EMBEDDINGS EXTENDING VARIOUS TYPES OF DISJOINT SETS

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1. Introduction. It is well known that a set S is C^*-embedded in a set X if and only if two disjoint zero sets of S can be extended to disjoint zero of X ([7] and [4]). In the Wallman compactification [7, p. 270] two disjoint closed sets of S are extended to two disjoint closed sets of X. Here we study analogous extensions based on the sets being cozero, open, cozero and open, closed and zero and extend results on those of losed sets. The last two are related to C^*-embeddings and the first three to z-embeddings. In doing this, new characterizations of Oz-spaces, extremally disconnected spaces and modifications of these spaces will be obtained. Also, Tychonoff spaces will be characterized such that every subset (every open subset [every closed set] has a certain type of embedding property and Tychonoff spaces will be characterized such that every embedding into a Tychonoff space is of a certain type. Mappings involving these embeddings are also discussed.

In general the notation and terminology of Gillman and Jerison [7] will be used. Most of the background material on F-spaces, basically and extremally disconnected spaces, and C- and C^*-embedding will be found is this reference. Background material on z-embeddings will be found in [3] and [4]. The term normal will not necessarily include T_1.

2. Basic results.

DEFINITION 1. A space X is CC-embedded (CG-embedded) [GG-embedded] in a space Y if given two disjoint sets, both cozero (one open, one [both open] in X, they can be extended to disjoint sets, both cozero (one open, one cozero) [both open] in Y. A set B in X is extended to a set $E(B)$ in Y if $E(B) \cap X = B$.

Analogously, we define FF-embedding and FZ-embedding where F stands for a closed set and Z for a zero set.

THEOREM 1. *The following are satisfied.*

(a) *Every closed subset of a space Y is FF-embedded and every open subset of a dense set of Y is GG-embedded in the space. A set S is FF-*

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embedded (GG-embedded) in a space Y if and only if disjoint closed (open) sets in S are contained in disjoint closed (open) sets in Y.

(b) *CC-embedding is equivalent to z-embedding in a space Y.*

(c) *If $S \subset X \subset Y$ and S is FF-embedded in X and X is FF-embedded in Y then S is FF-embedded in Y. Analogous results hold for FZ-embedding, GG-embedding and CG-embedding.*

(d) *For a dense subset of a Tychonoff space Y, FF-embedding or FZ-embedding → C*-embedding. If Y is normal, for a subset, FF-embedding → FZ-embedding → C*-embedding. The converses are true in completely normal spaces. A space is normal if and only if every FF-embedded subset is C*-embedded.*

(e) *If M is an open subset of a dense set of a space Y, z-embedding of M in Y → CG-embedding of M in Y. For any set the converse is true. If M is closed, z-embedding → FZ-embedding and the converse is always true.*

(f) *A set that is FF-embedded and C-embedded in a Tychonoff space Y is also FZ-embedded.*

(g) *A set S is z-embedded in a space X if and only if (*) disjoint cozero sets of S are contained in disjoint cozero sets of X.*

PROOF. (a), (c) and (d) are immediate. (b) Clearly every CC-embedding is a z-embedding. Suppose C and W are disjoint cozero sets of a set X which is z-embedded in Y. There exists continuous functions f and g on Y such that $f(x) = 0$ if $x \in X \sim C$, $f(x) > 0$ if $x \in C$ and $g(x) = 0$ if $x \in X \sim W$, $g(x) < 0$ for $x \in W$. Set $E(C) = \{y \in Y, f(y) + g(y) > 0\}$ and $E(W) = \{y \in Y, f(y) + g(y) < 0\}$. Then $E(C)$ and $E(W)$ are disjoint cozero extensions of C and W respectively. (e) is immediate or follows from (b). (f) Let S be FF-embedded and C-embedded in Y. By the FF-embedding, $\bar{F}$ and $\bar{Z}$ in $\bar{S}$ are disjoint and $\bar{Z}$ is a zero set in $\bar{S}$ by the C-embedding. Since $\bar{S}$ is z-embedded in Y, $\bar{Z}$ has a zero set extension in Y, $E(Z)$ such that $E(Z) \cap \bar{F} = \emptyset$. (g) From part (b), z-embedding → CC-embedding → (*). Let Z be a zero set of S; then $S \sim Z = \bigcup Z_n$ where each Z_n is a zero set. Since Z and Z_n are contained in disjoint cozero sets of S they are contained in disjoint cozero sets of X by (*). Then there exists a zero set of X, H_n such that $Z \subset H_n \subset \sim Z_n$. Then $H = \bigcap H_n$ is a zero set of X and $H \cap S = Z$.

We note that we that we can characterize the other types of embedding in an analogous manner to that of (g)

REMARK 1. We do not know if we can replace C-embedding by C^*-embedding in (f). Nor, do we know if a set that is both z-embedded and GG-embedded in a space is necessarily CG-embedded in that space.

We conclude this section with a mapping theorem.

THEOREM 2. *Let* S *be* *GG-embedded* (*CG-embedded*) [*FF-embedded*] {*FZ-embedded*} *in* X *and let* f *be a closed* (*closed and zero set preserving*) [*open*] {*open and cozero set preserving*} *map such that* $S = f^{-1}(f(S))$. *Then* $f(S)$ *is GG-embedded* (*CG-embedded*) [*FF-embedded*] {*FZ-embedded*} *in* $f(X)$.

PROOF. Suppose G, $H \subset f(S)$, $G \cap H = \varnothing$, G and H open in $f(S)$. Since S is GG-embedded in X, $f^{-1}(G)$ and $f^{-1}(H)$ have disjoint open extensions U and V respectively in X. Then $f(X) \sim f(\sim U)$ and $f(X) \sim f(\sim V)$ are disjoint open extensions of G and H respectively. The other cases are similar.

We note in the FZ-embedding case the open mapping would be redundant if X were Tychonoff and that C^*-embeddings (z-embeddings) are preserved under cozero (zero) set preserving maps. These last two results may be known. As a result weak perfect normality and Oz properties are preserved under zero set preserving maps and normality and perfect normality are preserved under cozero set preserving maps.

3. GG- and CG-embeddings. Blair [3] has studied Tychonoff spaces in which every subset is z-embedded and designated such spaces as weakly perfectly normal, a condition strictly between perfectly normal and completely normal. He designated Tychonoff spaces such that every open subset is z-embedded as Oz-spaces and along with Hager [4] proved that every closed subspace of a Tychonoff space is z-embedded if and only if the space is normal.

THEOREM 3. *The following are equivalent for a space* X
(a) X *is completely normal.*
(b) *Every subset of* X *us GG-embedded.*
(c) *Every closed subset of* X *is GG-embedded.*

PROOF. (a) → (b). Two disjoint open subsets of a subset S of X are separated in X and hence are contained in disjoint open subsets of X and hence by Theorem 1(a), S is GG-embedded. (b) → (a). If A and B are separated in X they are disjoint open subsets of $A \cup B$ and by the GG-embedding property are contained in disjoint open sets of X. (b) → (c) is immediate. (c) → (b). Suppose $S \subset X$. By Theorem 1(a) S is GG-embedded in $\bar{S}$ and hence GG-embedded in X by Theorem 1(c).

COROLLARY 3. *Complete normality is preserved under closed maps* [6].

DEFINITION 2. (Zenor [10]) A space X is a Z-space if given two disjoint subsets of X, one closed and the other a zero set, then they are completely separated in X.

Mack [8] called a Z-space, a δ-normally separated space.

DEFINITION 2A. A subset $S \subset X$ is relative Z if given two disjoint subsets of S, one closed and the other a zero set extendable to X, then they are completely separated in X.

Any space that is normal, countably compact [10] or a P-space is a Z-space.

LEMMA 4. *The following are equivalent for a space X (a set $X \subset Y$).*

(a) *X is a Z-space (X is a relative Z-space with respect to Y).*

(b) *For F closed, Z a zero set (Z an extendable zero set to Y), $F \cap Z = \emptyset$, there exists disjoint cozero sets (of Y) containing F and Z.*

(c) *For F closed, Z a zero set (Z an extendable zero set to Y), $F \cap Z = \emptyset$ there exists an open set G (of Y), a cozero set C (of Y), $G \cap C = \emptyset$, $F \subset G$, $Z \subset C$.*

(d) *Same as (c) except $F \subset C$, $Z \subset G$.*

PROOF. (a) $\rightarrow$ (b) $\rightarrow$ (c) $\rightarrow$ (d) are immediate. (d) $\rightarrow$ (a). There exists a cozero set C (extendable to Y) such that $F \subset C \subset \bar{C} \subset \sim Z$. We then continue the proof, analogously to that of Urysohn's Lemma to establish the existence of a continuous function g to the reals such that $g(F) = 0$ and $g(Z) = 1$.

THEOREM 4. *Every closed subset of a Tychonoff space X is CG-embedded if and only if X is normal and hereditarily relatively Z with respect to X (i.e., every subset is relatively Z with respect to X).*

PROOF. Suppose $C, G \subset B$ closed in X, C cozero, G open in B, $C \cap G = \emptyset$. Set $Q = (\sim B) \cup (C \cup G)$. Then C is closed in Q and G is contained in a zero disjoint from C. If Q is relatively Z with respect to X, then C and G are contained in disjoint cozero sets of X, C' and G' respectively. Since B is z-embedded in X there exists a cozero set $E(C)$ of X such that $E(C) \cap B = C$. Then $E(C) \cap C'$ is a cozero of X disjoint from the open set G' of X. Suppose $F, Z \subset H$, F closed in H, Z a zero set of H extendable to X, $F \cap Z = \emptyset$. Let $A = E(Z) \cup \bar{F}$ where $E(Z)$ is a zero set extension of Z and $\bar{F}$ is the closure of F in X. Then $A \sim E(Z)$ and $A \sim \bar{F}$ are cozero sets and open sets respectively with disjoint extensions to X, $E(A \sim E(Z))$ and $E(A \sim \bar{F})$ which are cozero and open respectively; furthermore $F \subset E(A \sim E(Z))$ and $Z \subset E(A \sim \bar{F})$. By Lemma 4, H satisfies the relative Z property.

EXAMPLE 1. Let D and D' be discrete sets such that $c \leq |D| < |D'|$. Let $L(L')$ be a one point extension of $D(D')$ such that open sets consist of subsets of $D(D')$ and complements of countable subsets of $D(D')$. The space $L \times L'$ is a normal P-space and hence is hereditarily Z but $L \times L'$ has a non-normal subspace. Thus every closed subset is CG-embedded but this space has a closed subset that is not GG-embedded [1].

THEOREM 5. *Every (open) subspace of a Tychonoff space X is CG-embedded if and only if X is weakly perfectly normal (Oz).*

PROOF. The result on open subspaces follows from Theorem 1(e). If every subspace of X is CG-embedded, then every subspace is z-embedded so X is weakly perfectly normal. Suppose X is weakly perfectly normal. Then every set is z-embedded in its closure and CG-embedded in its closure by Theorem 1(c). By Theorem 4, using Blair's results that weakly perfectly normal spaces are completely normal and hence satisfy the hereditary relative Z property closed sets are CG-embedded so by the transitivity property (Theorem 1(c)) every set is CG-embedded.

COROLLARY 5. $a \leftrightarrow b \rightarrow c \leftrightarrow f \rightarrow e \leftrightarrow d$ *where* a(b) [c] *is the condition that every subset of a set X is z- (CG-) [GG-] embedded in X and* d(e) [f] *is the condition that every closed subset of X is z- (CG-) [GG-] embedded in X.*

DEFINITION 3. A space is weak Oz if the closure of any cozero set is a zero set.

LEMMA 6. *A space is weak Oz if and only if given two disjoint sets, one open and one cozero, they are contained in disjoint cozero sets. A space is Oz if and only if given two disjoint open sets, they are contained in disjoint cozero sets.*

The latter result is due to Terada [9] and is easily obtained from Blair's [3] characterization of Oz spaces as spaces where the regular closed sets are zero sets.

We note that basically disconnected spaces are weak Oz, weak Oz F-spaces are basically disconnected and that Oz F-spaces are extremally disconnected and extremally disconnected spaces are Oz. The last two results are due to Blair and Hager [4].

THEOREM 6. *A z-embedded, Oz-space (weak Oz-space) is GG-embedded (CG-embedded). A GG-embedded (CG-embedded) subset of an Oz-space (weak Oz-space) is Oz (weak Oz). A subspace of an Oz-space is z-embedded if it is GG-embedded.*

PROOF. We prove only the GG-embedding case since the CG-embedding case is similar. Let S be a z-embedding Oz-subspace of a space X and let G and H be open in S, $G \cap H = \varnothing$. By Lemma 6, G and H are contained in disjoint cozero sets of S which can be extended to disjoint cozero sets $C(G)$ and $C(H)$ of X by Theorem 1 with $G \subset C(G)$ and $H \subset C(H)$. Let G and H be disjoint open sets of $S \subset X$, where S is GG-embedded and X is Oz. Then the disjoint open extensions of G and H are contained in disjoint cozero sets $C(G)$ and $C(H)$ respectively. Then $G \subset S \cap C(G)$ and $H \subset S \cap C(H)$ which are both cozero sets of S. So S is Oz. We have

also from the second to the last sentence that disjoint cozero sets of S are contained in disjoint cozero sets of X so that S is z-embedded in X by theorem 1(g).

COROLLARY 6. *A Tychonoff space is Oz if and only if every GG-embedded set is z-embedded.*

We can replace z-embedding by CG-embedding in the above statement.

THEOREM 7. (a) *Let Tychonoff X be extremally disconnected (basically disconnected) [an F-space] and let $S \subset X$ such that S is GG-embedded [CG-embedded) [z-embedded] in X. Then S is extremally disconnected (basically disconnected) [an F-space].*

(b) *A Tychonoff space is extremally disconnected (an F-space) [an F-space] if and only if every GG-embedded (CG-embedded) [z-embedded] subset is C^*-embedded.*

PROOF. In any case the pairs of disjoint subsets of S will be extended to pairs of disjoint subsets of X that are contained in disjoint zero sets of X and thus the original sets are then contained in disjoint zero sets of X. Blair and Hager [4] have shown that a space is an F-space if and only if every z-embedded set is C^*-embedded. The result of CG-embedding would follow from the result of Blair and Hager [4] that cozero sets are z-embedded in any space and thus C^*-embedded by Theorem 1 and characterization of F-spaces in [7] as spaces where cozero sets are C^*-embedded. If X is extremally disconnected every GG-embedded set will be z-embedded by Theorem 6 and hence C^*-embedded, since X is an F-space. If X is such that every GG-embedded set is C^*-embedded then every dense set is C^*-embedded since by Theorem 1 every dense set is GG-embedded. So X is extremally disconnected [7].

4. Some examples. Example 1 established that a space may be CG-embedded but not GG-embedded.

EXAMPLE 2. Since $\beta N \sim N$ is not even basically disconnected [7], then $\beta N \sim N$ is neither CG-embedded or GG-embedded in βN even though βN is extremally disconnected and thus Oz and $\beta N \sim N$ is C-embedded in βN. Thus a z-embedded subset of an Oz space is not necessarily GG-embedded.

We may also show that a GG-embedded Oz-space is not necessarily z-embedded thus showing, using Theorem 6 that a z-embedded Oz-subspace is GG-embedded and GG-embedded subset of an Oz-space is z-embedded but neither converse is true.

EXAMPLE 3. We may use the construction of Problem $9K$ in [7] to obtain a GG-embedded closed set that is not z-embedded. If X is not Lindelöf or

almost compact, set $KX = \beta Y \sim Y$ and let $Z = X \cup Y$ where KX is a compactification in which X is not z-embedded. Then X is not z-embedded in Z but is GG-embedded. We may choose X to be Oz in particular a discrete space of uncountable cardinal.

EXAMPLE 4. The space Ω of Problem 8L of [7], where $\Omega = [\beta W \times \beta W) \sim (w)$ where W is the set of uncountable ordinals and w is the corner point. The union of the top edge and right edge is GG-embedded in Ω but not z-embedded. The union of an edge with the diagonal is not even GG-embedded in Ω.

5. Absolute embedding.

DEFINITION 4. A Tychonoff space X is absolutely GG-embedded (CG-embedded) [z-embedded] if it is GG-embedded (CG-embedded) [z-embedded] in any Tychonoff space in which it is embedded.

Blair and Hager [4] have shown that a Tychonoff space X is absolutely z-embedded if and only if X is Lindelöf or almost compact.

THEOREM 8. *A space X is absolutely GG-embedded (CG-embedded) if and if it is either Lindelöf or almost compact, and Oz (and weak Oz).*

PROOF. The sufficiency of the conditions follow from Theorem 6. Suppose a space is not Oz (not weak Oz), then X may be embedded in an Oz space Y. (This was pointed out to the author by E. Van Douwen as a consequence of Blair's [3] result that the product of separable metric space and in particular the product of closed intervals of the reals is Oz and every Tychonoff space can be embedded in a space of this type.) So by Theorem 6, X is not GG-embedded (CG-embedded) in Y. Since a CG-embedded space is z-embedded the result for CG-embeddings have been established. If X is a space that is not almost compact or Lindelöf and KX is a compactification in which X is not z-embedded we embed KX in an Oz-space W. Since X is not z-embedded in W, X can not be GG-embedded in W by Theorem 6.

Analogously we define absolute FF-embeddings and absolute FZ-embeddings of Tychonoff spaces. We note Hewitt's [7] result that a Tychonoff space X is absolutely C^*-embedded if and only if X is almost compact.

THEOREM 9. *A Tychonoff space X is absolutely FF-embedded (FZ-embedded) if and only if X is almost compact and normal (and countably compact).*

The above Theorem follows from Theorem 1(d), Hewitt's result and the following Lemma and Zenor's [10] result that a pseudocompact space is a Z-space if and only if it is countably compact. We note Lemma 9 is

dual to Theorem 7 with the roles of open (cozero) and closed (zero) sets interchanged.

LEMMA 9. *Let X be normal (a Z-space) and let $S \subset X$ such that S is FF-embedded (FZ-embedded) in X. Then S is normal (satisfies Z).*

From Lemma 9 it is clear that the Tychonoff plank [7] is neither *FF*-embedded nor *FZ*-embedded in its compactification and the space of Example 4 is *FZ*-embedded but not *FF*-embedded in its compactification.

The result in Lemma 9 on *FF*-embeddings is closely related to the result that a Wallman compacitification is T_2 if and only if the space is normal. This latter result can be also used in part of the proof of Lemma 11.

6. FF-and FZ-Embeddings.

THEOREM 10. *The following are equivalent for a space X.*
(a) *Every subspace of X is extremally disconnected.*
(b) *Every subspace is FF-embedded.*
(c) *Every dense subspace is FF-embedded.*
(d) *Every open subspace is FF-embedded.*
(e) *Every dense open subspace is FF-embedded.*

PROOF. (a) → (b). Let S be a subspace of X. It suffices to prove that S is *FF*-embedded is $\bar{S}$. Suppose $A, B \subset S$, A and B closed in S, $A \cap B = \emptyset$. Let $\bar{A}$ and $\bar{B}$ be the closures of A and B respectively in $\bar{S}$. Since $\bar{A} = \bar{A} \sim \bar{B}$ in $\bar{A} \cup \bar{B}$, $\bar{A}$ is open in $\bar{A} \cup \bar{B}$ so that $\bar{A} \cap \bar{B} = \emptyset$ in S. (b) → (c) → (e) and (b) → (d) → (e) are immediate. (b) → (a). Let G and H be disjoint open subsets of a subspace S. Since G and H have disjoint closures in $E(G) \cup E(H)$ where $E(G)$ and $E(H)$ are open extensions in X of G and H respectively, they have disjoint closures in X and hence in S so S is extremally disconnected. (c) → (b). Let A and B be closed in S with $A \cap B = \emptyset$; then A and B are closed in the dense set $S \cup (\sim \bar{S})$ and hence have disjoint closed extensions in X. (e) → (d). Let S be an open set then $S \cup (\sim \bar{S})$ is a dense open set and we continue with the argument in (c) → (b). (d) → (b). Let $S \subset X$ and suppose A and B are disjoint closed sets of S. Let $\bar{A}$ and $\bar{B}$ be the closures of A and B in X. The sets $\bar{A} \sim (\bar{A} \cap \bar{B})$ and $\bar{B} \sim (\bar{A} \cap \bar{B})$ are closed and disjoint in $\sim (\bar{A} \cap \bar{B})$ and thus $\bar{A} \cap \bar{B} = \emptyset$. So S is *FF*-embedded in X.

We note there is a duality relationship between Theorems 3(a), (c) and 10(a), (d) obtained by interchanging roles of open and closed sets.

COROLLARY 10A. *Every subset of a space X is FF-embedded and GG-embedded if and only if X is normal and hereditarily extremally disconnceted.*

COROLLARY 10B. *Hereditarily extremally disconnectedness is preserved by open mappings.*

THEOREM 11. *Every subspace of a Tychonoff space X is FZ-embedded if and only if X is normal and hereditarily extremally disconnected.*

PROOF. If every subspace is FZ-embedded then every closed subset is z-embedded and hence every closed subset is C^*-embedded and every subset is C^*-embedded in its closure by Theorem 1 so every subset is C^*-embedded and the space is normal and hereditarily extremally disconnected. Conversely a normal and hereditarily extremally disconnected space is completely normal [6]; so by Lemma 11 below, every subset is FZ-embedded, since every subset is C^*-embedded.

LEMMA 11. *A Tychonoff space X is FF-embedded (FZ-embedded) in every Tychonoff space in which it is C*-embedded if and only if X is normal (satisfies Z).*

DEFINITION 5. Let $S \subseteq X$. We say that S is relatively basically disconnected with respect to a space X if for G open in S, C, a cozero set in S which is extendable to X, $G \cap C = \emptyset$, there is a zero of Z of X and a closed set F of X such that $G \subset Z$, $C \subset F$, $F \cap Z = \emptyset$.

We note that a closed set S of a normal space X is relatively basically disconnected if and only if S is basically disconnected.

THEOREM 12. *The following are equivalent for a Tychonoff space X.*
(a) *Every dense set is FZ-embedded.*
(b) *Every open set is FZ-embedded.*
(c) *X is extremally disconnected and every set is relatively basically disconnected.*
(d) *X is extremally disconnected and every closed set is relatively basically disconnected.*
(e) *X is extremally disconnected and every C*-embedded set is FZ-embedded.*

PROOF. (a) → (b). Every dense set is C^*-embedded by Theorem 1 so that X is extremally disconnected and every open set is C^*-embedded [5]. So if $F, Z \subset G$, $F \cap Z = \emptyset$ F closed, G open and Z a zero set $F \cap E(Z) = \emptyset$ where $E(Z)$ is a zero set extension of Z in $G \cup (\sim \bar{G})$ and F is also closed in $G \cup (\sim \bar{G})$. The FZ-embedding of dense $G \cup (\sim \bar{G})$ completes the proof. (b) → (c). Let C be an extendable cozero set and G an open set of a set M, $C \cap G = \emptyset$. Let $E(C)$ and $E(G)$ be open and cozero extensions of C and G respectively to X. Set $Q = E(C) \cup E(G)$. Let $E(Q \sim E(C))$ be a zero set extension of $Q \sim E(C)$ to X and let $E(Q \sim E(G))$ be the closure of $Q \sim E(G)$ in X. These can be considered as disjoint since Q is FZ-embedded in X. Since $C \subset E(Q \sim E(G))$ and $G \subset E(Q \sim E(C))$, M is relatively basically disconnected. (c) → (d) is immediate. (d) → (e). Let $F, Z \subset H$, $F \cap Z = \emptyset$, F closed, Z a zero set,

and let H be C^*-embedded in X. Let $E(F)$ be the closure of F in X and $E(Z)$ a zero set extension of Z to X guaranteed by the C^*-embeddings of H. Set $B = E(F) \cup E(Z)$. The disjoint sets $B \sim E(F)$ and $B \sim E(Z)$ are contained in a zero set Z' and a closed set F' respectively of X, with $Z' \cap F' = \varnothing$. Then $Z \subset Z' \cap E(Z)$, $F \subset F' \cap E(F)$ so that H is FZ-embedded in X. (e) $\rightarrow$ (a) is immediate.

LEMMA 13. *Every C^*-embedded Z-subspace of a space X is FZ-embedded. A space Y is a Z-space if every FZ-embedded subset is C^*-embedded.*

PROOF. If F, $Z \subset S \subset X$, F closed in S, Z a zero set in S, $F \cap Z = \varnothing$, they are completely separated in S and hence in X so that $\bar{F}$ in X is disjoint from a zero set of X containing Z. Then there exists an extension of Z to a zero set in X disjoint from $\bar{F}$. Suppose Y is a Z-space and Z_1 and Z_2 are disjoint zero sets of a subspace S. Then $\bar{Z}_1$ is disjoint from $E(Z_2)$ a zero set extension of Z_2 to Y. By the Z property $E(Z_2)$ and $\bar{Z}_1$ are completely separated in X and so are Z_2 and Z_1.

THEOREM 13. *For a T_4 space X the following are equivalent.*
(a) *Every dense set is FZ-embedded.*
(b) *Every open set is FZ-embedded.*
(c) *X is extremally disconnected and every closed set is basically disconnected.*
(d) *X is extremally disconnected and every closed set is CG-embedded.*
(e) *X is extremally disconnected and hereditarily relative Z, with respect to X.*
(f) *X is extremally disconnected and every dense subset satisfies Z.*
(g) *X is extremally disconnected and every open subset satisfies Z.*

PROOF. From Theorem 12 (a) $\leftrightarrow$ (b) $\leftrightarrow$ (c), we note that condition (d) Theorem 12 is equivalent to (c) Theorem 13 for normal spaces. (c) $\leftrightarrow$ (d) follows from Theorem 6 and 7 and the fact that closed sets are z-embedded [3]. (d) $\leftrightarrow$ (e) follows from Theorem 4. (e) $\rightarrow$ (f), (e) $\rightarrow$ (g) are immediate. (f) $\rightarrow$ (a), (g) $\rightarrow$ (b). The dense and open sets are C^*-embedded and a C^*-embedded Z-space is FZ-embedded by Lemma 13.

EXAMPLE 5. (E. Van Douwen) Balčar and Simon, Kunen and Shelah, have shown that a one point Lindelöf extension of discrete space of cardinal 2^K as in Example 1 embeds in βK. More recently E. Van Douwen [5] has shown that every P-space of weight 2^K embeds in βK. So that the space $P \cup K$ where P is a P-space satisfies the conditions of Theorem 13 without being hereditarily extremally disconnected unless P is extremally disconnected.

THEOREM 13A. *For a normal Oz-space X, the following are equivalent*
(a) *Every dense set satisfies Z.*

(b) *Every open set satisfies Z.*

(c) *Every closed set is CG-embedded.*

(d) *Every closed set is weak Oz.*

7. Some characterization theorems. In Corollary 6 *Oz*-spaces are characterized as spaces where every *GG*-embedded subset is *z*-embedded. Theorem 7 gives other results of this type. We summarize these results with others without proof.

THEOREM 14. *Every A-embedded subset of a Tychonoff space X is B-embedded if and only if X satisfies R where A, B and R are one of the following combinations.*

A	B	R	Proof indication or source of result
(a) *FF*	*C**	*Normal*	Theorem 1 (d)
(b) *FF*	*FZ*	*Normal*	Theorem 1 (d) and (e)
(c) *FF*	*GG*	*Completely Normal*	Theorem 3
(d) *FF*	*CG*	*Completely Normal & Hereditarily Relatively Z with respect to X.*	Theorem 4
(e) *FF*	*z*	*Normal*	Theorem 1 and [4]
(f) *z*	*C**	*F-space*	[4]
(g) *z*	*C*	*P-space*	[4]
(h) *CG*	*C**	*F-space*	Theorem 1 and (g)
(i) *CG*	*C*	*P-space*	Theorems 1 and 14
(j) *GG*	*z*	*Oz-space*	Corollary 6
(k) *GG*	*CG*	*Oz-space*	Like Corollary 6
(l) *GG*	*C**	*extremally disconnected*	Theorem 7
(m) *GG*	*C*	*extremally disconnected P-space*	[5] and [1]
(n) *GG*	*FF*	*hereditary extremally disconnected*	Theorem 10
(o) *GG*	*FZ*	*extremally disconnected & every closed set is relatively basically disconnected*	Theorem 12

In regard to (m) if there exist measurable cardinals, there exist extremally disconnected *P*-spaces that are not discrete [7].

In [2] it was shown that a Tychonoff space is pseudocompact if and only if every *z*-embedding of *X* in a Tychonoff space is a *C**-embedding. We can obtain similar type theorems involving the embeddings studied here.

THEOREM 15. *A Tychonoff space X is Oz (weak Oz) if and only if X is GG(CG) embedded in every Tychonoff space in which it is z-embedded or C*-embedded. A Tychonoff space X is FZ-embedded (FF-embedded) in every*

*Tychonoff space in which it is CG- or z-embedded if and only if X is count-
ably compact (and normal).*

PROOF. The proof of the first statement can be obtained by modifying
the proof of Theorem 8. If X is countably compact (and normal) and
z-embedded in a Tychonoff space Y then X is C^*-embedded in Y by a
result of Blair and Hager [4] and by Lemma 11 is FZ-embedded (FF-
embedded). If every z-embedding of X in a Tychonoff space is a FZ-
embedding (FF-embedding) then every z-embedded compactification
is βX so X is pseudocompact [2]. So by Lemma 9, X satisfies Z (is normal)
and by a result of Zenor [8] [(see paragraph following Theorem 9)] X is
countably compact (and normal).

Finally we note that for FF-embedding any family of disjoint closed
sets in a subspace S is extended to a family of disjoint closed sets in a
space X. For C^*-embedding (GG-embeddings) [z-embeddings] a denumer-
able family of disjoint zero sets (open sets) [cozero sets] may be extended.

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