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FINITE HARMONIC AND GEOMETRIC INTERPOLATION

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1. Introduction. In the works [4] and [5], the authors have been developing the theory of finite harmonic interpolation in the unit disk. The basic idea is to express the value of a real-valued harmonic function u in the disk as a finite weighted mean

$$u(z) = \frac{1}{N}\sum_{k=1}^{N} \frac{R^2 - |z|^2}{|\zeta_k - z|^2} u(\zeta_k), \tag{1}$$

for $|z| < R < 1$, $\zeta_1, \zeta_2, \ldots, \zeta_N$ points equally spaced on $|z| = R$, and N a fixed positive integer.

In the present work, we also consider the notion of finite harmonic interpolation on a general domain Ω with an exhaustion $\{\Omega_n\}$ such that the boundary of each Ω_n, $\partial\Omega_n$, is an analytic Jordan curve. The Green's function $g_n(z, \zeta)$ of Ω_n with pole z has an inner normal derivative $\partial g/\partial\eta$ and each Ω_n has length L_n.

If u is a real-valued harmonic function on Ω and z is in Ω_n then

$$u(z) = \frac{1}{2\pi}\int_{\partial\Omega_n} u(\zeta)\frac{\partial g_n(z, \zeta)}{\partial\eta}\,|d\zeta|\,, \tag{2}$$

and $\partial g_n(z, \zeta)/\partial\eta$ is continuous on the analytic Jordan curve $\partial\Omega_n$. Since $\int_{\partial\Omega_n} u(z)\,|d\zeta| = L_n u(z)$ we can rewrite equation (2) to obtain

$$\int_{\partial\Omega_n}\left[u(\zeta)\,\frac{L_n}{2\pi}\,\frac{\partial g_n(z, \zeta)}{\partial\eta} - u(z))\right]|d\zeta| = 0. \tag{3}$$

Let $F(\zeta) = u(\zeta)\,(L_n/2\pi)(\partial g_n(z, \zeta)/\partial\eta - u(z))$ and parametrize ζ in terms of arc length s, say $\zeta = \phi(s)$. Also let $\partial\Omega_n = \bigcup_{k=1}^{N}\gamma_k$, where each segment γ_k has length L_n/N, and denote by $F_k(s)$, $0 \leqq s \leqq L_n/N$, the restriction of $F(\phi(s))$ to γ_k. Then from (3),

$$\int_0^{L_n/N}\left[\sum_{k=1}^{N} F_k(s)\right]ds = 0.$$

By the continuity of F there exists s_0 such that $\sum_{k=1}^{N} F_k(s_0) = 0$. That is, there exist N equally spaced points $\zeta_1, \zeta_2, \ldots, \zeta_N$ on $\partial\Omega_n$ such that

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$$u(z) = \frac{L_n}{2\pi N} \sum_{k=1}^{N} \frac{\partial g_n(z, \zeta_k)}{\partial \eta} u(\zeta_k). \tag{4}$$

In the papers [4] and [5] these points were called harmonic interpolation points. A repetition of the above argument shows that there are at least two distinct sets of harmonic interpolation points on each $\partial\Omega_n$. The reader is referred to [4] and [5] for further results in the case that Ω is the unit disk.

If u is a real-valued harmonic function on the complex plane then a set of harmonic interpolation points $R \exp i(\theta + 2\pi j/N)$ with respect to $z = 0$ exist on each circle $|z| = R$ and we have a discrete mean value theorem $u(0) = (1/N) \sum_{j=0}^{N-1} u(R \exp i(\theta + 2\pi j/N))$. It is shown in [4] that in this case either all points are harmonic interpolation points or the number of distinct sets of harmonic interpolation points is finite. In the latter case we shall denote the number of distinct sets of points by $p(u, R, N)$.

If we set $F(Re^{i\theta}) = u(0) - (1/N) \sum_{j=0}^{N-1} u(R \exp i(\theta + 2\pi j/N))$ then F is a real-valued harmonic function and the harmonic interpolation points are traced by the level curves of F corresponding to $F(Re^{i\theta}) = 0$.

We denote the harmonic conjugate for u (unique up to a constant) in the complex plane by v and set $g(z) = g(x + iy) = u(x + iy) + iv(x + iy)$. Then setting the Nth root of unity $\omega = \exp(2\pi i/N)$, we define $h(z) = g(z) + g(\omega z) + \cdots + g(\omega^{N-1}z) - Ng(0)$. It is clear that $F(z) = 0$ if and only if $\operatorname{Re} h(z) = 0$. A considerable simplification follows from the Taylor expansion of g in that (see [4] for details)

$$h(z) = N\left[\frac{g^{(N)}(0)}{N!} z^N + \frac{g^{(2N)}(0)}{(2N)!} z^{2N} + \cdots + \frac{g^{(mN)}(0)}{(mN)!} z^{mN} + \cdots\right].$$

Hence a necessary and sufficient condition for all points in the plane to be harmonic interpolation points is that $g^{(mN)}(0) = 0$ for all positive integers m. The objective of §2 is to show that if h has an essential singularity at infinity then $\lim_{R\to\infty} p(u, R, N) = \infty$. The technique used will depend on the theory of autonomous systems of differential equations in the plane.

In §3 we shall consider a Jordan domain Ω and the concept of a normal exhaustion.

DEFINITION. A normal exhaustion $\{\Omega_n\}$ of a Jordan domain Ω with rectifiable boundary is an exhaustion such that the boundary of each Ω_n, $\partial\Omega_n$, is an analytic Jordan curve, and such that the Green's function $g_n(z, \zeta)$ of Ω_n with pole z satisfies $0 < m \leqq \partial g_n(z, \zeta)/\partial\eta \leqq M < \infty$ for $n = 1, 2, 3, \ldots$, where $\zeta \in \partial\Omega_n$, and z belongs to Ω_1. The constants m and M may depend on z. We also assume that the lengths L_n of $\partial\Omega_n$ are uniformly bounded.

Conditions for Ω to have a normal exhaustion are given in §4.

In §3 this notion is applied to $\log |f(z)|$ where f is an analytic function which is not zero in Ω. Then $|f(z)|$ may be written as a product of geometric interpolation points. For fixed z and fixed N, we consider the geometric interpolation points of $|f(z)|$ on each $\partial\Omega_n$. Theorem 4 shows that if the exhaustion is normal and f is a bounded analytic function, then f cannot tend to zero along a sequence of geometric interpolation points which approach $\partial\Omega$.

2. Level curves and harmonic interpolation. In order to prove the main theorem concerning harmonic interpolation points we first consider level curves of harmonic functions. Let $f(z) = f(x + iy) = \phi(x, y) + i\psi(x, y)$ be a non-constant entire function and define a gradient system by $dx/dt = \psi_x(x, y)$, $dy/dt = \psi_y(x, y)$.

LEMMA 1. *Every path of the gradient system is a level curve of ϕ and conversely.*

PROOF. We apply the Cauchy-Riemann equations to the equation of an orbit to obtain $dy/dx = \psi_y/\psi_x = -\phi_x/\phi_y$. It follows that $d\phi = \phi_x dx + \phi_y dy = 0$ and that $\phi(x, y) = c$ is a solution. To prove the converse the argument is reversed.

LEMMA 2. *The function ψ is monotone along any level curve of ϕ.*

PROOF. Suppose ψ were not monotone along a level curve $\phi(x, y) = c$. Since f and hence ψ is non-constant it follows that ψ must attain a local maximum say, (the situation is similar for a local minimum) at a point (x_0, y_0) on the level curve. But the level curves of ψ are orthogonal to the level curve $\phi(x, y) = c$. Hence by the orthogonality (transversality is enough) and smoothness properties of $\phi(x, y) = c$ it is possible to construct a disk with centre (x_0, y_0) such that ψ attains its maximum value on the disk at the centre. This is a contradiction since ψ is harmonic. Hence ψ is monotone along $\phi(x, y) = c$.

Now let us consider u harmonic on the complex plane and as in the introduction set $g(z) = g(x + iy) = u(x, y) + iv(x, y)$ and

$$h(z) = N\left[\frac{g^{(N)}(0)}{N!} z^N + \frac{g^{(2N)}(0)}{(2N)!} z^{2N} + \cdots + \frac{g^{(mN)}(0)}{(mN)!} z^{mN} + \cdots\right].$$

THEOREM 3. *If $h(z)$ has an essential singularity at infinity, then* $\lim_{R\to\infty} p(u, R, N) = \infty$.

PROOF. Since the total number of harmonic interpolation points on $|z| = R$ is $N\,p(u, R, N)$ it suffices to show that the number of such points tends to infinity. We note that a harmonic interpolation point occurs when a level curve of $\operatorname{Re} h(z) = 0$ cuts $|z| = R$. We shall now set $\phi(z) = \operatorname{Re} h(z)$, $\psi(z) = \operatorname{Im} h(z)$ and apply Lemmas 1 and 2.

The singularities of the gradient system correspond to points $z_0 = x_0 + iy_0$ where $\psi_x(x_0, y_0) = 0$, $\psi_y(x_0, y_0) = 0$, and hence $h'(z_0) = 0$. If q is the least positive integer such that $h^{(q)}(z_0) \neq 0$ then the local mapping theorem [1, pp. 130–133] describes the bifurcation of the level curves into $2q$ arms at z_0. There are tangents equally spaced at angles of π/q and a level curve entering z_0 is identified with the unique branch levaing z_0 along the same tangent line.

In view of Lemma 1 we shall use the Poincare-Bendixson theorem to show that no level curve can remain bounded. Firstly, closed level curves and hence paths are impossible since ϕ is a nonconstant harmonic function. Secondly, a half path entering a singularity is identified with a half path leaving it. (The direction of the gradient field may be different on the two half paths. This does not affect the argument concerning level curves.) A level curve consists of a succession of half paths. Since there can only be a finite number of singularities in each finite region and no closed paths, the Poincare-Bendixon theorem does not allow any level curve to remain bounded.

In fact the level curves can be countably indexed. This is done by considering each circle $|z| = R$ for increasing R. Each level curve has a point of closest approach to the origin and is counted when it first appears. There can only be finitely many level curves in any bounded region so the indexing can always be well defined.

We now show that the level curves indexed in this way are infinite. By Lemma 2, ψ is monotone along each level curve and hence for a fixed value ψ_0, the complex number $i\psi_0$ is a value taken on by h at only one point on the level curve $\phi(x, y) = 0$. But h has an essential singularity at ∞ and by the big Picard theorem there certainly exists a complex number $i\psi_0$ which is taken on by h in every neighbourhood of infinity. Hence there must be infinitely many level curves.

It remains to show that the number of harmonic interpolation points on $|z| = R$ becomes infinite as R tends to infinity. The indexing of the level curves was necessary because distinct level curves can intersect at a bifurcation point leading to a reduction of harmonic interpolation points on that particular circle. However, this can only occur once for a given pair of level curves as a second intersection would lead go a closed level curve. Hence for sufficiently large R the first m indexed level curves give rise to m harmonic interpolation points. This completes the proof.

REMARKS. We believe that there should exist examples which show that the limit $\lim_{R\to\infty} p(u, R, N) = \infty$ is not montone. We give several interesting open questions:

(i) Suppose N remains fixed and harmonic interpolation points are defined with respect to $z = 0$ for a general exhaustion $\{\Omega_n\}$. Does the assumption that h has an essential singularity at infinity imply the analogoue of Theorem 3 for the exhaustion $\{\Omega_n\}$?

(ii) If the analogue of Theorem 3 holds for a single exhaustion (assume N fixed), does this imply that the result holds for a class of exhaustions?

(iii) What happens under the hypothesis of Theorem 3 if harmonic interpolation points are defined with respect to $z \neq 0$?

3. Bounded analytic functions. Let Ω be a Jordan domain. If $f(z)$ is an analytic function which is not zero in Ω then $\log |f(z)|$ is harmonic on Ω and there exist harmonic interpolation points $\zeta_1^n(z), \zeta_2^n(z), \ldots, \zeta_N^n(z)$ on each $\partial\Omega_n$ such that

$$\log |f(z)| = \frac{L_n}{2\pi N} \sum_{k=1}^{N} \frac{\partial g_n(z, \zeta_k^n(z))}{\partial\eta} \log |f(\zeta_k^n(z))|$$

and

$$(5) \qquad |f(z)| = \left(\prod_{k=1}^{N} |f(\zeta_k^n(z))|^{\frac{\partial g_n(z, \zeta_k^n(z))}{\partial\eta}} \right)^{L_n/2\pi N}.$$

Following [3] and [4], the points $\zeta_k^n(z)$, $1 \leqq k \leqq N$, are called geometric interpolation points of $|f(z)|$. We denote a particular point on Ω_n by $\zeta_{k(n)}^n(z)$.

THEOREM 4. *Let f be a bounded analytic function which is not zero on a Jordan domain Ω with a rectifiable boundary and a normal exhaustion $\{\Omega_n\}$. Let $(\zeta_{k(n)}^n(z))_{n=1,2,\ldots}$,* be a sequence of geometric interpolation points of $|f(z)|$. Then $\lim_{n\to\infty} f(\zeta_{k(n)}^n(z)) \neq 0$.

PROOF. We shall show that if $\lim_{n\to\infty} f(\zeta_{k(n)}^n(z)) = 0$, then $f(z) = 0$ giving a contradiction.

For each n, equation (5) holds for the geometric interpolation points on $\partial\Omega_n$. Hence

$$|f(z)| = \lim_{n\to\infty} \left(\prod_{k=1}^{N} |f(\zeta_k^n(z))|^{\frac{\partial g_n(z, \zeta_k^n(z))}{\partial\eta}} \right)^{L_n\backslash 2\pi N}.$$

Also for each n, the term containing $|f(\zeta_{k(n)}^n(z))|$ appears somewhere in this product.

Since $\partial g_n(z, \zeta)/\partial\eta \geqq m > 0$ and L_n is bounded it follows that

$$\lim_{n\to\infty} \left(|f(\zeta_{k(n)}^n(z))|^{\frac{\partial g_n(z, \zeta_{k(n)}^n(z))}{\partial\eta}} \right)^{L_n/2\pi N} = 0.$$

It then suffices to observe that since f is bounded and $\partial g_n(z, \zeta)/\partial\eta \leqq M$, all other terms in the product are bounded above as $n \to \infty$. Hence, by taking the limit, $|f(z)| = 0$. This completes the proof.

It is an immediate corollary that f cannot tend to zero along a level curve generated by geometric interpolation points. We give an example to show that the theorem does not hold if f is not bounded.

EXAMPLE. Let $f(z) = (1 + z)/(1 - z)$ be defined on the unit disk. Then R and $-R$ are geometric interpolation points for $0 \leqq R < 1$ since

$$|f(R)|\,|f(-R)| = \left|\frac{1+R}{1-R}\right| \left|\frac{1-R}{1+R}\right| = 1 = |f(0)|.$$

Also $\lim_{R\to 1} f(-R) = 0$.

4. Normal Exhaustions. The following theorem gives certain conditions for a domain to have a normal exhaustion.

THEOREM 5. *Let Ω be a Jordan domain with a rectifiable boundary. Then the following are equivalent.*

(a) *There exists an exhaustion $\{\Omega_n\}$ of Ω and a point $z_0 \in \Omega_1$ such that the boundary of Ω_n, $\partial\Omega_n$, is an analytic Jordan curve and the conformal mapping ψ_n of Ω_n onto the unit disk ($\psi_n(z_0) = 0$, $\psi_n'(z_0) > 0$) can be extended to a conformal mapping defined on $\bar{\Omega}_n$ with $0 < m \leqq \partial g_n(z_0, \zeta)/\partial\eta = |\psi_n'(\zeta)| \leqq M < \infty$.*

(b) *There exists a conformal mapping ϕ of $|w| < 1$ onto the domain Ω and positive constants a and b such that $a \leqq |\phi'(w)| \leqq b$, $|w| < 1$.*

(c) *Let ϕ be an arbitrary conformal mapping of $|w| < 1$ onto the domain Ω. Then there exist positive constants a and b such that $a \leqq |\phi'(w)| \leqq b$, $|w| < 1$.*

(d) *There exists a normal exhaustion $\{\Omega_n\}$ of Ω which has the property that any conformal mapping ψ_n of Ω_n onto the unit disk can be extended to a conformal mapping defined on $\bar{\Omega}_n$.*

PROOF. It is clear that (d) implies (a) and straightforward to show that (b) implies (c). It is also routine to show that (c) implies (d) by constructing an appropriate normal exhaustion using the images under ϕ of circles $|w| = r$ where ϕ is a conformal mapping of the unit disk onto Ω.

It remains to show that (a) implies (b). By the maximum principle $m \leqq |\psi_n'(z)| \leqq M$ on $\bar{\Omega}_n$ and if we define ϕ_n to be the inverse conformal mapping of the unit disk $|w| < 1$ onto Ω_n then $1/M \leqq |\phi_n'(z)| \leqq 1/m$. Now let ϕ be the conformal mapping of $|w| < 1$ onto Ω ($\phi(0) = z_0$, $\phi'(0) > 0$). By the Carathéodory convergence theorem (see [3]) ϕ_n tends to ϕ uniformly in each disk $|w| \leqq R < 1$. By the corresponding uniform convergence of ϕ_n' to ϕ' it follows that $1/M \leqq |\phi'(w)| \leqq 1/m$. This completes the proof.

We remark that condition (c) is necessary and sufficient for $H^p(\Omega) = E^p(\Omega)$ where $E^p(\Omega)$ is the space first considered by Smirnov (see Duren [2]).

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