

AN INVERSE PROBLEM FOR A PARABOLIC PARTIAL DIFFERENTIAL EQUATION

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Introduction. In this paper we shall study the problem of finding the coefficient $a(x)$ as well as the temperature $u(x, t)$ in the initial value problem

$$(0.1) \quad u_t(x, t) - u_{xx}(x, t) + a(x)u(x, t) = 0, \quad 0 < x < 1, \quad 0 < t < T,$$

$$(0.2) \quad u(0, t) = u(1, t) = 0, \quad 0 \leq t \leq T,$$

$$(0.3) \quad u(x, 0) = f(x), \quad 0 \leq x \leq 1,$$

$$(0.4) \quad u_x(0, t) = g(t), \quad 0 \leq t \leq T.$$

If the coefficient $a(x)$ were known, then (0.1)–(0.3) would constitute a well-posed problem for $u(x, t)$, but the indeterminate nature of the differential equation demands that we impose some additional boundary conditions and we have chosen to prescribe the flux, condition (0.4), at one end of the region.

Our methods will lead us to the classical inverse Sturm-Liouville problem, namely that of determining the potential $a(x)$ in the operator

$$(0.5) \quad Ly = -y''(x) + a(x)y(x), \quad 0 \leq x \leq 1$$

where y satisfies the boundary condition

$$(0.6) \quad y(0) = y(1) = 0.$$

Typically in this problem one is given the spectrum $\{\lambda_n\}_{n=1}^{\infty}$ of L (which in itself is insufficient to determine $a(x)$) plus some additional “Tauberian condition”. This problem has received considerable attention and the Tauberian condition has taken a variety of forms. For example in [1], [5] a second complete spectrum $\{\mu_n\}_{n=1}^{\infty}$, arising from alternative self-adjoint boundary conditions, linearly independent to (0.6), was given. In [4], [5] it was assumed a priori that $a(x)$ is symmetric, that is $a(x) = a(1-x)$. In [3], the spectral function $\rho(\lambda)$ was specified, that is, that monotonic function with jump discontinuities at the points $\lambda = \lambda_1, \lambda_2, \dots, \lambda_n, \dots$ with the value of the jumps equal to $[\int_0^1 \phi_n^2(x) dx]^{-1}$ where $\phi_n(x)$

is some suitably chosen eigenfunction for L . All of the above conditions are sufficient to reconstruct the operator L .

Our equations (0.1) and (0.2) will yield the Sturm-Liouville operator with its boundary conditions. Condition (0.4) will provide the spectrum $\{\lambda_n\}_{n=1}^{\infty}$ and (0.3) will be utilized to give a Tauberian condition rather different from any of the above.

In this paper we shall give conditions on the functions $f(x)$ and $g(t)$ that will guarantee the existence of a unique solution to the undetermined coefficient problem. Our approach will follow that of Gelfand and Levitan, in obtaining a representation for the eigenfunctions of the Sturm-Liouville operator (0.5) with the boundary conditions (0.6). We will then use the representation to obtain a nonlinear integral equation for the coefficient $a(x)$.

We note that uniqueness for the coefficient $a(x)$ can be obtained from the Gelfand-Levitan approach quite directly, cf. [6]. Our contribution is to show the existence and the continuous dependence of the function $a(x)$ on the data.

Our work also has a similarity with that of Suzuki and Murayama [7] and Suzuki [8]. In these papers the authors use the methods of Gelfand and Levitan to obtain uniqueness and nonuniqueness results for the problem of determining both the coefficient $a(x)$ and the initial function $f(x)$ (as well as $u(x, t)$) from a specification of Cauchy data on each of the boundaries $x = 0$ and $x = 1$.

Preliminaries. By $C^k[a, b]$ we mean the class of functions whose derivatives of order up to and including k are continuous on $[a, b]$. The supremum norm we shall denote by $\| \cdot \|_{\infty}$, that is $\sup_{x \in [0, 1]} |f(x)| = \|f\|_{\infty}$.

The pair of functions $(a(x), u(x, t))$ is said to be a solution of the initial value problem (0.1)–(0.4) provided that

- (i) $a(x)$ is continuous on $[0, 1]$,
- (ii) $u(x, t)$ is continuous for $0 \leq x \leq 1, 0 \leq t \leq T$,
- (iii) $\lim_{x \rightarrow 0^+} u_x(x, t)$ exists for $0 \leq t \leq T$,
- (iv) u_t, u_{xx} are continuous for $0 < x < 1, 0 < t < T$, and
- (v) (0.1)–(0.4) are satisfied.

Without loss of generality we may assume that $a(x) \geq 0$ on $[0, 1]$ since this can always be achieved by the change of dependent variable $v(x, t) = u(x, t)e^{-\alpha t}$ in (0.1)–(0.4) provided α is sufficiently large.

Throughout this paper we shall make the following assumptions on the data.

(A1) The function $f(x) \in C^2[0, 1]$ is positive for $0 < x < 1$ and satisfies $f(0) = f(1) = 0$. Also $f'(0) = c_0 \neq 0, f'(1) = c_1 \neq 0$ and $f''(x)/f(x)$ is bounded for $0 \leq x \leq 1$.

(A2) The function $g(t) \in C^1[0, T]$ and is analytic on the half plane

$\operatorname{Re}\{t\} > 0$, having Dirichlet series $g(t) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n t)$ where $\{\lambda_n\}_{n=1}^{\infty}$ is a positive increasing sequence of real numbers satisfying the asymptotic formula

$$(1.1) \quad \sqrt{\lambda_n} n = n\pi + o(1/n).$$

The compatibility condition $g(0) = f'(0) = c_0$ is also required.

REMARK 1. The assumption $\lambda_n > 0$ can be made without any loss of generality since this can be achieved by the change of dependent variable mentioned previously.

REMARK 2. Since $g'(0)$ exists, the series $\sum_{n=1}^{\infty} a_n \lambda_n$ is convergent.

REMARK 3. Many of the above conditions can probably be relaxed, in particular those in (A1). However for continuous, positive $a(x)$ the solution $u(x, t)$ to (0.1)–(0.3) must for each x be analytic in t and with an asymptotic behavior precisely that obtained from the conditions imposed on $g(t)$ by λ_n and a_n . Thus assumption (A2) is essentially sharp.

For later reference we define the function $h(x)$ by

$$(1.2) \quad h(x) = \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{\lambda_n}} \sin \sqrt{\lambda_n} x.$$

It follows from Remark 2 that $h \in C^2[0, \infty)$ and in fact

$$(1.3) \quad \frac{h'(x)}{x} = c_0 + o(x) \quad \text{at } x = 0.$$

We can relate the function $g(t)$ and $h(x)$ by the following approach. If $v(x, t)$ is defined by

$$(1.4) \quad v(x, t) = \sum_{n=1}^{\infty} \frac{a_n}{\sqrt{\lambda_n}} e^{-\lambda_n t} \sin \sqrt{\lambda_n} x$$

then it is immediate that

$$(1.5) \quad v_t - v_{xx} = 0, \quad x > 0, \quad t > 0,$$

$$(1.6) \quad v(0, t) = 0, \quad v_x(0, t) = g(t),$$

$$(1.7) \quad v(x, t) \text{ bounded for } x \geq 0, \quad t \geq 0, \text{ and}$$

$$(1.8) \quad v(x, 0) = h(x).$$

Equations (1.5)–(1.7) represent a Cauchy problem for the heat equation and given the assumptions on $g(t)$ it possesses a unique solution. The value of this solution at $t = 0$ is the function $h(x)$. By elementary transform techniques we can thus relate $h(x)$ and $g(t)$ by

$$(1.9) \quad \frac{1}{\sqrt{4\pi t^3}} \int_0^\infty x e^{-x^2/4t} h(x) dx = g(t).$$

Existence and uniqueness of a solution. We are now in a position to state and prove our main theorem.

THEOREM: *Let (A1) and (A2) hold. Then the undetermined coefficient problem (0.1)–(0.4) possesses a unique solution.*

PROOF. Corresponding to the sequence $\{\lambda_n\}_{n=1}^\infty$ generated by the function $g(t)$ we define a set \mathcal{A} as follows: $\mathcal{A} = \{a(x) \in C[0, 1]: -\phi_n''(x) + a(x)\phi_n(x) = \lambda_n\phi_n(x), \phi_n(0) = \phi_n(1) = 0 \text{ for some set of eigenfunctions } \{\phi_n(x)\}_{n=1}^\infty\}$. Thus \mathcal{A} consists of those continuous potentials appearing in Sturm-Liouville operators of the form (0.5) with Dirichlet boundary conditions that have the set $\{\lambda_n\}_{n=1}^\infty$ as spectrum.

We shall assume that the eigenfunctions possess a common slope at $x = 0$ choosing

$$(2.1) \quad \phi_n'(0) = 1.$$

For any $a(x) \in \mathcal{A}$ and a corresponding eigenfunction basis $\{\phi_n(x)\}_{n=1}^\infty$ the function

$$(2.2) \quad u(x, t) = \sum_{n=1}^\infty a_n e^{-\lambda_n t} \phi_n(x)$$

clearly satisfies the partial differential equation (0.1) together with the boundary conditions (0.2), and in view of (2.1), the condition (0.4) also holds. We have to prove that for any $f(x)$ satisfying conditions (A1) there exists a unique $a(x) \in \mathcal{A}$ and an associated basis $\{\phi_n(x)\}_{n=1}^\infty$ such that f can be written as

$$(2.3) \quad f(x) = \sum_{n=1}^\infty a_n \phi_n(x).$$

Our approach will be to obtain an expression that relates the function $a(x)$ to its associated eigenfunctions, and then use this to obtain a nonlinear Volterra integral equation for $a(x)$.

Following Gelfand and Levitan we consider the relation

$$(2.4) \quad \phi_n(x) = \frac{1}{\sqrt{\lambda_n}} \left\{ \sin \sqrt{\lambda_n} x + \int_0^x K(x, t) \sin \sqrt{\lambda_n} t dt \right\}$$

and determine conditions on $K(x, t)$ for $0 \leq t \leq x \leq 1$ in order that $\phi_n(x)$ be a set of eigenfunctions corresponding to an $a \in \mathcal{A}$.

If we differentiate (2.4) twice with respect to x , we obtain

$$(2.5) \quad \phi_n''(x) = \frac{1}{\sqrt{\lambda_n}} \left\{ -\lambda_n \sin \sqrt{\lambda_n} x + \sin \sqrt{\lambda_n} x \frac{d}{dx} K(x, x) \right. \\ \left. + \sqrt{\lambda_n} \cos \sqrt{\lambda_n} x K(x, x) + \sin \sqrt{\lambda_n} x K_x(x, x) \right. \\ \left. + \int_0^x K_{xx}(x, t) \sin \sqrt{\lambda_n} t dt \right\}.$$

Integrating by parts twice gives

$$(2.6) \quad \lambda_n \int_0^x K(x, t) \sin \sqrt{\lambda_n} t dt \\ = -\sqrt{\lambda_n} \cos \sqrt{\lambda_n} x K(x, x) + \sqrt{\lambda_n} K(x, 0) \\ + \sin \sqrt{\lambda_n} x K_t(x, x) \\ - \int_0^x K_{tt}(x, t) \sin \sqrt{\lambda_n} t dt.$$

Thus,

$$(2.7) \quad \phi_n'' - a\phi_n + \lambda_n\phi_n \\ = \frac{1}{\sqrt{\lambda_n}} \left[K_t(x, x) + K_x(x, x) + \frac{d}{dx} K(x, x) - a(x) \right] \sin \sqrt{\lambda_n} x \\ + K(x, 0) + \frac{1}{\sqrt{\lambda_n}} \int_0^x (K_{xx} - aK - K_{tt}) \sin \sqrt{\lambda_n} t dt.$$

From the uniqueness of the representation of a function as a Fourier-Stieltjes integral it follows that $-\phi_n'' + a\phi_n = \lambda_n\phi_n$ on $(0, 1)$ if and only if $K(x, t)$ satisfies the hyperbolic equation

$$(2.8) \quad K_{tt} = K_{xx} - aK, \quad 0 \leq t \leq x \leq 1,$$

with the conditions $K(x, 0) = 0$, $dK(x, x)/dx = a(x)/2$, which can be put in the form

$$(2.9) \quad K(x, 0) = 0, \quad \text{and}$$

$$(2.10) \quad K(x, x) = \frac{1}{2} \int_0^x a(s) ds.$$

Equations (2.8)–(2.10) represent a Goursat problem for the above hyperbolic equation. If $a(x)$ is continuously differentiable, then (2.8)–(2.10) possesses a unique solution $K(x, t)$ that is twice continuously differentiable in x and t . We are however, only assuming $a(x)$ continuous and we must therefore take $K(x, t)$ to be a weak solution of (2.8)–(2.10) that possesses only one continuous derivative. This will be sufficient, however, for our needs.

Returning to (2.4) it is clear that $\phi_n(0) = 0$ and $\phi_n'(0) = 1$. To see the

boundary condition at $x = 1$ we need a lemma, the proof of which can be found in [3, §11].

LEMMA. *The functions $\phi_n(x)$ defined by (2.4) form a complete orthogonal system on the interval $(0, 1)$.*

If we now consider the equations

$$\begin{aligned} -\phi_n''(x) + a(x)\phi_n(x) &= \lambda_n\phi_n(x), \\ -\phi_m''(x) + a(x)\phi_m(x) &= \lambda_m\phi_m(x), \end{aligned}$$

and multiply the first by $\phi_m(x)$, the second by $\phi_n(x)$, and subtract, we obtain the identity

$$\phi_n''(x)\phi_m(x) - \phi_n(x)\phi_m''(x) = (\lambda_m - \lambda_n)\phi_n(x)\phi_m(x).$$

Integrating this between $x = 0$ and $x = 1$ and using the boundary conditions at $x = 0$ we obtain

$$(2.11) \quad \phi_n'(1)\phi_m(1) - \phi_n(1)\phi_m'(1) = 0.$$

Thus $\phi_n(x)$ satisfies the boundary condition

$$(2.12) \quad \alpha\phi_n'(1) + \phi_n(1) = 0, \quad n = 1, 2, 3, \dots,$$

for some constant α . Now if $\phi_n(x)$ satisfies

$$(2.13) \quad \begin{aligned} -\phi_n'' + a\phi_n &= \lambda_n\phi_n, \\ \phi_n(0) = 0, \quad \alpha\phi_n'(1) + \phi_n(1) &= 0, \end{aligned}$$

with $\alpha \neq 0$, then the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ satisfy the asymptotic estimate

$$(2.14) \quad \sqrt{\lambda_n} = (n + 1/2)\pi + o(1/n)$$

(see [3], [9]), which is in violation of (A2).

Thus $\alpha = 0$ and ϕ_n satisfies the condition

$$(2.15) \quad \phi_n(1) = 0, \quad n = 1, 2, \dots$$

By applying McLaurin's Theorem to $\phi_n(x)$ and using (2.1) we obtain

$$(2.16) \quad \frac{\phi_n(x)}{x} = 1 + \frac{1}{2}x\phi_n''(\eta_n) \quad \text{for } 0 \leq \eta_n < 1.$$

At the point $x = 1$ we have the similar asymptotic behavior

$$(2.17) \quad \frac{\phi_n(x)}{1-x} = \gamma_n + \frac{1}{2}(1-x)\phi_n''(\eta_n), \quad \text{for } 0 \leq \eta_n < 1,$$

and some constant γ_n , uniformly bounded in n , cf. [9, p. 10].

It will be convenient at this point to obtain some properties of the function $K(x, t)$ that will be useful later.

The solution to (2.8)–(2.10) can be written in the form

$$(2.18) \quad K(x, t) = \frac{1}{2} \int_0^{(x+t)/2} a(y)dy + \iint_{D_{x,t}} a(y)K(y, s)dyds$$

where $D_{x,t}$ is the region bounded by the characteristic lines of slope ± 1 through the origin and the point (x, t) . We may consider (2.18) as an integral equation for $K(x, t)$ with free term $(1/2)\int_0^{(x+t)/2} a(y)dy$ and kernel $a(y)$, both of which are nonnegative. Thus the solution to (2.18) will be nonnegative,

$$(2.19) \quad K(x, t) \geq 0, \quad 0 \leq t \leq x.$$

If we now define $\omega(x, t) = K_t(x, t)$, then it follows that

$$(2.20) \quad \omega_{tt} = \omega_{xx} - a(x)\omega.$$

The initial condition $K(x, 0) = 0$ and the hyperbolic equation satisfied by K gives $\omega_t(x, 0) = 0$ while the condition $K(x, 0) = 0$ and (2.19) implies that $\omega(x, 0) \geq 0$. If we now extend $a(x)$ and hence $K(x, t)$ into the region $1 \leq x \leq 2, t \geq 0$ as an even function, then $\omega_t(x, 0) = 0$ and $\omega(x, 0) \geq 0$ for $0 \leq x \leq 2$. Thus $\omega(x, t)$ is obtained as the solution to the integral equation

$$(2.21) \quad \begin{aligned} \omega(x, t) = & \frac{1}{2}[\omega(x - t, 0) + \omega(x + t, 0)] \\ & + \frac{1}{2} \iint_{\Delta_{x,t}} a(y)\omega(y, s)dyds, \end{aligned}$$

where $\Delta_{x,t}$ is the triangle bounded by the line $t = 0$ and the characteristic lines through the point (x, t) . Again the free term and kernel of this equation are nonnegative for $t \leq x$ and it follows by Picard iteration that $\omega(x, t) \geq 0$ for $0 \leq t \leq x \leq 1$. This means that $K(x, t)$ is increasing in t for each fixed x and thus we have the inequality

$$(2.22) \quad 0 \leq K(x, t) \leq \frac{1}{2} \int_0^x a(s)ds, \quad 0 \leq t \leq x \leq 1.$$

If we now take $a(x), b(x) \in \mathcal{A}$ and denote by $K(x, t; a(x))$ and $K(x, t; b(x))$ the solutions of (2.8)–(2.10) with undetermined coefficients and boundary conditions determined by $a(x)$ and $b(x)$ respectively, then we obtain

$$(2.23) \quad \begin{aligned} K(x, t; a(x)) - K(x, t; b(x)) = & \frac{1}{2} \int_0^x (a(y) - b(y))dy \\ & + \frac{1}{2} \iint_{D_{x,t}} [a(y) - b(y)]K(y, s, a(y))dyds \\ & + \frac{1}{2} \iint_{D_{x,t}} b(y)[K(y, s; a(y)) - K(y, s; b(y))]dyds, \end{aligned}$$

where $D_{x,t}$ is the parallelogram-shaped region as in equation (2.18). Let us now restrict $a(x)$ and $b(x)$ so that $0 \leq a(x) \leq M$ and $0 \leq b(x) \leq M$ for some fixed M . Also note that the region $D_{x,t}$ is a subset of the triangular region $0 \leq t \leq x$. We thus obtain from (2.23) the estimate

$$(2.4) \quad \begin{aligned} & |K(x, t; a(x)) - K(x, t; b(x))| \\ & \leq \frac{1}{2} \int_0^x |a(y) - b(y)| dy + \frac{1}{4} M \int_0^x |a(y) - b(y)| dy \\ & \quad + \frac{1}{2} M \int_0^x \int_0^y |K(y, s; a) - K(y, s; b)| dy ds, \end{aligned}$$

which in turn yields

$$(2.25) \quad \begin{aligned} \sup_{0 \leq t \leq x} |K(x, t; a(x)) - K(x, t; b(x))| & \leq \frac{1}{4} (M + 2) \int_0^x |a(y) - b(y)| dy \\ & \quad + \frac{1}{2} M \int_0^x \sup_{0 \leq s \leq x} |K(y, s; a(y)) - K(y, s; b(y))| dy. \end{aligned}$$

An application of Gronwall's inequality to the above yields for some constant $C = C(M)$,

$$(2.26) \quad \sup_{0 \leq t \leq x} |K(x, t; a(x)) - K(x, t; b(x))| \leq C \sup_{0 \leq y \leq x} |a(y) - b(y)|.$$

Let us now return to the representation (2.4). If we multiply both sides by a_n and sum from $n = 1$ to infinity using the properties of $\{a_n\}_{n=1}^{\infty}$ to justify the interchange of summation and integration, we obtain

$$(2.27) \quad f(x) = h(x) + \int_0^x K(x, t; a(x)) h(t) dt.$$

As it stands (2.27) is a Volterra integral equation of the first kind for the unknown $a(x)$. We shall convert this to an equation of the second kind by differentiating (2.27) twice, using equations (2.8)–(2.10) and then integrating by parts twice to obtain

$$(2.28) \quad f''(x) - h''(x) = f(x)a(x) + \int_0^x K(x, t, a(x)) h''(t) dt.$$

Since $f(x) > 0$ for $0 < x < 1$, the only possible degeneracies are at $x = 0$ and $x = 1$.

From condition (A1), $f''(x)/f(x)$ remains bounded at both endpoints. From (2.4) and the definition of $h(x)$ we have

$$(2.29) \quad h''(x) + \int_0^x K(x, t) h''(t) dt = - \sum_{n=1}^{\infty} \sqrt{\lambda_n} a_n \phi_n(x).$$

since $\sum_{n=1}^{\infty} \sqrt{\lambda_n} |a_n| < \infty$, it follows from (2.16) and (2.17) that

$$(2.30) \quad h''(x) + \int_0^x K(x, t)h''(t)dt = \beta_0x + o(x),$$

and

$$(2.31) \quad h''(x) + \int_0^x K(x, t)h''(t)dt = \beta_1(1 - x) + o(1 - x).$$

Equations (2.30) and (2.31) along with the fact that $f''(0)$ and $f''(1)$ are nonzero, show that the equation (2.28) is nonsingular in the range $0 \leq x \leq 1$.

The contraction mapping theorem and the estimate (2.26) gives the existence of a unique solution $a(x)$ to (2.28) in the space $C[0, \delta]$ for δ sufficiently small. Indeed the Lipschitz estimate (2.26) is sufficient to guarantee uniqueness in the range $0 \leq x \leq 1$. To get existence for all x in $0 \leq x \leq 1$ we have to obtain a global bound on $a(x)$ for this range, but this follows directly by using the estimate (2.22) in the integral equation (2.28) to obtain

$$(2.32) \quad f(x)a(x) \leq |f''(x) - h''(x)| + \frac{1}{2} \left\{ \int_0^x |h''(t)| dt \right\} \int_0^x a(s) ds.$$

The required bound on $a(x)$ is now a consequence of Gronwall's inequality applied to the inequality (2.32).

This completes the proof of the theorem.

A natural question is whether the coefficient $a(x)$ depends continuously on the data $f(x)$ and $g(t)$. Our Volterra equation (2.28) would yield the continuous dependence of $a(x)$ on the functions $f(x)$, $f''(x)$ and $h''(x)$ in the supremum norm. From §1 it is easy to see that $h''(x)$ represents the value of $v(x, t)$ at $t = 0$ where $v(x, t)$ satisfies $v_t = v_{xx}$, $0 < x < \infty$, $t > 0$; $v(0, t) = 0$, $v_x(0, t) = g'(t)$, and where $v(x, t) \leq \sum_{n=1}^{\infty} \sqrt{\lambda_n} a_n = M < \infty$. Under this a priori bound on v the non-characteristic Cauchy problem for the heat equation is well-posed in the sense of Hadamard [2], and one can obtain Holder continuity of $v(x, t)$ in terms of the data $g'(t)$. Thus we have that

$$\sup_{x \geq 0} |h''(x)| \leq C \left\{ \sup_{t \geq 0} |g'(t)| \right\}^\varepsilon$$

for some $\varepsilon = \varepsilon(M) > 0$. However it seems unlikely that the inverse problem (0.1)–(0.4) is sufficiently ill-posed to require control on two derivatives of the initial data, and one on the boundary data in order to obtain control on the supremum norm of $a(x)$.

ACKNOWLEDGEMENT. The author wishes to thank John R. Cannon for several improvements in this paper.

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