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Q-ANALOGUE OF A TRANSFORMATION OF WHIPPLE

A. VERMA AND V.K. JAIN

1. Whipple [18] obtained two transformations between nearly-poised hypergeometric series (second kind) ${}_4F_3$ and Saalschützian hypergeometric series ${}_5F_4$.

$$(1.1)\quad {}_4F_3\left[\begin{matrix}a, b, c, -m;\\ 1+a-b, 1+a-c, w\end{matrix}\right]$$
$$= \frac{(w-a)_m}{(w)_m}\, {}_5F_4\left[\begin{matrix}1+a-w, \frac{a}{2}, \frac{1+a}{2}, 1+a-b-c, -m;\\ 1+a-b, 1+a-c, \frac{1}{2}(1+a-w-m), \frac{1}{2}(2+a-w-m)\end{matrix}\right]$$

and

$$(1.2)\quad {}_4F_3\left[\begin{matrix}-n, b, c, d;\\ 1-n-b, 1-n-c, w\end{matrix}\right]$$
$$= \frac{(w-d)_n}{(w)_n}\, {}_5F_4\left[\begin{matrix}d, 1-n-b-c, \frac{-n}{2}, \frac{1-n}{2}, 1-n-w;\\ 1-n-b, 1-n-c, \frac{1}{2}(1+d-w-n), \frac{1}{2}(2+d-w-n)\end{matrix}\right].$$

It is easy to note that (1.2) follows from (1.1) (and vice-versa). Indeed in (1.1) setting $a = -n$ and $m = -d$ we get (1.2) with the restriction $d = -m$. The condition on d can be waived by analytic continuation because both sides are polynomials in d.

In 1929 Bailey [6] obtained a transformation between a nearly-poised series ${}_5F_4$ and Saalschützian ${}_5F_4$ which is analogous to (1.1), viz.,

$$(1.3)\quad {}_5F_4\left[\begin{matrix}a, 1+\frac{a}{2}, b, c, -m\\ \frac{a}{2}, 1+a-b, 1+a-c, w\end{matrix}\right] = \frac{(w-a-1-m)\,(w-a)_{m-1}}{(w)_m}$$
$$\cdot\, {}_5F_4\left[\begin{matrix}1+\frac{a}{2}, \frac{1+a}{2}, 1+a-b-c, 1+a-w, -m;\\ 1+a-b, 1+a-c, \frac{1}{2}(2+a-w-m), \frac{1}{2}(3+a-w-m)\end{matrix}\right].$$

Using (1.1) and (1.2) Bailey [7] obtained identities of Cayley-Orr type. In fact there are many other interesting applications of (1.1) and (1.2). In an earlier paper [17] we obtained q-analogues of (1.1) and (1.3). The q-analogues of the transformations of hypergeometric identities add substantially to the analytic side of the theory of partitions and the partition identities. One may of course ask whether either partition identities or their analytic counterparts are of any interest to the rest of mathematics. The answer is surely yes. Indeed the recent rapid advance of Lie algebras starting with Ian Macdonald's [14] major breakthrough has produced significant results. Illustrating the relation between Lie algebras and partition identities (see Lepowsky and Milne [13], Kac [11]). Indeed it appears now that this work has produced a fully Lie algebraic proof of the Rogers-Ramanujan identities (Lepowsky [12]).

A proof of the weak form of the beautiful conjecture by I. Macdonald concerning rotationally invariant plane partitions relies on (1.2) (see Andrews [5]).

Indeed the q-analogue of (1.2) can be written out from the q-analogue of (1.1) by following the method described above for deducing (1.2) from (1.1). However in §3 we give an alternative simple proof for a q-analogue of (1.2) by using the following q-analogue of Gauss summation theorem (which is in fact a special case of ${}_6\phi_5$ [16; 3.3.1.3])

$$(1.4)\qquad \phi\begin{bmatrix} a^2,\ -aq^2 :\ -aq/e,\ f;\ e/af \\ q^2,\ -a : e,\ -aq/f \end{bmatrix} = \Pi\begin{bmatrix} -aq,\ -q/f,\ e/a,\ e/f\,;q \\ -q,\ -aq/f,\ e,\ e/af \end{bmatrix}.$$

It may be remarked that (1.4) is different in nature than the known q-analogue of Gauss' theorem [16; 3.3.2.5]. We also discuss in §3 some of the interesting special cases of the transformations proved in this section. Furthermore as applications of the transformations of §3, we obtain in §4 a q-analogue of a formula of Howell [9], viz.,

$$(1.5)\qquad [L_n^a(z)]^2 = \frac{(1+a)_n}{2^{2n}(1)_n}\sum_{r=0}^{n}\frac{(1)_{2r}(1)_{2n-2r}}{(1)_r\{(1)_{n-r}\}^2(1+a)_r}\, L_{2r}^{2a}(2z),$$

where

$$L_n^a(z) = \frac{(1+a)_n}{(1)_n}\,{}_1F_1\begin{bmatrix} -n\,;z \\ 1+a \end{bmatrix}$$

is a Laguerre polynomial.

In this sequel, we also obtain the q-analogues of the identities of Cayley-Orr type due to Bailey [7].

2. Definitions and notations. If we let $|q| < 1$,

$$[a; q]_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}),\ [a; q]_0 = 1,$$

$$\text{and } [a; q]_\infty = \prod_{j=0}^{\infty} (1 - aq^j),$$

then we may define the basic hypergeometric series as

$$\begin{aligned} {}_{p+1}\phi_{p+r}&\left[\begin{matrix} a_1, a_2, \ldots, a_{p+1} ; q; x \\ b_1, b_2, \ldots, b_{p+r} \end{matrix}\right] \\ &= \sum_{n=0}^{\infty} \frac{[a_1; q]_n \cdots [a_{p+1} ; q]_n x^n(-)^{nr} q^{rn(n-1)/2}}{[q; q]_n [b_1; q]_n \cdots [b_{p+r}; q]_n}, \end{aligned}$$

where the series ${}_{p+1}\phi_{p+r}(x)$ converges for all positive integral values *of* r and for all x, except when $r = 0$, it converges only for $|x| < 1$.

Further, we shall denote the following infinite product by

$$\Pi\left[\begin{matrix} a_1, \ldots, a_r ; q \\ b_1, \ldots, b_s \end{matrix}\right] = \prod_{j=0}^{\infty} \left[\frac{(1 - a_1q^j) \cdots (1 - a_rq^j)}{(1 - b_1q^j) \cdots (1 - b_sq^j)}\right].$$

Following Agarwal and Verma **[3]**, we define the bi-basic hypergeometric series as

$$\phi\left[\begin{matrix} (a): (b); x \\ (c): (d); q^\lambda \end{matrix}\right] = \sum_{n=0}^{\infty} \frac{[(a); q^2]_n[(b); q]_n x^n q^{\lambda n(n-1)/2}}{[(c); q^2]_n [(d); q]_n},$$

The series converges for all values of x if $\lambda > 0$, and when $\lambda = 0$, we require the additional convergence condition $|x| < 1$.

3. We begin this section by proving the following formula

$$\begin{aligned} \phi&\left[\begin{matrix} a^2, -aq^2, b^2, c^2: -\frac{aq}{w}, d; \frac{awq^2}{b^2c^2d} \\ q^2, -a, \frac{a^2q^2}{b^2}, \frac{a^2q^2}{c^2}: w, -\frac{aq}{d} \end{matrix}\right] = \Pi\left[\begin{matrix} -aq, w/a, -q/d, w/d; q \\ -q, -aq/d, w, w/ad \end{matrix}\right] \\ &\cdot {}_5\phi_4\left[\begin{matrix} \frac{a^2q^2}{b^2c^2}, a, aq, d^2, \frac{a^2q^2}{w^2}; q^2; q^2 \\ \frac{a^2q^2}{b^2}, \frac{a^2q^2}{c^2}, \frac{adq}{w}, \frac{adq^2}{w} \end{matrix}\right] \end{aligned} \tag{3.1}$$

where a or d is of the form q^{-n}. Formula (3.1) for $a = q^{-n}$ yields a q-analogue of (1.2), whereas for $d = q^{-n}$, (3.1) becomes a q-analogue of (1.1) (see for alternative proof **[17]**).

Proof of (3.1.) Multiplying both sidse of the q-analogue of Saalschütz summation theorem **[16**; 3.3.2.2]

$$
{}_3\phi_2\left[\begin{matrix} \dfrac{a^2q^2}{b^2c^2}, a^2q^{2m}\; q^{-2m};\; q^2;\; q^2 \\ \dfrac{a^2q^2}{b^2}, \dfrac{a^2q^2}{c^2} \end{matrix}\right] = \frac{[b^2;\; q^2]_m[c^2;\; q^2]_m\,(aq)^{2m}}{\left[\dfrac{a^2q^2}{b^2};\; q^2\right]_m\left[\dfrac{a^2q^2}{c^2};\; q^2\right]_m(bc)^{2m}}
$$

by

$$
\frac{[a^2;\; q^2]_m[-aq^2;\; q^2]_m\left[-\dfrac{aq}{w};\; q\right]_m[d;\; q]_m\, w^m}{[q^2;\; q^2]_m[-a;\; q^2]_m[w:\; q]_m\left[-\dfrac{aq}{d};\; q\right]_m(ad)^m}
$$

and summing with respect to n (where a or d is of the form q^{-n}), we get

$$
\phi\left[\begin{matrix} a^2, -aq^2, b^2, c^2:\; -\dfrac{aq}{w}, d;\; \dfrac{awq^2}{b^2c^2d} \\ q^2, -a, \dfrac{a^2q^2}{b^2}, \dfrac{a^2q^2}{c^2}:\; w, -\dfrac{aq}{d} \end{matrix}\right] = \sum_{r\geq 0}\frac{\left[\dfrac{a^2q^2}{b^2c^2};\; q^2\right]_r[a^2;\; q^2]_{2r}}{[q^2;\; q^2]_r\left[\dfrac{a^2q^2}{b^2};\; q^2\right]_r\left[\dfrac{a^2q^2}{c^2};\; q^2\right]_r}
$$

$$
\cdot\frac{[-aq^2;\; q^2]_r\left[-\dfrac{aq}{w};\; q\right]_r[d;\; q]_r\, w^r(-)^rq^{2r}}{[-a,\; q^2]_r[w:\; q]_r\left[-\dfrac{aq}{d};\; q\right]_r(ad)^rq^{r(r+1)}}
$$

$$
\cdot\phi\left[\begin{matrix} a^2q^{4r}, -aq^{2+2r}:\; -\dfrac{aq^{1+r}}{w}, dq^r;\; \dfrac{wq^{-2r}}{ad} \\ q^2, -aq^{2r}:\; wq^r, -\dfrac{aq^{1+r}}{d} \end{matrix}\right],
$$

Summing the inner ϕ-series by (1.4), we get (3.1) on some simplification.

In an earlier paper [17], we discussed some of interesting special cases of (3.1) for $d = q^{-n}$, now ew discuss some other interesting special cases of (3.1) for $a = q^{-n}$.

(i) In (3.1) setting $w = c = dq$, we get

$$
\phi\left[\begin{matrix} q^{-2n}, -q^{2-n}, b^2:\; d, -dq;\; \dfrac{q^{1-n}}{b^2d^2} \\ q^2, -q^{-n}, \dfrac{q^{2-2n}}{b^2}:\; \dfrac{q^{-n}}{d}, -\dfrac{q^{1-n}}{d} \end{matrix}\right]
$$

$$
\text{(3.3)}\qquad = \frac{[q;\; q]_n[-1;\; q]_nd^n}{[dq;\; q]_n[-d;\; q]_n}\cdot{}_2\phi_1\left[\begin{matrix} \dfrac{q^{-2n}}{b^2d^2}, d^2;\; q^2;\; q^2 \\ \dfrac{q^{2-2n}}{b^2} \end{matrix}\right]
$$

$$
= \frac{[q;q]_n[-1;\; q]_n[b^2d^2;\; q^2]_n[d^2q^2;\; q^2]_m[b^2;\; q^2]_{n-m}d^{n-2m}}{[dq;\; q]_n[-d;\; q]_n[b^2;\; q^2]_n[q^2;\; q^2]_m[b^2d^2;\; q^2]_{n-m}}.
$$

to $(m + 1)$ terms (where m is the greatest integer less than or equal to $n/2$) on using [**2**; p. 443 (iii)].

(ii) Setting $w = c$, $d = q^{1-n}/b^2c$ in (3.1) and then summing ${}_3\phi_2$ on the right-hand side by the q-analogue of Saalschütz summation theorem [**16**; 3.3.2.2] we get

$$\text{(3.4)}\quad \phi\begin{bmatrix} q^{-2n}, -q^{2-n}, b^2: \frac{q^{1-n}}{b^2c}, -c; q \\ \\ q^2, -q^{-n}, \frac{q^{2-2n}}{b^2}: -b^2c, \frac{q^{1-n}}{c} \end{bmatrix} = \frac{[-1; q]_n[b^2; q]_n[b^2c^2; q^2]_n}{[-b^2 c: q]_n[c; q]_n[b^2; q^2]_n}.$$

(iii) On the other hand for $w = c$, $d = q^{-n}/b^2c$, (3.1) yields the summation formula (using [**16**; 3.3.2.2]),

$$\phi\begin{bmatrix} q^{-2n}, -q^{2-n}, b^2: \frac{q^{-n}}{b^2c}, -c; q^2 \\ \\ q^2, -q^{-n}, \frac{q^{2-2n}}{b^2}: -b^2cq, \frac{q^{1-n}}{c} \end{bmatrix} = \frac{[-1; q]_n[b^2; q]_n[b^2c^2q; q]_n[b^2c^2; q^2]_n}{[-b^2cq; q]_n[c; q]_n[b^2c^2; q]_n[b^2; q^2]_n}.$$

(iv) Furthermore (3.1) for $w = c = q^{2-n}/d$ gives the summation forumla

$$\text{(3.6)}\quad \phi\begin{bmatrix} q^{-2n}, -q^{2-n}, b^2: d, -\frac{q^{2-n}}{d}; \frac{1}{b^2} \\ \\ q^2, -q^{-n}, \frac{q^{2-2n}}{b^2}: -\frac{q^{1-n}}{d}, \frac{d}{q} \end{bmatrix} = \frac{[b^2 q: q]_n\left[\frac{d^2}{q}; q\right]_n\left[\frac{d^2}{q^2}; q^2\right]_n[-1; q]_n}{\left[\frac{d}{q}; q\right]_n\left[\frac{d^2}{q^2}; q\right]_n[b^2q; q^2]_n[-d; q]_n}.$$

(v) In (3.1), replacing w by aq/w, setting $a = q^{-n}$ and $b \to 1$, we get

$$\text{(3.7)}\quad {}_4\phi_3\begin{bmatrix} q^{-n}, q^{1-n}, d^2, w^2; q^2; q^2 \\ q^{2-2n}, wd, wdq \end{bmatrix} = \frac{[d; q]_n[-w; q]_n + [-d; q]_n[w; q]_n}{[-1; q]_n\,[dw; q]_n},$$

provided $n \geqq 1$.

Now, we prove the following forumla which is different in nature from (3.7); viz.,

$$\text{(3.8)}\quad {}_4\phi_3\begin{bmatrix} q^{-n}, q^{1-n}, c^2, d^2; q^2; q^2 \\ q^{-2n}, cdq, cdq^2 \end{bmatrix} = \frac{[c; q]_{n+1}[-d; q]_{n+1} - [-c; q]_{n+1}[d; q]_{n+1}}{(d-c)\,[-1; q]_{n+1}\,[cdq; q]_n}.$$

PROOF OF (3.8). In the transformation [**10**]

$$(3.9)\qquad {}_4\phi_3\left[\begin{matrix}a^2, b^2, c^2, d^2; q^2; q^2\\ a^2b^2q, cd, cdq\end{matrix}\right] = {}_4\phi_3\left[\begin{matrix}a^2, b^2, -c, d; q, q\\ ab\sqrt{q}, -ab\sqrt{q}, cd\end{matrix}\right].$$

(provided either a, b, c or d is of the form q^{-n} with n a non-negative integer), setting $a^2 = q^{-2n}$ and then transforming the ${}_4\phi_3$ on the left-hand side by using [15]

$$(3.10)\qquad {}_4\phi_3\left[\begin{matrix}a, b, c, q^{-n}; q; q\\ d, e, f\end{matrix}\right] = \frac{[e/c; q]_n[de/ab; q]_n}{[e; q]_n[de/abc; q]_n}\,{}_4\phi_3\left[\begin{matrix}\frac{d}{a}, \frac{d}{b}, c, q^{-n}; q; q\\ d, \frac{c}{e}q^{1-n}, \frac{c}{f}q^{1-n}\end{matrix}\right],$$

provided $abcq^{1-n} = def$ (with first replacing q by q^2 and then replacing a, b, c, d, e, f by $b^2, c^2, d^2, b^2q^{1-2n}$, cd and cdq respectively), we get on some simplification

$$(3.11)\qquad {}_4\phi_3\left[\begin{matrix}q^{1-2n}, \frac{b^2}{c^2}q^{1-2n}, d^2, q^{-2n}; q^2; q^2\\ b^2q^{1-2n}, \frac{d}{c}q^{2-2n}, \frac{d}{c}q^{1-2n}\end{matrix}\right] = \frac{[cd; q]_{2n}}{d^{2n}[c/d; q]_{2n}}\,{}_4\phi_3\left[\begin{matrix}q^{-2n}, b^2, -c, d; q; q\\ bq^{\frac{1}{2}-n}, -bq^{\frac{1}{2}-n}, cd\end{matrix}\right].$$

However, in (3.11) replacing c by q^{-2n}/c and $b^2 \to q^{-2n-1}$, we get on some manipulation

$$(3.12)\qquad {}_4\phi_3\left[\begin{matrix}q^{-2n}, q^{1-2n}, c^2, d^2; q^2; q^2\\ q^{-4n}, cdq, cdq^2\end{matrix}\right] = \frac{[-d; q]_{2n+1}[c; q]_{2n+1} - [d; q]_{2n+1}[-c; q]_{2n+1}}{(d-c)[-1; q]_{2n+1}[cdq; q]_{2n}}.$$

On the otherhand it is easy to verify that

$$(3.13)\qquad {}_4\phi_3\left[\begin{matrix}q^{2-2n}, q^{1-2n}, c^2, d^2; q^2; q^2\\ q^{2-4n}, cdq, cdq^2\end{matrix}\right] = {}_4\phi_3\left[\begin{matrix}q^{-2n}, q^{1-2n}, c^2, d^2; q^2; q^2\\ q^{-4n}, cdq, cdq^2\end{matrix}\right] + \frac{(1-c^2)(1-d^2)q^{2-2n}}{(1-cdq)(1-cdq^2)(1+q^{-2n})(1+q^{1-2n})} \cdot {}_4\phi_3\left[\begin{matrix}q^{2-2n}, q^{3-2n}, c^2q^2, d^2q^2; q^2; q^2\\ q^{4-4n}. cdq^3, cdq^4\end{matrix}\right].$$

In (3.13) summing the two ${}_4\phi_3$'s on the right hand sidd by (3.12), we get

$$(3.14)\qquad {}_4\phi_3\left[\begin{matrix}q^{2-2n}, q^{1-2n}, c^2, d^2; q^2; q^2\\ q^{2-4n}, cdq, cdq^2\end{matrix}\right] = \frac{[-d; q]_{2n}[c; q]_{2n} - [d; q]_{2n}[-c; q]_{2n}}{(d-c)[-1; q]_{2n}[cdq; q]_{2n-1}}.$$

Now, combining (3.12) and (3.14), we get (3.8).

In (3.7) replacing d, w by q^d and q^w respectively and then let $q \to 1$, we get a summation formula for Saalschültzian ${}_4F_3$, different in nature than Bailey's [6; p. 512, c] summation formula for a terminating Saalschützian ${}_4F_3$:

$$(3.15)\quad {}_4F_3\left[\begin{matrix}-n/2,\ (1-n)/2,\ d,\ w; \\ 1-n,\ (w+d)/2,\ (1+d+w)/2\end{matrix}\right] = \frac{1}{(d+w)_n}\{(d)_n + (w)_n\},$$

whereas in (3.8) proceeding to the limit, as above, we get (cf., Carlitz [8]),

$$(3.16)\quad {}_4F_3\left[\begin{matrix}-n/2,\ (1-n)/2.\ c,\ d; \\ -n,\ (1+c+d)/2,\ (2+c+d)/2\end{matrix}\right] = \frac{(c)_{n+1} - (d)_{n+1}}{(c-d)(1+c+d)_n}.$$

It may be worthwhile to remark that Abiodum [1; 1] has recently given a nonterminating version of the formula of Carlitz [8] in the form

$$(3.17)\quad {}_4F_3\left[\begin{matrix}(1+a)/2,\ 1+a/2,\ 2b-c,\ c; \\ 1+a,\ 1/2+b,\ 1+b\end{matrix}\right] = \frac{b\Gamma(2b)\Gamma(2b-a-c)}{(b-c)\Gamma(2b-c)\Gamma(2b-a)},$$

provided $\operatorname{Re}(2b - a - c) > 0$. Formula (3.17) is clearly incorrect as can be seen by first setting $c = 1 + a$ and then $a \to -N - 1$ (or by setting $a = 2b$ and summing the left-hand side by Gauss' summation theorem).

(vi) Lastly, in (3.1) setting $a = q^{-n}$, $w = c$, $b = q$ and using (3.8), we get an interesting summation formula, summing to $(n + 1)$ terms:

$$(3.18)\quad \phi\left[\begin{matrix}-q^{-2n}:\ d,\ -c;\ q^{-n}/cd \\ -q^{-n}:\ \dfrac{-q^{1-n}}{d},\ \dfrac{q^{1-n}}{c}\end{matrix}\right] = \frac{[-d;q]_{n+1}[c;q]_{n+1} + (-1)^n[d;q]_{n+1}[-c;q]_{n+1}}{(1+q^n)(1-cdq^n)[-d;q]_n[c;q]_n}.$$

However on using (3.8), we prove the following interesting transformation

$$(3.19)\quad \begin{aligned} &{}_5\phi_4\left[\begin{matrix}q^{-2n}/b^2c^2,\ q^{-n},\ q^{1-n},\ d^2,\ e^2;\ q^2;\ q^2 \\ deq,\ deq^2,\ q^{-2n}/b^2,\ q^{-2n}/c^2\end{matrix}\right] = \frac{[-d;q]_{n+1}[e;q]_{n+1}}{(d-e)[-1;q]_{n+1}[deq;q]_n} \\ &\cdot \sum_{r=0}^{n+1} \frac{[q^{-2n-2};q^2]_r[b^2;q^2]_r[c^2;q^2]_r[-e;q]_r[d;q]_r}{[q;^2q^2]_r[q^{-2n}/b^2;q^2]_r[q^{-2n}/c^2;q^2]_r[q^{-n}/e;q]_r[-q^{-n}/d;q]_r} \\ &\cdot \frac{(1-q^{4r-2n-2})q^{-2nr}}{(1-q^{-2n-2})(b^2c^2de)^r}.\end{aligned}$$

PROOF OF (3.19). Multiplying both the sides of the summation theorem [16, 3.3.1.4]

$$(3.20)\quad {}_6\phi_5\left[\begin{matrix} a^2, aq^2, -aq^2, b^2, c^2, q^{-2r}; q^2; a^2q^{2+2r}/b^2c^2 \\ a, -a, a^2q^2/b^2, a^2q^2/c^2, a^2q^{2+2r} \end{matrix}\right] = \frac{[a^2q^2; q^2]_r[a^2q^2/b^2c^2; q^2]_r}{[a^2q^2/b^2; q^2]_r[a^2q^2/c^2; q^2]_r},$$

by

$$\frac{[d; q^2]_r[e^2; q^2]_r[x; q]_{2r}q^{2r}}{[q^2; q^2]_r[a^2q^2; q^2]_r[deq; q]_{2r}}$$

and summing with respect to r from 0 to ∞, we get

$$(3.21)\quad \begin{aligned} &{}_5\phi_4\left[\begin{matrix} a^2q^2/b^2c^2, x, xq, d^2, e^2; q^2; q^2 \\ deq, deq^2, a^2q^2/b^2, a^2q^2/c^2 \end{matrix}\right] \\ &= \sum_{s=0}^{\infty} \frac{[a^2; q^2]_s[b^2; q^2]_s[c^2; q^2]_s[d^2; q^2]_s}{[q^2; q^2]_s[a^2q^2/b^2; q^2]_s[a^2q^2/c^2; q^2]_s} \\ &\cdot \frac{[e^2; q^2]_s[x; q]_{2s}(-)^s a^{2s}q^{s(s+3)}}{[a^2; q^2]_{2s}[deq; q]_{2s}(b^2c^2)^s}\ {}_4\phi_3\left[\begin{matrix} xq^{2s}, xq^{1+2s}, d^2q^{2s}, e^2q^{2s}; q^2; q^2 \\ a^2q^{2+4s}, deq^{1+2s}, deq^{2+2s} \end{matrix}\right] \end{aligned}$$

Summing the inner ${}_4\phi_3$ on the right-hand side of (3.21) using (3.12) by specializing the parameters x and a, we get (3.19) on some manipulation.

A number of summation formulas could be obtained from (3.19) by specializing the parameters suitably. For instance we mention below some of the interesting summation formulas.

(i) In (3.19) setting $e = q^{-n}/c$ and then $c = dq$, we get

$$(3.22)\quad \begin{aligned} &\sum_{r=0}^{n+1} \frac{[q^{-2n-2}; q^2]_r(1-q^{4r-2n-2})[b^2; q^2]_r[d; q]_r[-dq; q]_r q^{r(1-n)}}{[q^2; q^2]_r(1-q^{-2n-2})[q^{-2n}/b^2; q^2]_r[-q^{-n}/d; q]_r[q^{-n-1}/d; q]_r(bd)^{2r}} \\ &= \frac{[-1; q]_{n+1}[q; q]_n(1-d^2q^{1+n})d^n}{[-d; q]_{n+1}[dq; q]_{n+1}}\ {}_2\phi_1\left[\begin{matrix} d^2, q^{-2n-2}/b^2d^2; q^2; q^2 \\ q^{-2n}/b^2 \end{matrix}\right] \\ &= \frac{[-1; q]_{n+1}[q; q]_n(1-d^2q^{1+n})[d^2q^2; q^2]_m[b^2d^2q^2; q^2]_n[b^2q^2; q^2]_{n-m}d^{n-2m}}{[-d; q]_{n+1}[dq; q]_{n+1}[q^2; q^2]_m[b^2q^2; q^2]_n[b^2d^2q^2; q^2]_{n-m}} \end{aligned}$$

to $(m + 1)$ terms (where m is the greatest integer less than or equal to $n/2$) on using [**2**; p. 433(iii)].

(ii) Setting $c = q^{-n}/e$, $b^2 = e/dq$ in (3.19) and then summing the resulting ${}_3\phi_2$ on the right hand side by the q-analogue of the Saalschütz summation theorem [**16**; 3.3.2.2], we get

$$(3.23)\quad \begin{aligned} &\sum_{r=0}^{n+1} \frac{[q^{-2n-2}; q^2]_r(1-q^{4r-2n-2})[eq/d; q^2]_r[-q^{-n}/e; q]_r[d; q]_r q^r}{[q^2; q^2]_r(1-q^{-2n-2})\left[\frac{d}{e}q^{1-2n}; q^2\right]_r[e; q]_r[-q^{-n}/d; q]_r} \\ &= \frac{(d-e)[-1; q]_{n+1}[eq/d; q]_n[deq; q^2]_n}{[-d; q]_{n+1}[e; q]_{n+1}[eq/d; q^2]_n}. \end{aligned}$$

(iii) On the otherhand for $c = q^{-n}/e$, $b^2 = e/dq^2$, (3.19) yields the summation formula (using [**16**; 3.3.2.2]),

$$\sum_{r=0}^{n+1} \frac{[q^{-2n-2}; q^2]_r(1-q^{-2n+4r-2})[e/dq^2; q^2]_r[-q^{-n}/e; q]_r[d; q]_r q^{2r}}{[q^2; q^2]_r(1-q^{-2n-2})\left[\frac{d}{e}q^{2-2n}; q^2\right]_r[e; q]_r[-q^{-n}/d; q]_r} \tag{3.24}$$
$$= \frac{(d-e)[-1; q]_{n+1}\,[deq; q]_n[de; q^2]_n[e/d; q]_n}{[-d; q]_{n+1}\,[e; q]_{n+1}[de; q]_n[e/d; q^2]_n}.$$

(iv) Furthermore (3.19) for $c = q^{-n}/e$, $d = eq$ gives the summation formula

$$\sum_{r=0}^{n+1} \frac{[q^{-2n-2}; q^2]_r(1-q^{4r-2n-2})[b^2; q^2]_r[-q^{-n}/e; q]_r[eq; q]_r}{[q^2; q^2]_r(1-q^{-2n-2})[q^{-2n}/b^2; q^2]_r[e; q]_r[-q^{-n-1}/d; q]_r b^{2r}q^r} \tag{3.25}$$
$$= \frac{e(q-1)[-1; q]_{n+1}[b^2q^2; q]_n}{(1-e)(1+eq^{1+n})[b^2q^2; q^2]_n}.$$

(v) Lastly, (3.19) for $c = q^{-n}/e$, $d = eq^2$ yields the summation formula

$$\sum_{r=0}^{n+1} \frac{[q^{-2n-2}; q^2]_r(1-q^{4r-2n-2})[b^2; q^2]_r[-q^{-n}/e; q]_r[eq^2; q]_r}{[q^2; q^2]_r(1-q^{-2n-2})[q^{-2n}/b^2; q^2]_r[e; q]_r[-q^{-n-2}/d; q]_r(bq)^{2r}} \tag{3.26}$$
$$= \frac{e(q^2-1)(1-e^2q^{2+n})[-1; q]_{n+1}[b^2q^2; q]_n}{(1-eq)(1-eq^{1+n})(1+eq^{2+n})[b^2q^2; q^2]_n}.$$

4. In this section using (3.1) for $a = q^{-n}$, we obtain the following q-analogue of (1.5):

$$L_N^{a^2}(z; q^2)L_N^{a^2}(zq; q^2) = \frac{[a^2q^2; q^2]_N[q; q^2]_N q^N a^{2N}}{[q^2; q^2]_N[q^2; q^2]_N} \tag{4.1}$$
$$\cdot \sum_{s=0}^{N} \frac{[q^{-2N}; q^2]_s[q; q^2]_s}{[a^2q^2; q^2]_s[q^{1-2N}; q^2]_s a^{2s}} L_{2s}^{a^2}(z; q),$$

where

$$L_N^{a}(z; q) = \frac{[aq; q]_N}{[q; q]_N}\sum_{r=0}^{N} \frac{[q^{-N}; q]_r z^r}{[q; q]_r[aq; q]_r}.$$

PROOF OF (4.1). In (3.1) setting $a = q^{-n}$ and then $d = q^{-n}/a$, $w = aq$, $b = q^{-N}$ $(N \geqq n)$, multiplying by

$$\frac{[q^{-2N}; q^2]_n[c^2; q^2]_n z^n}{[a^2q^2; q^2]_n[q; q]_n[-1; q]_n}$$

and summing with respect to n from 0 to N we get

$${}_2\phi_1\begin{bmatrix} q^{-2N}, c^2; q^2; z \\ a^2q^2 \end{bmatrix} {}_2\phi_1\begin{bmatrix} q^{-2N}, c^2; q^2; zq \\ a^2q^2 \end{bmatrix} = \sum_{n=0}^{N} \frac{[q^{-2N}; q^2]_n[c^2; q^2]_n}{[q; q]_n[a^2q^2; q^2]_n} \tag{4.2}$$
$$\cdot \frac{[a^2q^2; q]_n z^n q^{-n(n-1)/2}}{[a^2q; q]_n} \sum_{r=0}^{n} \frac{[q^{-n}; q]_r[c^2q^{2n-2N}; q^2]_r z^r}{[q^2; q^2]_r[a^2q^{1+n}; q]_r}.$$

Formula (4.2) for $c \to 0$ yields

$$\begin{aligned}(4.3)\qquad &\frac{[q^2;q^2]_N[q^2;q^2]_N}{[a^2q^2;q^2]_N[a^2q^2;q^2]_N} L_N^{a^2}(z;q^2)L_N^{a^2}(zq;q^2)\\ &= \sum_{n=0}^{N} \frac{[q^{-2N};q^2]_n[a^2q;q^2]_n q^{-n(n-1)/2}}{[q;q]_n]a^2q^2;q^2]_n[a^2q;q]_n} \sum_{r=0}^{n} \frac{[q^{-n};q]_r z^{n+r}}{[q^2;q^2]_r[a^2q^{1+n};q]_r}.\end{aligned}$$

However, we have

$$(4.4)\qquad z^n q^{-n} = [a^2q;q]_n \sum_{s=0}^{n} \frac{[q^{-n};q]_s q^{ns}}{[a^2q;q]_s} L_s^{a^2}(z;q),$$

which could be obtained by writing the series definition of $L_s^{a^2}(z;q)$ and then rearranging the series and summing the inner ${}_1\phi_0$.

Now, denoting the left-hand side of (4.3) by S, subtituting for $(zq^{-1})^{n+r}$ in (4.3) from (4.4) and rearranging the inner two series, we get

$$\begin{aligned}S &= \sum_{n=0}^{N} \frac{[q^{-2N};q^2]_n[a^2q;q^2]_n q^{-n(n-3)/2}}{[q;q]_n[a^2q^2;q^2]_n}\\ &\cdot \sum_{s=0}^{n} \frac{[q^{-n};q]_s\, q^{ns} L_s^{a^2}(z;q)}{[a^2q;q]_s}\, {}_3\phi_2\left[\begin{matrix} q^{-n}, q^{1+n}, 0; q; q\\ -q, q^{1+n-s}\end{matrix}\right].\end{aligned}$$

Summing the innermost ${}_3\phi_2$ by the following special case of Andrews **[4]** q-analogue of Whipple's theorem.

$${}_3\phi_2\left[\begin{matrix} a, q/a, 0; q; q\\ -q, e\end{matrix}\right] = \frac{q^{n(n+1)/2}[ea;q^2]_\infty[eq/a;q^2]_\infty}{[e;q]_\infty},$$

$a = q^{-n}$, we get

$$\begin{aligned}S &= \sum_{n=0}^{N} \frac{[q^{-2N};q^2]_n[a^2q;q^2]_n q^{2n}}{[a^2q^2;q^2]_n} \sum_{s=0}^{n} \frac{[q;q^2]_s(-)^s\, q^{s(s-1)} L_{2s}^{a^2}(z;q)}{[a^2q;q]_{2s}[q^2;q^2]_{n-s}}\\ &= \sum_{s=0}^{N} \frac{[q^{-2N};q^2]_s[q;q^2]_s(-)^s q^{s(s+1)} L_{2s}^{a^2}(z;q)}{[a^2q^2;q^2]_s[a^2q^2;q^2]_s}\, {}_2\phi_1\left[\begin{matrix} q^{-2N+2s}, a^2q^{1+2s}; q^2; q^2\\ a^2q^{2+2s}\end{matrix}\right],\end{aligned}$$

summing the innermost ${}_2\phi_1$ by the q-analogue of Vandermonde's theorem **[16**; 3.3.2.7], we get (4.1).

In this sequel we also prove the following two theorems.

THEOREM I. *If*

$$(4.5)\qquad \sum_{n=0}^{\infty} a_n t^n = {}_4\phi_3\left[\begin{matrix} b, -b, c, -c; q; tq\\ -q, d, -w\end{matrix}\right] {}_4\phi_3\left[\begin{matrix} b, -b, c, -c; q; t\\ -q, -d, w\end{matrix}\right]$$

and

$$(4.6)\qquad \sum_{n=0}^{\infty} b_n t^n = {}_4\phi_3\left[\begin{matrix} b, -b, c, -c, q; t\\ -q, d, -w\end{matrix}\right] {}_4\phi_3\left[\begin{matrix} b, -b, c, -c; q; tq\\ -q, -d, w\end{matrix}\right],$$

then

$$(4.7)\quad {}_6\phi_5\left[\begin{matrix} b^2, c^2, f^2, h^2, wd/q, wd; q^2; q^2/fh \\ b^2c^2, d^2, w^2, fh, fhq \end{matrix}\right] = \frac{1}{2}\sum_{n=0}^{\infty}\frac{[f;q]_n[-h;q]_n[wd/q;q]_n q^n}{[fh;q]_n[b^2c^2;q^2]_n}\{a_n + b_n\},$$

provided either b, c or f is of the form q^{-N} (N a non-negative integer).

THEOREM II. *If*

$$(4.8)\quad \sum_{n=0}^{\infty} a_n z^n = {}_4\phi_3\left[\begin{matrix} c, -c, d, -w; q; z \\ -q, b, -b \end{matrix}\right]{}_4\phi_3\left[\begin{matrix} c, -c, -d, w; q; zq \\ -q, b, -b \end{matrix}\right]$$

and

$$(4.9)\quad \sum_{n=0}^{\infty} b_n z^n = {}_4\phi_3\left[\begin{matrix} c, -c, d, -w; q; zq \\ -q, b, -b \end{matrix}\right]{}_4\phi_3\left[\begin{matrix} c, -c, -d, w; q; z \\ -q, b, -b \end{matrix}\right],$$

then

$$(4.10)\quad \frac{[c^2z^2;q^2]_\infty}{[-c^2z;q]_\infty[z;q]_\infty}\,{}_4\phi_4\left[\begin{matrix} b^2/c^2, c^2, d^2, w^2; q^2; c^2z^2/q \\ b^2, wd, wdq, c^2z^2 \end{matrix}\right] = \frac{1}{2}\sum_{n=0}^{\infty}\frac{[b^2;q^2]_n z^n}{[wd;q]_n[-c^2z;q]_n}\{a_n + b_n\}.$$

PROOF OF THEOREM (I). In (3.1) setting $a = q^{-n}$ and then replacing d by q^{1-n}/d, one may prove the theorem easily on some manipulation.

However, in Theorem (I) replacing b, c, d, f, h, w by q^b, q^c, q^d, q^f, q^h and q^w respectively and then letting $q \to 1$, we get that

$$(4.11)\quad \sum_{n=0}^{\infty} a_n t^n = {}_2F_1\left[\begin{matrix} b, c; t \\ d \end{matrix}\right]{}_2F_1\left[\begin{matrix} b, c; t \\ w \end{matrix}\right],$$

implies

$$(4.12)\quad {}_6F_5\left[\begin{matrix} b, c, f, h, (w+d)/2, (w+d-1)/2 \\ b+c, d, w, (f+h)/2, (f+h+1)/2 \end{matrix}\right] = \sum_{n=0}^{\infty}\frac{(f)_n(w+d-1)_n}{(f+h)_n(b+c)_n}a_n.$$

provided either b, c or f is a negative integer.

In (4.11) and (4.12) on replacing h, f by Nx and $-N$ respectively and then letting $N \to \infty$, we get a known result of Baily [7; Th IV] on some manipulation.

Proof of Theorem (II) follows on the lines of Theorem (I) (on setting $a = q^{-n}$ and then replacing b, w, by q^{1-n}/b and q^{1-n}/w respectively in (3.1)). Theorem (II) is a q-analogue of a result of Baily [7; Th. V].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ROORKEE, ROORKEE (U.P.), INDIA