/M$ or $BM/M \neq G/M$, since otherwise $M = AM \cap BM = G$. By Theorem 3.1, $G/M$ is cyclic of prime order. Hence $AM = M$ or $BM = M$, so that $A \subseteq M$ or $B \subseteq M$.

REMARK 3.3. (a) Corollary 3.2 implies the main result of Knop [**5**].

(b) The last part of the proof of Theorem 3.1 uses an argument due to Sesekin [**10**]. If $M \neq 1$ is an abelian normal subgroup of the group $G = AB$, which is the product of two abelian subgroups $A$ and $B$, then this argument shows that for every $1 \neq m = a^{-1}b \in M$ with $a \in A$ and $b \in B$, the normal subgroup $[M, a] = [M, b]$ of $G$ is properly contained in $M$. This leads to the following observation:

*The factorizer of a minimal normal subgroup of a product of two abelian subgroups is always abelian.*

## References


**1.** B. Amberg, *Artinian and noetherian factorized groups*, Rend. Sem. Mat. Univ. Padova **55** (1976), 105–122.

**2.** ———, *Lokal endlich-auflösbare Produkte von zwei hyperzentralen Gruppen*, Arch. Math. **35** (1980), 228–238.

**3.** J.D. Gillam, *A finite p-group $P = AB$ with* Core(A) = Core(B) = 1, Rocky Mountain J. of Math. **3** (1973), 15–17.

**4.** N. Itô, *Über das Produkt von zwei abelschen Gruppen*, Math. Z. **62** (1955), 400–401.

**5.** L.E. Knop, *Normal subgroups of groups which are the product of two abelian subgroups*, Proc. AMS **40** (1973), 37–41.

**6.** D.J.S. Robinson, *Residual properties of some classes of infinite soluble groups*, Proc. London Math. Soc (3), **18** (1968), 495–520.

**7.** ———, *A property of the lower central series of a group*, Math. Z. **107** (1968), 225–231.

**8.** ———, *Finiteness conditions and generalized soluble groups*, Part **1**, Springer, Berlin (1972).

**9.** ———, *Finiteness conditions and generalized soluble groups*, Part **2**, Springer, Berlin (1972).

**10.** N.F. Sesekin, *On the product of two finitely connected abelian groups*, Sibir. Mat. Zurn. **9** (1968), 1427–1430, Sibir. Math. J. **9** (1968), 1070–1072.

**11.** B.A.F. Wehrfritz, *Frattini subgroups in finitely generated linear groups*, J. London Math. Soc. **43** (1968), 619–622.

**12.** H. Wielandt, *Über das Produkt von zwei nilpotenten Gruppen*, Illinois J. Math. **2** (1958), 611–618.

**13.** D.I. Zaičev, *Nilpotent approximations of metabelian groups*, Algebra i Logika **20** (1981), 638–653 = Algebra and Logik **20** (1982), 413–423.


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