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UNIFORM APPROXIMATION BY RANDOM FUNCTIONS

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1. **Introduction.** In this paper we prove a number of stochastic analogues of deterministic uniform approximation theorems.

One possible approach to a stochastic analogue of a deterministic theorem is a careful examination of the parameters involved in the proof of the deterministic case. If one can show that these parameters can be chosen in a measurable way then the probabilistic version should follow with little difficulty. However, there are drawbacks to this line of attack. Firstly, with each new deterministic theorem comes a new set of parameters and their corresponding measurability problems; this is often a long and tedious process. Secondly, one is rarely able to prove much more than a direct analogue to the deterministic case; when in fact one has in mind more general results.

As is true with many problems in analysis, probabilistic approximation problems can be attacked with greater ease when they are placed within a suitably chosen abstract framework. The theory of measurable selection affords the degree of abstraction we are seeking.

In §2 we present the necessary preliminaries, including a selection theorem. In §3 we prove a selection theorem and stochastic versions of several deterministic approximation theorems.

2. **Definitions, properties, and a selection theorem.** If $(\Omega, \mathscr{A})$ and $(Z, \mathscr{B})$ are two measurable spaces and X is an arbitrary non-empty set, then a mapping f: $\Omega \times X \to Z$ is a *random function* if and only if the function $f(\cdot, x)$ is measurable for each x in X. In our applications Z is a metric space and, in this case, we choose the σ-algebra $\mathscr{B}$ as the class of Borel subsets of Z.

If X is a metric space, $(\mathscr{C}(X), H)$ is the space of compact non-empty subsets of X with the Hausdorff metric.

A relation F: $\Omega \to X$ is a subset of $\Omega \times X$. Alternatively, F may be regarded as a function from Ω to the set of all subsets of X. When we want to emphasize the properties of F as a subset of $\Omega \times X$, we will refer to its graph $\mathrm{Gr}(F)$ rather than F. If domain $(F) = \Omega$, then F is called a

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multi-function from Ω to X. For $B \subset X$, $F^{-1}(B) = \{\omega \in \Omega: F(\omega) \cap B \neq \emptyset\}$.

If $(\Omega, \mathscr{A})$ is a measurable space and X is a topological space, a relation $F: \Omega \to X$ is *weakly measurable* (*measurable*) if and only if $F^{-1}(B) \in \mathscr{A}$ for every open (closed) subset B of X. An excellent source for measurability properties of relations is Himmelberg [6]. While we often make use of this source we do not necessarily mean to imply that the referred result originated with Himmelberg. We will often be concerned with relations having closed values. In this case the following proposition is useful (for a proof see Himmelberg [6], Theorem 3.2).

PROPOSITION 1. (a) *Let X be a separable metric space and $F: \Omega \to X$ a relation with closed values. Then measurability of F implies weak measurability of F.*

(b) *If, in* (a), *X is also σ-compact, then the two measurability concepts are equivalent.*

We shall need the notion of a generalized random function, which is defined as follows. Let X be a T_1-topological space, Z a measurable space, and $P: \Omega \to X$ a weakly measurable relation. A function f: Gr$(P) \to Z$ is called a *generalized random function* if for each $x \in X, f(\cdot, x)$, a mapping from $P^{-1}(x)$ into Z, is measurable.

A function $f: \Omega \to X$ is a *selector* for a multi-function $F: \Omega \to X$ if and only if $f(\omega) \in F(\omega)$ for all ω in Ω. If X and Y are topological spaces, we say $f: \Omega \times X \to Y$ is a *Caratheodory map* if $f(\omega, \cdot)$ is continuous for $\omega \in \Omega$ and $f(\cdot, x)$ is measurable for $x \in X$. A Caratheodory map is a measurable map (see e.g., Theorem 6.1 of Himmelberg [6] or Kuratowski [7], p. 378).

A topological space X is *Polish* if X is separable and is metrizable by a complete metric; X is *Souslin* if it is metrizable and the continuous image of a Polish space. The definition of Souslin space given here is according to Bourbaki [3]. Currently, many authors do not require metrizability in the definition of Souslin space.

In proving our results, we use the following selection theorem (a proof of which may be found in Himmelberg [6], Theorem 5.7).

PROPOSITION 2. *If Ω is a σ-finite measure space, X a Souslin space, and $F: \Omega \to X$ is a multi-function with measurable graph, then there is a measurable function $f: \Omega \to X$ such that $f(\omega) \in F(\omega)$* a.e.

3. Stochastic analogues of deterministic approximation theorems. In this section our aim is to extend, using the theory of measurable selection, the classical uniform approximation theorems of Runge, Arakelyan, Mergelyan, and Vitushkin.

We have the following selection theorem.

THEOREM 1. *Let* $(\Omega, \mathscr{A}, \mu)$ *be a* σ*-finite measure space. Let* Y *be a Souslin subset of a Polish space* (X, d)*. Suppose that* $\varphi: \Omega \to X$ *and* $\varepsilon: \Omega \to (0, \infty)$ *are both measurable functions. Define a relation* $F: \Omega \to Y$ *by* $F(\omega) = \{x \in Y: d(x, \varphi(\omega)) < \varepsilon(\omega)\}$, *for* $\omega \in \Omega$*. If* $F(\omega) \neq \varnothing$ a.e., *then there exists a measurable map* $\sigma: \Omega \to Y$ *such that* $d(\sigma(\omega), \varphi(\omega)) < \varepsilon(\omega)$ a.e.

PROOF. Let Λ be the domain of the relation F, i.e., $\Lambda = \{\omega \in \Omega: F(\omega) \neq \varnothing\}$. By hypothesis Λ is a measurable set, with $\mu(\Omega \backslash \Lambda) = 0$.

Consider the graph of F,

$$\begin{aligned} \mathrm{Gr}(F) &= \{(\omega, x) \in \Lambda \times X: x \in F(\omega)\} \\ &= \{(\omega, x) \in \Lambda \times Y: d(\varphi(\omega), x) < \varepsilon(\omega)\}. \end{aligned}$$

Next define a function $q: \Lambda \times Y \to (0, \infty)$ by setting $q(\omega, x) = \varepsilon(\omega) - d(\varphi(\omega), x)$, for $(\omega, x) \in \Lambda \times Y$. Since q is measurable in ω and continuous in x, it is a Carathéodory map and hence q is measurable with respect to the product σ-algebra $\mathscr{A}_\Lambda \times \mathscr{B}(Y)$.

Since, $\mathrm{Gr}(F) = q^{-1}((0, \infty))$, the measurability of q implies $\mathrm{Gr}(F) \in \mathscr{A}_\Lambda \times \mathscr{B}(Y)$.

And now according to Proposition 2 there is a measurable function $f: \Lambda \to Y$ such that $f(\omega) \in F(\omega)$ a.e.

Finally, define $\sigma: \Omega \to Y$ by

$$\sigma(\omega) = \begin{cases} f(\omega), & \text{if } \omega \in \Lambda, \\ y_0, & \text{if } \omega \in \Omega \backslash \Lambda, \end{cases}$$

where y_0 is some point in Y.

Then σ is measurable and $d(\sigma(\omega), \varphi(\omega)) < \varepsilon(\omega)$ a.e. This completes the proof of the theorem.

If R is a rational function, and if we define $R(z) = \infty$ whenever z is a pole of R, then R is in $C(\mathbf{C}, S^2)$. Recall that $C(\mathbf{C}, S^2)$ is metrized by the spherical metric ρ.

Before developing our generalizations we prove two lemmas.

LEMMA 1. (a) *Let* $\mathscr{R}$ *be the subset of* $C(\mathbf{C}, S^2)$ *consisting of those rational functions which have all of their poles in some given countable set* $A = \{\alpha_j\} \subset S^2$*. Then* $\mathscr{R}$ *is an* F_σ *subset of* $C(\mathbf{C}, S^2)$.

(b) *Let* $K \subset \mathbf{C}$ *be compact and* $A \subset S^2 \backslash K$ *be countable. Denote by* $S(K)$ *the subspace of* $C(K)$ *consisting of the restrictions to* K *of rational functions having all of their poles in* A*. Then* $S(K)$ *is an* F_σ *subset of* $C(K)$.

PROOF. For positive integers k, m, and n, set $R(k, m, n)$ equal to the set of rational functions $f(z) = \sum_{i=1}^{k} p_i((z - \alpha_i)^{-1})$, where the p_i's are polynomials of degree $\leqq m$ with coefficients bounded by n and for each i, $p_i(0) = 0$. The set $R(k, m, n)$ is compact in $C(\mathbf{C}, S^2)$ and thus $\mathscr{R} = \bigcup_{k,m,n} R(k, m, n)$ is an F_σ set.

The mapping T: $C(\mathbf{C}, S^2) \to C(K)$, $Tf = f|_K$ is continuous so $T(R(k, m, n))$ is compact and hence $S(K)$ is an F_σ subset of $C(K)$.

The next lemma demonstrates how to measurably extend functions in $S(K)$ to functions in $\mathscr{R}$.

LEMMA 2. *Suppose* $(\Omega, \mathscr{A}, \mu)$ *is a σ-finite measure space. Let* $K \subset \mathbf{C}$ *be compact and* $A \subset S^2\backslash K$ *be countable. Denote by* $\mathscr{R}$ *the class of all rational functions whose only poles lie in A and set* $S(K) = \{R|_K: R \in \mathscr{R}\}$. *If* $r: \Omega \to S(K)$ *is a measurable function, then there exists a measurable function* R: $\Omega \to \mathscr{R}$ *such that* $R(\omega)|_K = r(\omega)$ a.e.

PROOF. We define a relation $F: \Omega \to \mathscr{R}$ by setting $F(\omega) = \{R \in \mathscr{R}: R|_K = r(\omega)\}$, for $\omega \in \Omega$. By the definition of $S(K)$, $F(\omega) \neq \emptyset$ for all $\omega \in \Omega$; that is domain F equals Ω. So F is a multifunction. Consider the graph of F:

$$\begin{aligned}\mathrm{Gr}(F) &= \{(\omega, R) \in \Omega \times \mathscr{R}: R|_K = r(\omega)\} \\ &= \{(\omega, R) \in \Omega \times \mathscr{R}: R(z) = [r(\omega)](z), \forall z \in K\} \\ &= \{(\omega, R) \in \Omega \times \mathscr{R}: \|R - r(\omega)\|_K = 0\}.\end{aligned}$$

Define $q: \Omega \times \mathscr{R} \to [0, \infty)$ by $q(\omega, R) = \|R - r(\omega)\|_K$. Fix $R \in \mathscr{R}$. Since r is measurable, $q(\cdot, R)$ is measurable (since it is the composition of a continuous function with a measurable function).

Now fix $\omega \in \Omega$ and consider the map $q(\omega, \cdot)$: $\mathscr{R} \to [0, \infty)$. Suppose that $\{R_n\}$ is a sequence in $\mathscr{R}$ converging to some $R \in \mathscr{R}$, i.e., $\rho(R_n, R) \to 0$ as $n \to \infty$. Therefore, $\|R_n - R\|_K \to 0$ as $n \to \infty$. It therefore follows from

$$|q(\omega, R_n) - q(\omega, R)| = |\,\|R_n - r(\omega)\|_K - \|R - r(\omega)\|_K| \leqq \|R_n - R\|_K,$$

that $q(\omega, R_n) \to q(\omega, R)$ as $n \to \infty$. This establishes the continuity of the function $q(\omega, \cdot)$ and thus q is measurable, since it is a Carathéodory map.

Since, $\mathrm{Gr}(F) = q^{-1}(0)$ it follows that $\mathrm{Gr}(F) \in \mathscr{A} \times \mathscr{B}(\mathscr{R})$.

By Lemma 1, $\mathscr{R}$ is a Borel subset of the Polish space $C(\mathbf{C}, S^2)$. In a Souslin space every Borel set is a Souslin set (see [3], p. 200, Proposition 11).

We are now in a position to apply Proposition 2 to conclude that there exists a measurable function R: $\Omega \to \mathscr{R}$ such that $R(\omega) \in F(\omega)$ a.e., that is $R(\omega)|_K = r(\omega)$ a.e.

A direct generalization of Runge's Theorem ([9]) follows easily from our Theorem 1, Lemma 1, and Lemma 2 and is subsumed by our next theorem in which we allow the compact set K to depend on ω.

THEOREM 2. *Let* $(\Omega, \mathscr{A}, \mu)$ *be a* σ*-finite measure space. Suppose that* $K: \Omega \to \mathbf{C}$ *is a weakly measurable, compact-valued, multifunction such that* $K(\Omega)$ *has cardinality less than or equal to* $\aleph_0$. *Let* $\{\alpha_j\}$ *be a sequence of complex numbers having the property that for each* $\omega \in \Omega$ *the set* $\{\alpha_j\}$ *contains one point from each component of* $S^2 \backslash K(\omega)$. *Let* ε *be a positive measurable function defined on* Ω *and let* φ: $\mathrm{Gr}(K) \to \mathbf{C}$ *be a generalized random function satisfying*

$$(*) \qquad \varphi(\omega, \cdot) \in H(K(\omega)) \text{ a.e.}$$

Then there exists a random rational function R *all of whose poles lie in* $\{\alpha_j\}$ *and* $\|\varphi(\omega, \cdot) - R(\omega, \cdot)\|_{K(\omega)} < \varepsilon(\omega)$ a.e.

PROOF. We assume without loss of generality, that (*) holds for all $\omega \in \Omega$.

Enumerate $K(\Omega)$, $K(\Omega) = \{K_j\}_{j=1}^{\infty}$, and partition the space Ω by setting $\Omega_j = \{\omega \in \Omega: K(\omega) = K_j\}$. It follows from the weak measurability of K and the separability of the plane that the Ω_j's are measurable.

We will use φ_j to denote φ restricted to $\mathrm{Gr}(K) \cap (\Omega_j \times \mathbf{C})$.

By the definition of a generalized random function $\varphi(\cdot, z)$: $K^{-1}(z) \to \mathbf{C}$ is measurable for all z in $\mathbf{C}$. Since $K^{-1}(z) \supset \Omega_j$ whenever $z \in K_j$, it follows that $\varphi_j(\cdot, z)$: $\Omega_j \to \mathbf{C}$ is measurable for each z in K_j. Therefore, φ_j: $\Omega_j \times K_j \to \mathbf{C}$ is a random function. Also, by (*) $\varphi_j(\omega, \cdot) \in H(K_j)$ for $\omega \in \Omega_j$. It is not difficult to show that the function $\hat{\varphi}_j$: $\Omega_j \to C(K_j)$ given by $\hat{\varphi}_j(\omega) = \varphi_j(\omega, \cdot)$ is measurable.

Denote by $S(K_j)$ the restrictions to K_j of all rational functions whose only poles lie in the set $A \backslash K_j$. By Lemma 1(b) $S(K_j)$ is a Borel subset of $C(K_j)$.

Since $\hat{\varphi}_j(\omega) \in H(K_j)$ for all $\omega \in \Omega_j$, it follows from Runge's Theorem that the set

$$\{r \in S(K_j): \|r - \hat{\varphi}_j(\omega)\|_{K_j} < \varepsilon(\omega)\}$$

is non-empty for all $\omega \in \Omega_j$. It is a consequence of Theorem 1 that we can now find a measurable function r_j: $\Omega_j \to S(K_j)$ such that $\|r_j(\omega) - \hat{\varphi}_j(\omega)\|_{K(\omega)} < \varepsilon(\omega)$ a.e. (on Ω_j).

Denoting by $\mathscr{R}_j$ the rational functions whose only poles lie in $A \backslash K_j$ and applying Lemma 2 yields a measurable map R_j: $\Omega_j \to \mathscr{R}_j$ such that $R_j(\omega)|_{K_j} = r_j(\omega)$ a.e. (on Ω_j).

We repeat the above procedure for each positive integer j and then define R: $\Omega \to C(\mathbf{C}, S^2)$ by setting $R(\omega) = R_j(\omega)$ for $\omega \in \Omega_j$.

The requirement that $K(\Omega)$ be countable can be removed, in this case less precision in the location of the poles of the approximating random rational function is achieved.

THEOREM 3. *Let* $(\Omega, \mathscr{A}, \mu)$ *be a* σ*-finite measure space. Suppose we are given the following*:

(a) $K: \Omega \to \mathbf{C}$ *a weakly measurable, compact-valued multifunction.*

(b) $J: \Omega \to \mathbf{C}$ *a weakly measurable, closed-valued multifunction such that* $G(\omega) = \mathbf{C}\backslash J(\omega) \supset K(\omega)$ *for almost all* ω *in* Ω.

(c) ε *a positive measurable function defined on* Ω.

(d) φ: $\mathrm{Gr}(G) \to \mathbf{C}$ *a generalized random function satisfying*, $\varphi(\omega, \cdot) \in H(G(\omega))$ a.e.

Then there exists a random rational function R such that for almost all $\omega \in \Omega$:

(1) $R(\omega, \cdot)$ *has no poles on* $K(\omega)$, *and*

(2) $\|\varphi(\omega, \cdot) - R(\omega, \cdot)\|_{K(\omega)} < \varepsilon(\omega)$.

PROOF. First observe that Proposition 1(b) implies that the multifunction J is measurable. Furthermore, J measurable implies that G is measurable (see [6], Theorem 4.4).

We may assume, without loss of generality, that (c) and (d) hold for all ω in Ω.

For F a finite subset of $\mathbf{C}$ and $\varepsilon > 0$ we define the following sets.

$$A_{F,\varepsilon} = \bigcap_{z\in F} \{\omega \in \Omega: \Delta(z, \varepsilon) \cap K(\omega) \neq \varnothing\},$$

$$B_{F,\varepsilon} = \{\omega \in \Omega: \bigcup_{z\in F} \overline{\Delta(z, \varepsilon)} \cap J(\omega) \neq \varnothing\},$$

$$C_{F,\varepsilon} = \{\omega \in \Omega: (\mathbf{C}\backslash \bigcup_{z\in F} \Delta(z, \varepsilon)) \cap K(\omega) \neq \varnothing\},$$

$$E_{F,\varepsilon} = A_{F,\varepsilon}\backslash(B_{F,\varepsilon} \cup C_{F,\varepsilon}).$$

The measurability of each of these sets follows immediately from the measurability of J and the weak measurability of K.

Observe that for all $\omega \in E_{F,\varepsilon}$,

$$G(\omega) \supset \bigcup_{z\in F} \overline{\Delta(z, \varepsilon)} \supset \bigcup_{z\in F} \Delta(z, \varepsilon) \supset K(\omega).$$

Let D be a countable dense subset of $\mathbf{C}$. The family $\mathscr{F}$ of all non-empty finite subsets of D is a countable dense set in $C(\mathbf{C})$.

Now suppose $\omega \in \Omega$. Since $d(K(\omega), J(\omega)) > 0$, there exists an $F \in \mathscr{F}$ and $q \in Q^+$(= the positive rationals) such that

$$H(F, K(\omega)) < \frac{1}{4} d(K(\omega), J(\omega)) < q \leqq \frac{1}{2} d(K(\omega), J(\omega)).$$

It follows that

$$\Delta(z, q) \cap K(\omega) \neq \emptyset, \forall z \in F,$$
$$\bigcup_{z \in F_i} \overline{\Delta(z, q)} \cap J(\omega) = \emptyset, \text{ and}$$
$$(\mathbf{C} \setminus \bigcup_{z \in F} \overline{\Delta(z, q)}) \cap K(\omega) = \emptyset;$$

that is $\omega \in E_{F,q}$.

We have established that $\Omega = \bigcup_{F,q} E_{F,q}$, where the union is taken over all F in $\mathscr{F}$ and q in Q^+. As a matter of notational convenience we let $\{\Omega_n\}$ be an enumeration of $\{E_{F,q}\}_{F,q}$. So each Ω_n is some member of $\{E_{F,q}\}_{F,q}$, say E_{F_n, q_n}. Define $\Lambda_1 = \Omega_1$, and $\Lambda_n = \Omega_n \setminus (\Lambda_1 \cup \cdots \cup \Lambda_{n-1})$ for $n \geqq 2$.

By φ_n is meant the function φ restricted to $\mathrm{Gr}(G) \cap (\Lambda_n \times \mathbf{C})$. The open set $\bigcup_{z \in F_n} \Delta(z, q_n)$ will be denoted by Δ_n.

From (1) it follows that

$$G(\omega) \supset \bar{\Delta}_n \supset \Delta_n \supset K(\omega) \tag{2}$$

for all $\omega \in \Omega_n \supset \Lambda_n$. As a result of the first inclusion in (2) we have that $G^{-1}(z) \supset \Omega_n \supset \Lambda_n$ for all $z \in \bar{\Delta}_n$. Therefore, by the definition of a generalized random function $\varphi_n(\cdot, z)\colon \Lambda_n \to \mathbf{C}$ is measurable for all $z \in \bar{\Delta}_n$. Thus $\varphi_n\colon \Lambda_n \times \bar{\Delta}_n \to \mathbf{C}$ is a random function. Also, utilizing (d) and (2),

$$\varphi_n(\omega, \cdot) \in H(\bar{\Delta}_n) \text{ for all } \omega \in \Lambda_n. \tag{3}$$

The function $\hat{\varphi}_n\colon \Lambda_n \to C(\bar{\Delta}_n)$ given by $\hat{\varphi}_n(\omega) = \varphi_n(\omega, \cdot)$ is measurable.

Not let $A_n = \{\alpha_1, \alpha_2, \ldots, \alpha_{n_k}\}$ be a set consisting of one point from each component of $S^2 \setminus \bar{\Delta}_n$. Observe that since $\bar{\Delta}_n \supset K(\omega)$ for all $\omega \in \Lambda_n$, $A \subset S^2 \setminus K(\omega)$ for all $\omega \in \Omega_n$. By Lemma 1(b) the set $S(\bar{\Delta}_n)$ of all restrictions to $\bar{\Delta}_n$ of rational functions having poles only in A_n is a Borel subset of $C(\bar{\Delta}_n)$.

From (3) and Runge's Theorem it follows that the set $\{r \in S(\bar{\Delta}_n)\colon \|r - \hat{\varphi}_n(\omega)\|_{\bar{\Delta}_n} < \varepsilon(\omega)\}$ is non-empty for each $\omega \in \Omega_n$. Therefore, by Theorem 1, there exists a measurable function

$$r_n\colon \Omega_n \to S(\bar{\Delta}_n) \text{ with} \tag{4}$$

$$\|r_n(\omega) - \hat{\varphi}_n(\omega)\|_{\bar{\Delta}_n} < \varepsilon(\omega) \text{ a.e.} \tag{5}$$

Denoting by $\mathscr{R}_n$ the set of all rational functions whose only poles lie in the set A_n, we apply Lemma 2 to obtain a measurable map

$$R_n\colon \Omega_n \to \mathscr{R}_n \text{ such that} \tag{6}$$

$$R_n(\omega)|_{\bar{\Delta}_n} = r_n(\omega) \text{ a.e.} \tag{7}$$

Finally, set $R(\omega) = R_n(\omega)$ for $\omega \in \Omega_n$. Then, $R\colon \Omega \to \bigcup_{n=1}^{\infty} \mathscr{R}_n \subset C(\mathbf{C}, \mathbf{S}^2)$ is measurable. From (5) and (7) it follows that the set

$$\Omega' = \bigcup_n \{\omega \in \Lambda_n : \|\Omega_n(\omega) - \hat{\varphi}_n(\omega)\|_{\bar{\Delta}_n} < \varepsilon(\omega) \text{ and } R_n(\omega)|_{\bar{\Delta}_n} = r_n(\omega)\}$$

is a measurable subset of Ω with $\mu(\Omega\backslash\Omega') = 0$.

If $\omega \in \Omega'$, then for some n, $\omega \in \Lambda_n$, $\|r_n(\omega) - \hat{\varphi}_n(\omega)\|_{\bar{\Delta}_n} < \varepsilon(\omega)$, and $R_n(\omega)|_{\bar{\Delta}_n} = r_n(\omega)$. Since $\bar{\Delta}_u \supset K(\omega)$ for all $\omega \in \Lambda_n$, it follows, using the definitions of R and $\hat{\varphi}_n$, that $\|R(\omega) - \varphi(\omega, \cdot)\|_{K(\omega)} < \varepsilon(\omega)$. So if we set $R(\omega, z) = R(\omega)(z)$, then R is a random rational function satisfying the conclusion of the theorem.

Stochastic analogues (similar to Theorem 2) to Vitushkin's theorem on rational approximation ([**11**];, Mergelyan's theorem on polynomial approximation ([**8**]), Arakelyan's theorem on approximation on closed sets by entire functions ([**1**], see also [**5**]), and Arakelyan's theorem on approximation on closed sets in a general domain ([**2**], see also [**4**]) can be proved.

For example, a stochastic version of Mergelyan's theorem is as follows.

THEOREM 4. *Let* $(\Omega, \mathscr{A}, \mu)$ *be a* σ*-finite measure space. Suppose that* $K: \Omega \to \mathbf{C}$ *is a weakly-measurable compact-valued, multifunction such that*:

(a) *cardinality* $K(\Omega)$ *is less than or equal to* $\aleph_0$, *and*

(b) $\mathbf{C}\backslash K(\omega)$ *is connected* a.e.

Let ε *be a positive random variable defined on* Ω *and let* φ: $\mathrm{Gr}(K) \to \mathbf{C}$ *be a generalized random function satisfying*:

(c) $\varphi(\omega, \cdot) \in A(K(\omega))$ a.e. (*The algebra* $A(K)$ *consists of the function in* $C(K)$ *which are analytic on the interior of* K.)

Then there exists a random polynomial P *such that* $\|\varphi(\omega, \cdot) - P(\omega, \cdot)\|_{K(\omega)} < \varepsilon(\omega)$ a.e.

The proof of Theorem 4 is similar to that of Theorem 2 and will therefore be omitted.

An alternative proof is to first prove a stochastic version of Arakelyan's theorem and then apply Theorem 1 to choose polynomials which approximate the entire function in a measurable fashion.

It is not clear how to remove condition (a) from this theorem. The method in which the same condition was dispensed with in Theorem 3 will not work.

In Theorems 1, 2, 3, and 4 the hypothesis Ω is a σ-finite measure space can be replaced by the hypothesis $(\Omega, \mathscr{A})$ is a "complete measurable space"; in these cases the conclusions hold everywhere on Ω. These results follow from a theorem of Sainte-Beuve ([**10**]).

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