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ISOLATORS IN SOLUBLE GROUPS OF FINITE RANK

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ABSTRACT. A group G is said to have the *isolator property* if for every subgroup H of G the set $\sqrt{H} = \{g \in G: h^n \in H$ for some $n \geqq 1\}$ is a subgroup. G is said to have the *strong isolator property* if in addition $|\sqrt{H}: H|$ is finite whenever $H/\bigcap_{g\in G}H^g$ is finitely generated. It is well known that locally nilpotent groups have the isolator property, that finitly generated nilpotent groups have the strong isolator property and that the infinite dihedral group D has neither. However D clearly has a subgroup of finite index with the strong isolator property.

Our purpose here is to show that for soluble groups and linear groups the isolator property is closely associated with the finiteness of the (Prüfer) rank, while the strong isolator property is closely associated with polycyclicity.

For subgroups $H \leqq K$ of a group G let $\sqrt[K]{H}$ denote the set $\{g \in K: g^n \in H$ for some $n \geqq 1\}$. If $K = G$ write $\sqrt{H}$ for $\sqrt[G]{H}$. The group G is said to have the *isolator property* if $\sqrt{H}$ is a subgroup of G for every subgroup H of G. If in addition $|\sqrt{H}: H|$ is finite for every H such that $H/\bigcap_{g\in G}H^g$ is finitely generated say that G has the *strong isolator property.* In his Edmonton lectures ([1]) P. Hall proved that locally nilpotent groups and finitely generated nilpotent groups have the isolator and the strong isolator property respectively. If G_0 is the infinite dihedral group and $H = \langle 1 \rangle$ then trivially $\sqrt{H}$ is not a subgroup. However the cyclic subgroup of G_0 of index 2 clearly does have the strong isolator property. For any class Σ of groups say that a group G is almost a Σ-group if it has a normal Σ-subgroup of finite index. Thus G_0 above almost has the strong isolator property. For soluble groups and for linear groups there is a close link between groups that almost have the isolator property and groups with finite (Prüfer) rank. Our main results are as follows.

THEOREM A. *Let G be a finitely generated soluble group. Then G is polycyclic if and only if G almost has the strong isolator property.*

THEOREM B. *A torsion-free soluble group of finite rank almost has the*

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isolator property. If G is a finitely generated, nilpotent-by-polycyclic-by-finite group with the isolator property, then G has finite rank.

THEOREM C. *Let G be a finitely generated linear group. Then G almost has the isolator property if and only if G is almost soluble-of-finite-rank.*

THEOREM A can be used to give a short proof of the following known result ([3]).

Suppose G is a group, H a subgroup of G and Ω a set of automorphisms of G Let (H, Ω) denote the set of all $\bigcap_{\omega\in\Phi} H^{\omega}$ as Φ ranges over all the finite subsets of Ω.

THEOREM D. *Let G be a polycyclic group. Then for every H and Ω as above the set (H, Ω) satisfies the minimal condition.*

A subgroup H of a group G is *isolated* if $H = \sqrt{H}$. Define the relation $\sim$ on the set of subgroups of G by the rule $H \sim K$ whenever $\sqrt{H} = \sqrt{K}$. Clearly $\sim$ is an equivalence relation and every $\sim$ equivalence class contains a maximal element. G has the isolator property if and only if every $\sim$ equivalence class has a unique maximal element that is isolated.

P. Hall has proved the following (see [1] Theorem 4.6). Let G be a locally nilpotent group and let θ be any word in n variables. Let H_i, K_i be subgroups of G such that $H_i \sim K_i$ for each $i = 1, 2, \ldots, n$. Then $\theta(H_1, H_2, \ldots, H_n) \sim \theta(K_1, K_2, \ldots, K_n)$. Possibly the same conclusion is valid for polycyclic groups with the isolator property. We show the following.

THEOREM E. *Let G be a nilpotent-by-abelian group of finite rank with the isolator property. If H_1, H_2, K_1 and K_2 are subgroups of G such that $H_i \sim K_i$ for $i = 1, 2$ then $[H_1, H_2] \sim [K_1, K_2]$. If ϕ is any outer commutator word then $\phi(H_1) \sim \phi(K_1)$.*

For a nilpotent group G it is well known that if $H \leqq G$ with $HG' = G$ then $H = G$. If G is soluble the same conclusion holds if H is also subnormal. It is also well known that for a nilpotent group G, if $HG' \sim G$ then $H \sim G$. This does not hold for polycyclic groups even if H is normal, for consider any infinite polycyclic group G with $|G: G'|$ finite and set $H = \langle 1 \rangle$. However any polycyclic group does have a subgroup of finite index which has this property. This is a consequence of Theorem E. For suppose G satisfies the hypothesis of Theorem E and let H be a normal subgroup of G with $HG' \sim G$. Set $H_1 = H_2 = HG'$ and $K_1 = K_2 = G$. Then by Theorem E $G' = [K_1, K_2] \sim [H_1, H_2] \leqq HG''$, so $G \sim HG' \sim HG''$.

By a simple induction we obtain $G \sim HG^{(i)}$ for each i and the solubility of G yields that $G \sim H$ as required. We prove below (Proposition 6) that in such G normalizers of isolated subgroups are isolated. Suppose now that H is merely subnormal in G with $HG' \sim G$. Then $\sqrt{H} \lhd K_1 \lhd$

$\cdots \lhd K_r \rhd G$ for some isolated subgroups K_i of G. If $\sqrt{H} \neq G$ we may choose $K_r \neq G$. But then $K_r \sim G$ by the normal case so $K_r = G$. This shows that $HG' \sim G$ implies $H \sim G$ whenever H is subnormal.

It would be interesting to determine those soluble groups G with the property that $H \sim G$ whenever $H \lhd\lhd G$ with $HG' \sim G$.

It is tempting to replace isolators in the above by π-isolators for some set of primes π. However in **[4]** it is proved that if G is a finitely generated soluble group and π any finite set of primes, then G has a subgroup of finite index with the π-isolator property if and only if G is nilpotent by finite.

PROPOSITION 1. *Let G_0 be a polycyclic group. Then G_0 has a subgroup of finite index with the strong isolator property.*

PROOF. Regard G_0 as a subgroup of $GL(n, \mathbf{Z})$ for some n. Then G_0 contains a triangularizable normal subgroup G of finite index such that the closed subgroups of G are all isolated (**[6]** Theorem 2.8). Let U be the unipotent radical of G and H any subgroup of G. If $\bar{H}$ is the (Zariski) closure of H in G then $H \subseteqq \sqrt{H} \subseteqq \bar{H}$. By Lemma 3.2 of **[8]** the subgroup $J = \sqrt[U]{U \cap H}$ is closed in U and hence in G. Now (3.3 of **[8]**) $U \cap H$ is of finite index in J, so $HJ \subseteqq \sqrt{H}$. Since $G' \leqq U$, we have $H' \leqq U \cap H$ and so the derived group of $\bar{H}$ lies in J by **[7]** Lemma 5.10. Hence $\bar{H}/J$ is a finitely generated abelian group and the result follows.

PROPOSITION 2. *Let G be a finitely generated, nilpotent-by-polycyclic-by-finite group with the isolator porperty. Then G has finite rank.*

PROOF. Let N be a nilpotent normal subgroup of G such that G/N is polycyclic-by-finite and suppose first that N is abelian. If G does not have finite rank then G has a section isomorphic to $\langle a \rangle \sim \langle b \rangle$ where a has prime order and b infinite order. But it is easily seen that $\sqrt{\langle b \rangle}$ in $\langle a, b \rangle$ is not a subgroup, so G has finite rank.

In general this proves that G/N' has finite rank. By **[5]** Theorem 2.26 every lower central factor of N has finite rank. Therefore G has finite rank.

COROLLARY. *Let G be a finitely generated soluble group with the strong isolator property. Then G is polycyclic.*

Theorem A follows from Proposition 1 and the Corollary to Proposition 2.

PROOF. By induction on the derived length we may assume that G has an abelian normal subgroup A such that G/A is polycyclic. Then A has finite rank by Proposition 2. Hence there exists a finitely generated subgroup B of A with $A \leqq \sqrt{B}$. By the strong isolator property $|\sqrt{B} : B|$ is finite, so A is finitely generated and G is polycyclic.

PROPOSITION 3. *Let G be a soluble subgroup of $GL(n, \mathbf{Q})$ of finite rank. Then G has a subgroup of finite index with each of its closed subgroups isolated.*

PROOF. We may assume that G is connected. If U is the unipotent radical of G then G/U is a finitely generated abelian group. By the Lie-Kolchin theorem G^x is triangular for some $x \in GL(n, \mathbf{C})$. For $g \in G$ and $i = 1, 2, \ldots, n$ let g^{ω_i} be the i-th diagonal entry of g^x. Then $X = \langle G^{\omega_i}: 1 \leqq i \leqq n\rangle$ is a finitely generated subgroup of $\mathbf{C}^*$ and as such has a torsion-free subgroup X_0 say of finite index. Let G_0 denote the intersection of the inverse images in G of X_0 under the ω_i. Then G_0 is a subgroup of G of finite index such that for each $g \in G$, no product of eigenvalues of g is a non-identity root of unity. The last part of the proof of Proposition 2 of [**10**] shows that every cyclic subgroup of G_0 is connected and so every closed subgroup of G_0 is isolated.

PROOF OF THEOREM B. If G_0 is a torsion-free soluble group of finite rank then for some n we can regard G_0 as a subgroup of $GL(n, \mathbf{Q})$ (e.g. [**8**]). Then the proof of Proposition 1 with Proposition 3 used in place of [**6**] Theorem 2.8 yields that G almost has the isolator property. (In fact the proof can be shortened somewhat.) The remainder of Theorem B is Proposition 2.

PROPOSITION 4. *Let G be a finitely generated group of automorphisms of a finitely generated module M over a commutative ring. Then G almost has the isolator property if and only if G is almost soluble-of-finite-rank.*

Theorem C is an immediate consequence of Proposition 4. If G is a finitely generated linear group of positive characteristic then G is almost soluble-of-finite-rank if and only if G is almost abelian by 10.9 of [7].

PROOF. Suppose G is soluble with finite rank. Then G is finite by almost torsion-free ([7] 13.4, 13.7, 13.11). But G is also residually finite ([**9**] 1.1) and so G is almost torsion-free. Therefore G almost has the isolator property by Theorem B. Conversely suppose G almost has the isolator property. By [7] 13.4 and 13.31 the group G is almost soluble. The result now follows from [7] 13.11 and Proposition 2.

PROOF OF THEOREM D. There exists by Proposition 1 a characteristic subgroup K of G of finite index that has the strong isolator property. Let $\omega_1, \omega_2, \ldots$ be an infinite sequence of elements of Ω and set $H_i = H^{\omega_1} \cap \cdots \cap H^{\omega_i}$ for each $i \geqq 1$. Now K satisfies the minimal condition on isolated subgroups so there exists r such that if $J_i = \sqrt[k]{H_i \cap K}$ then $J_i = J_r$ for all $i \geqq r$. By the choice of K the index $|\sqrt[k]{H \cap K}: H \cap K|$ is finite so for some integer m we have $(\sqrt[k]{H \cap K})^m \subseteqq H \cap K$. Consequently for $i \geqq r$,

$$\begin{aligned} J_r^m = J_i^m &= (\textstyle\bigcap_{j=1}^{i} (\sqrt[k]{H \cap K^{\omega_i}})^m \\ &\subseteq \textstyle\bigcap_{j=1}^{i} H^{\omega_i} \cap K = H_i \cap K. \end{aligned}$$

But $(J_r : J_r^m)$ is finite so $(H_r \cap K : H_i \cap K)$ and hence $(H_r : H_i)$ is bounded for $i \geqq r$ and the result follows.

REMARK. Let G be a finitely generated soluble group. If $(H, \operatorname{Inn} G)$ satisfies the minimal condition for every subgroup H of G then a straightforward induction on the derived length of G yields that G is polycyclic (cf. [3]). This gives a partial converse to Theorem D.

A finitely generated torsion-free soluble group of finite rank has a subgroup G of finite index such that if H is any torsion-free *section* of G and if $F(H)$ is the Fitting subgroup of H then $H/F(H)$ is torsion-free abelian. This was shown in [**2**]. The proof there was long if elementary. It is an immediate consequence of Theorem B and the following.

PROPOSITION 5. *Let G be a nilpotent-by-abelian torsion-free group o finite rank with the isolator property. Then the Fitting subgroup $F[G]$ is isolated.*

PROOF. Suppose $F = F(G)$ is not isolated in G. Then there exists a prime p and an element $g \in G\backslash F$ such that $g^p \in F$. Since $\langle g \rangle F$ is normal in G we may assume that $G = \langle g \rangle F$. Clearly there exists $x \in F$ with $t = [g, x] \neq 1$. We induct on the nilpotency class of F.

Suppose F is abelian. By the isolator property $\langle g, g^x \rangle \sim \langle g^p \rangle$. Then $t^n \in \langle g^p \rangle$ for some integer $n > 0$, say $t^n = g^{pm}$, where $m \neq 0$ as G is torsion-free. But $t^n = [g, x^n]$, so $g^{x^n} = g^{1+pm}$. Thus $g^p = (g^p)^{x^n} = g^{(1+pm)}$. Consequently $g^{p^2m} = 1$, which contradicts the fact that G is torsion-free.

Now consider the general case. Let Z denote the centre of F. By the above case $\langle g \rangle Z$ is nilpotent and so abelian. In particular Z is the centre of G. If G/Z is torsion-free it is nilpotent by induction, which contradicts $G \neq F$. But F/Z is torsion-free. Thus we may assume that $g^p \in Z$. As in the abelian case $t^n = g^{pm}$ for some non-zero integers m and n. But Z is isolated in F, so $t \in Z$ and the proof can be completed exactly as in the abelian case.

COROLLARY. *Let G be as in Propostion 5. Then G is an R-group and centralizers in G are isolated.*

For suppose n is a positive integer and x, y, g elements of G. If $x^n = y^n$ then $\langle x, y \rangle$ is nilpotent by Proposition 5, so $x = y$ by the nilpotent case. If $[g^n, x] = 1$ then $g^n = (g^x)^n$ so $[g, x] = 1$.

PROPOSITION 6. *Let G be as in Proposition 5. If H is any isolated subgroup of G then $N_G(H)$ is also isolated.*

PROOF. Of all possible counter examples choose a pair G, H in which the Hirsch number of H is minimal. In particular there will exist $g \in G \backslash N_G(H)$ and a prime p with $g^p \in N_G(H)$. We may assume that $G = \langle g, H \rangle$. The Fitting subgroup F of G is isolated by Proposition 5. Suppose $F \cap H = \langle 1 \rangle$. Now G/F is abelian. Hence g^p centralizes H, so g does too by the Corollary to Proposition 5. This contradiction shows that $F \cap H \neq 1$.

Suppose $F \cap H \neq H$. Then both $F \cap H$ and $F/F \cap H$ have smaller Hirsch number than H. By the minimal choice of H the element g normalizes $F \cap H$. Thus $F \cap H \lhd G$ and passing to $G/(F \cap H)$ the minimal choice of H yields the contradiction that $H \lhd G$. Consequently $H \leqq F$.

Therefore $G = \langle g \rangle F$. Also, being a counter example, G is not nilpotent by the nilpotent case and F is isolated. Thus $\langle g \rangle \cap F = \langle 1 \rangle$. Clearly $G \sim \langle g^p, H \rangle$ so if $x \in F$ there exists $n > 0$ with $x^n \in F \cap \langle g^p \rangle H = (F \cap \langle g^p \rangle)H = H$. But H is isolated so $x \in H$ and $H = F \lhd G$. This final contradiction completes the proof.

REMARK. If H is any subgroup of the polycyclic group G such that $\sqrt{H}$ is a subgroup, then $N_G(H) \leq N_G(\sqrt{H}) \subseteqq \sqrt{N_G(H)}$, the second containment following from the fact that $\sqrt{H}$ has only a finite number of subgroups of index $|\sqrt{H} : H|$. Thus if G also satisfies the hypotheses of Proposition 5 then $N_G(\sqrt{H}) = \sqrt{N_G(H)}$ by Proposition 6. However if $G = \langle x, y : x^y = x^2 \rangle$ and $H = \langle x \rangle$ then G is a finitely generated group satisfying the hypotheses of Proposition 5 such that $N_G(\sqrt{H}) = G \neq \langle x^G \rangle = \sqrt{N_G(H)}$.

PROOF OF THEOREM E. We may assume that H_1 and H_2 are isolated. Let K denote the isolator of $[K_1, K_2]$ in $H = \langle H_1, H_2 \rangle$. Since $\langle K_1, K_2 \rangle$ normalizes K, so does H by Proposition 6 applied to $H/^H\sqrt{1}$. Thus H/K is torsion-free. In it centralizers are isolated by the Corollary to Proposition 5 and trivially K_1 and K_2 commute modulo K. Consequently so do H_1 and H_2, that is $[H_1, H_2] \leqq K$ as required. The second part of Theorem E follows from this and a very easy induction.

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