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ON THE PICARD GROUP OF A COMPACT COMPLEX NILMANIFOLD

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1. Introduction. This paper deals with compact complex nilmanifolds. By a nilmanifold we mean a homogeneous space of a nilpotent Lie group. The nilmanifold we consider arises as the compact quotient of a simply connected nilpotent Lie group G by a lattice Γ of G. We write G/Γ to denote such a space. In general, G/Γ is a non-Kähler manifold, and in fact, it is Kähler if and only if it is a complex torus (see [5]). However, G/Γ is a generalization of the torus, and to this end, there is a canonically associated torus T given by

$$T = G/[G, G]/\pi(\Gamma), \tag{1.1}$$

where $G/[G, G]$ is a vector group and $\pi(\Gamma)$ is a lattice of $G/[G, G]$, π: $G \to G/[G, G]$ being the projection map. T plays an important role in the analysis of G/Γ. We point out that there is a holomorphic fibration of G/Γ over T where the fibre is the compact complex nilmanifold $N_1 = [G, G]/\Gamma_1$, $\Gamma_1 = \Gamma \cap [G, G]$. We let π: $G/\Gamma \to T$ also denote the bundle map.

Our main purpose is to give a description of the Picard group of G/Γ; that is, $\mathrm{Pic}(G/\Gamma)$, the group of holomorphic isomorphism classes of holomorphic line bundles on G/Γ. To this end, we obtain a partial generalization of the Appell-Humbert Theorem from the case of the complex torus to the case of G/Γ. Sakane [4] has shown that the first Chern class of any holomorphic line bundle $\mathscr{L}$ on G/Γ, $c_1(\mathscr{L})$, is represented by a unique hermitian form H defined on $G/[G, G]$. As a consequence of the Appell-Humbert Theorem, we know that H corresponds to the first Chern class of a line bundle on the complex torus T if and only if the imaginary part of H, A, is integral on the lattice $\pi(\Gamma)$. Consequently, we can factor $\mathscr{L}$ as

$$\mathscr{L} = \mathscr{L}_\lambda \otimes \pi^*\mathscr{L}_1, \tag{1.2}$$

where $\mathscr{L}_\lambda$ is the line bundle associated to some character λ of the lattice Γ and $\pi^*\mathscr{L}_1$ is the pullback of a line bundle $\mathscr{L}_1$ on T with $c_1(\mathscr{L}_1)$ determined by H. See Theorem 3 for details.

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Let $\mathrm{Pic}^\circ(G/\Gamma) = \{\mathscr{L} \in \mathrm{Pic}(G/\Gamma) \mid \delta(\mathscr{L}) = 0\}$ where $\delta\colon H^1(G/\Gamma, \mathcal{O}^*) \to H^2(G/\Gamma, Z)$ is the Bockstein map. In §3 we prove the following theorem.

THEOREM 2. *$\mathrm{Pic}^\circ(G/\Gamma)$ is a compact complex manifold.*

In fact, we show that $\mathrm{Pic}^\circ(G/\Gamma)$ is a finite sheeted disconnected covering of the dual torus of T, $\mathrm{Pic}^\circ(T)$. So $\mathrm{Pic}^\circ(G/\Gamma)$ is itself a compact complex manifold whose connected components are each biholomorphic to $\mathrm{Pic}^\circ(T)$. We note also that the number of components is dependent upon the order of $\Gamma_1/[\Gamma, \Gamma]$, where $\Gamma_1 = \Gamma \cap [G, G]$ and in fact, if the order is one, then $\mathrm{Pic}^\circ(G/\Gamma)$ and $\mathrm{Pic}^\circ(T)$ are holomorphically isomorphic. However, this order need not be one, and we present an example at the end of the paper showing this.

2. A review of the classical Appell-Humbert theorem. Let $T = V/\Lambda$ be a complex torus, where V is an n-dimensional complex vector space and Λ a lattice of V. Let H be a hermitian form on V and A its imaginary part. We consider all hermitian forms H on V such that A is integral valued on the lattice Λ. Then to such an H we associate a factor

$$e_\lambda(z) = \alpha(\lambda)\varepsilon\left[\frac{1}{2i} H(z, \lambda) + \frac{1}{4i} H(\lambda, \lambda)\right] \tag{2.1}$$

where $\lambda \in \Lambda$, $z \in V$, $\varepsilon[\cdot] = \exp 2\pi i[\cdot]$ and $\alpha\colon \Lambda \to \mathbf{C}_1^* = \{z \in \mathbf{C} \mid |z| = 1\}$ satisfies $\alpha(\lambda_1 + \lambda_2) = \alpha(\lambda_1)\alpha(\lambda_2)\varepsilon[A(\lambda_1, \lambda_2)/2]$; the function α being called a semi-character of Λ for A. In fact, the correspondence $\lambda \to e_\lambda$ is a group 1-cocycle on Λ with coefficients in H_V^* (see Appendix *I* of **[2]**). The factor $e_\lambda(z)$ defines a holomorphic action of the lattice Λ on $V \times \mathbf{C}$ via the rule

$$(z, a) \cdot \lambda = (z + \lambda, e_\lambda(z)a). \tag{2.2}$$

The action is free and the quotient of $V \times \mathbf{C}$ by Λ has a natural structure of a holomorphic line bundle over T. We shall denote this bundle by $\mathscr{L}(H, \alpha)$. Note that the map $(H, \alpha) \to \{e_\lambda\}$ satisfies the condition that if $\{e_\lambda^{(i)}\}$ corresponds to (H_i, α_i), then $\{e_\lambda^{(1)}e_\lambda^{(2)}\}$ corresponds to $(H_1 + H_2, \alpha_1\alpha_2)$. Thus, we have an isomorphism of line bundles $\mathscr{L}(H_1, \alpha_1) \otimes \mathscr{L}(H_2, \alpha_2) \cong \mathscr{L}(H_1 + H_2, \alpha_1\alpha_2)$.

The main theorem in the case of complex tori is the following theorem.

THEOREM 1. (APPELL-HUMBERT). *Any line bundle $\mathscr{L}$ on the complex torus T is isomorphic to an $\mathscr{L}(H, \alpha)$ for a uniquely determined (H, α) as indicated by the previous paragraph. We have isomorphic exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\Lambda, \mathbf{C}_1^*) \arrow[r] \arrow[d, "\beta"] & \{\text{Group of data } (H, \alpha)\} \arrow[r] \arrow[d, "\mu"] & \mathscr{H} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Pic}^\circ(T) \arrow[r] & \mathrm{Pic}(T) \arrow[r, "c_1"] & \mathrm{Ker}[H^2(T, Z) \to H^2(T, \theta_T)] \arrow[r] & 0
\end{tikzcd}$$

where $\mathrm{Pic}(T)$ *is the group of line bundles on* T, $\mathrm{Pic}^{\circ}(T)$ *the subgroup of those which are topologically trivial,* $\mathscr{H}$ *is the group of hermitian forms* $H\colon V \times V \to \mathbf{C}$ *with* $A(\Lambda \times \Lambda) \subset \mathbf{Z}$, *and the last vertical map is given by* $H \to A$ (*with the usual identification of* $H^2(T, Z)$ *with alternating 2-forms on* Λ).

REMARK. $\mu(H, \alpha) = \mathscr{L}(H, \alpha)$. Let $\hat{\Lambda} := \mathrm{Hom}(\Lambda, \mathbf{C}_1^*)$. Then for each $\alpha \in \hat{\Lambda}$, $\alpha \to \mathscr{L}(0, \alpha)$ is the first map in the top sequence. Finally, $c_1(\mathscr{L})$ is the first Chern class of the line bundle $\mathscr{L}$. See [2] for details.

3. $\mathrm{Pic}^{\circ}(G/\Gamma)$. In this section we give a geometric description of $\mathrm{Pic}^{\circ}(G/\Gamma)$ (see Theorem 2). To this end let $\alpha \in \hat{\Gamma} := \mathrm{Hom}(\Gamma, \mathbf{C}_1^*)$. Then α is a 1-dimensional holomorphic representation of Γ; and similar to (2.1), one obtains the group 1-cocycle given by $e_\gamma(g) = \alpha(\gamma)$ for each $\gamma \in \Gamma$. Thus, following (2.2) α defines a holomorphic line bundle $\mathscr{L}_\alpha$ over G/Γ. We now have the following proposition.

PROPOSITION 3.1. *The map* $\beta\colon \hat{\Gamma} \to \mathrm{Pic}^{\circ}(G/\Gamma)$ *defined by* $\beta(\alpha) = \mathscr{L}_\alpha$ *is an isomorphism.*

PROOF. Firstly, we can choose a finite cover $\mathscr{U} = \{U_j\}$ of G/Γ such that (1) $\bar{U}_j$ is compact, and (2) U_j is evenly covered by p; i.e., if $U_{j0} \subset p^{-1}(U_j)$ is a connected component, then p maps U_{j0} biholomorphically onto U_j and $p^{-1}(U_j) = \bigcup_{\gamma \in \Gamma} U_{j\gamma}$ is a disjoint union where $U_{j\gamma} := R_\gamma(U_{j0}) = \{g\gamma \mid g \in U_{j0}\}$. Let

$$\rho_j\colon U_j \to U_{j0} \tag{3.1}$$

be the inverse of $p\colon U_{j0} \to U_j$. It follows that for each pair of indices (j, k) such that $U_j \cap U_k \neq \varnothing$, there exists a unique $\gamma_{jk} \in \Gamma$ such that

$$\rho_k(x) = \rho_j(x)\gamma_{jk} \tag{3.2}$$

for all $x \in U_j \cap U_k$. Note also that the γ_{jk}'s satisfy the cocycle condition $\gamma_{j\ell} = \gamma_{jk}\gamma_{k\ell}$. Then relative to $\mathscr{U}$, the transition functions of $\mathscr{L}_\alpha$ are given by

$$g_{jk}(x) = \alpha(\gamma_{jk}) \tag{3.3}$$

for $x \in U_j \cap U_k$. The collection $\check{g} = \{g_{jk}\} \in \check{Z}'(\mathscr{U}, \mathcal{O}^*)$; i.e., $\check{g}$ is a Čech 1-cocycle. Computing the image of $\check{g}$ under the Bockstein map, we get $(\delta\check{g})_{jk\ell} = \alpha(\gamma_{jk}\gamma_{k\ell}\gamma_{j\ell}^{-1}) = 1$. So $\delta(\mathscr{L}_\alpha) \in H^2(G/\Gamma, \mathbf{Z})$ is represented by the Čech 2-cocycle $(\delta\check{g})_{jk\ell} = 1 \in Z^2(\mathscr{U}, \mathbf{Z})$ which is in fact a Čech 2-coboundary. Thus, it follows that $\delta(\mathscr{L}_\alpha) = 0$; i.e., the map β is well defined. Moreover, it is clear that β is a homomorphism.

Next, we show that β is injective. To this end, let $\alpha \in \hat{\Gamma}$ with $\beta(\alpha) \cong 1$, the trivial holomorphic line bundle on G/Γ. In group cocycle language, $\beta(\alpha) \cong 1$ means that $e = \{e_\gamma(g) = \alpha(\gamma)\}$ is cohomologous to the 1-cocycle $e' = \{e'_\gamma(g) = 1\}$. Thus, there exists a nonzero holomorphic function f

on G such that $\alpha(\gamma) = f(g\gamma)/f(g)$ ($\forall g \in G$, $\forall \gamma \in \Gamma$). We claim that f is a constant function on G. Relative to the cover $\mathscr{U}$ from above, the argument goes as follows. $\bar{U}_j$ compact implies that $\bar{U}_{j0}$ is compact. Moreover, $\bar{U}_{j\gamma} = R_\gamma(\bar{U}_{j0})$. Clearly, f is bounded on each $\bar{U}_{j0}$. Since there is only a finite number of the U_{j0}'s, it follows that f is a bounded entire function on G and hence is constant. Thus, $\alpha(\gamma) = 1$ for each $\gamma \in \Gamma$ implying that β is injective.

Finally, to show that β is surjective, it suffices to show that any $\mathscr{L} \in \mathrm{Pic}^\circ(G/\Gamma)$ can be realized by constant multipliers. In other words, if $\delta(\mathscr{L}) = 0$, then we will show that there is an element $C \in H^1(G/\Gamma, \mathbf{C}^*)$ such that $i^*C = \mathscr{L}$, where $i^*: H^1(G/\Gamma, \mathbf{C}^*) \to H^1(G/\Gamma, \mathscr{O}^*)$ is the induced map obtained from the inclusion of the constant sheaves $\mathbf{C}^* \subset \mathscr{O}^*$. Assuming the truth of the latter, we obtain from C a group 1-cocycle $e = \{e_\gamma\}_{\gamma \in \Gamma}$ of constant functions. In turn, we define $\alpha(\gamma) = e_\gamma(e)$ where e is the identity of G. The cocycle condition implies that $\alpha: \Gamma \to \mathbf{C}^*$ is a homomorphism. Using an argument similar to one in Weil [6, p. 93], we can then adjust C so that $\alpha \in \hat{\Gamma}$.

Lastly, we prove the existence of $C \in H^1(G/\Gamma, \mathbf{C}^*)$ such that $i^*C = \mathscr{L}$. Classically, it is known that for any complex manifold M one can compute Chern classes of line bundles by using the Bockstein operator δ. In fact, it is a theorem (cf. [7, pp. 106–109]) that the following diagram commutes:

(3.4)
$$\begin{tikzcd} H^1(M, \mathscr{O}^*) \arrow[r, "\delta"] \arrow[dr, "c_1"'] & H^2(M, \mathbf{Z}) \arrow[d, "j"] \\ & H^2(M, \mathbf{R}). \end{tikzcd}$$

Moreover, $c_1(H^1(M, \mathscr{O}^*)) = j\delta(H_1(M, \mathscr{O}^*))$, the cohomology classes in $jH^2(M, \mathbf{Z}) \subset H^2(M, \mathbf{R})$ which admit a d-closed differential form of type (1,1) as a representative. So let $\mathscr{L} \in \mathrm{Pic}^\circ(G/\Gamma)$. Then $c_1(\mathscr{L}) = j\delta(\mathscr{L}) = 0$ in $H^2(G/\Gamma, \mathbf{R})$ and hence by Proposition 3.4 of Sakane [4] there is a connection $\eta = (\eta_j)$ of type (1, 0) relative to $\mathscr{U}$ such that $d\eta_j = 0$ on U_j. Since $d\eta = 0$ on U_j, $\eta_j = df_j$ where $f_j: U_j \to \mathbf{C}$ is a holomorphic function. Thus we have

$$\eta_k - \eta_j = \frac{i}{2\pi} d \log g_{jk} \tag{3.5}$$

on $U_j \cap U_k \neq \emptyset$ where $\{g_{jk}\}$ is a set of transition functions for $\mathscr{L}$ relative to $\mathscr{U}$. Define

$$C_{jk} = g_{jk} \exp 2\pi i(f_k - f_j) \text{ on } U_j \cap U_k. \tag{3.6}$$

Then C_{jk} is constant on $U_j \cap U_k$. More importantly, the Čech 1-cocycle $\check{C} = \{C_{jk}\} \in Z^1(\mathscr{U}, \mathbf{C}^*)$ differs from the Čech 1-cocycle $\check{g} = \{g_{jk}\} \in Z^1(\mathscr{U}, \mathscr{O}^*)$ by the Čech coboundary $\check{\delta}(h)$, where $h = \{h_j = \exp 2\pi i f_j\} \in$

$\check{C}^\circ(\mathscr{U}, \mathscr{O})$. Hence, $\check{C}$ and $\check{g}$ define the same element $\mathscr{L}$ in $H^1(G/\Gamma, \mathscr{O}^*)$ and we have our assertion.

We now state and prove the main theorem of the section.

THEOREM 2. $\mathrm{Pic}^\circ(G/\Gamma)$ *is a compact complex manifold.*

As mentioned in the introduction, we show that $\mathrm{Pic}^\circ(G/\Gamma)$ is a finite sheeted disconnected covering of $\mathrm{Pic}^\circ(T)$, the latter already known to be a complex torus having the same dimension as T. Firstly, we have need of the following lemma.

LEMMA 3.1. *Let Λ denote the lattice Γ/Γ_1. There exists a surjective homomorphism $\hat{D}\colon \hat{\Gamma} \to \hat{\Lambda}$ having a finite kernel.*

PROOF. Firstly, note that $\Gamma/[\Gamma, \Gamma]$ is the direct sum of a free Abelian group of rank $2r$ and a finite Abelian group. The group $\Gamma_1/[\Gamma, \Gamma]$ where $\Gamma_1 = \Gamma \cap [G, G]$ is finite because $\Gamma/\Gamma_1 \cong (\Gamma/[\Gamma, \Gamma])/(\Gamma_1/[\Gamma, \Gamma])$ is a free Abelian group of rank $2r$ (cf. [**4**, p. 206]). Let k be the order of $\Gamma_1/[\Gamma, \Gamma]$. Note also that $\pi(\Gamma) = \Gamma[G, G]/[G, G] \cong \Gamma/\Gamma_1$. We now define a homomorphism $\hat{D}\colon \hat{\Gamma} \to \hat{\Lambda}$ by

$$\hat{D}(\alpha)(\bar{\gamma}) = \alpha(\gamma)^k \tag{3.7}$$

where $\alpha \in \hat{\Gamma}$, $\gamma \in \bar{\gamma} \in \Gamma/\Gamma_1$. That $\hat{D}$ is well defined follows immediately from the fact that k is the order of $\Gamma_1/[\Gamma, \Gamma]$. It is also clear that $\hat{D}$ is a homomorphism. To see that $\hat{D}$ has finite kernel, note that Ker $\hat{D} = \{\alpha \in \Gamma \mid$ image α is a subgroup of K, the k-th roots of unity group$\}$. Secondly, the lattice Γ is generated by a set of $2n = \dim_{\mathbf{R}} G$ elements (cf. [**3**, p. 42]) and K is a finite cyclic subgroup of $\mathbf{C}_1^*$. The cardinality of Ker $\hat{D}$ is the number of homomorphisms from Γ into K and this number is finite by the previous sentence.

To show that $\hat{D}$ is surjective, we use that Γ/Γ_1 is a free Abelian group of finite rank. Let $e_1, \ldots, e_{2r}$ be a Z-basis for Γ/Γ_1. Then each $\gamma \in \Gamma/\Gamma_1$ can be expressed uniquely as $\bar{\gamma} = \sum_i \lambda_i e_i$. Let $\alpha \in \hat{\Lambda}$. Then

$$\alpha(\bar{\gamma}) = \prod_i \alpha(e_i)^{\lambda_i}; \ \alpha(e_i) \in \mathbf{C}_1^*.$$

So choose a k-th root of $\alpha(e_i)$, say $\alpha(e_i)^{1/k}$, and fix it. Define

$$(\alpha(\gamma))^{1/k} = \prod_i (\alpha(e_i)^{1/k})^{\lambda_i}$$

for each $\gamma \in \Gamma$. Then $\alpha^{1/k} \in \hat{\Gamma}$ and $\hat{D}(\alpha^{1/k}) = \alpha$.

Using the previous lemma and Proposition 3.1 we proceed with the proof of the theorem by constructing a holomorphic homomorphism D: $\mathrm{Pic}^\circ(G/\Gamma) \to \mathrm{Pic}^\circ(T)$which forces the following diagram to commute:

(3.7)
$$\begin{array}{ccc} \hat{\Gamma} & \xrightarrow{D} & \hat{\Lambda} \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} \\ \mathrm{Pic}^\circ(G/\Gamma) & \xrightarrow{D} & \mathrm{Pic}^\circ(T); \end{array}$$

that is, $D = \beta\circ D\circ\beta^{-1}$. Explicitly, if $\mathscr{L}=\mathscr{L}_\alpha\in \mathrm{Pic}^\circ(G/\Gamma)$, then $D(\mathscr{L}) = \mathscr{L}(0, \hat{D}(\alpha))$ and the result is now clear.

COROLLARY 3.1. *Let* $\mathscr{L}_\alpha \in \mathrm{Pic}^\circ(G/\Gamma)$. *Then* $\mathscr{L}_\alpha^k = \mathscr{L}_\alpha \otimes \cdots \otimes \mathscr{L}_\alpha = \mathscr{L}_{\alpha^k} \cong \pi^*\mathscr{L}(0, \hat{D}(\alpha))$.

4. Pic(G/Γ). In this section we establish the factorization given by (1.2). To this end, let $\mathfrak{g}$ denote the Lie algebra of right invariant vector fields on G; I denotes the complex structure of $\mathfrak{g}$, and $\mathfrak{g}^+$ (resp. $\mathfrak{g}^-$) denotes the vector space of $\sqrt{-1}$ (resp. $-\sqrt{-1}$) eigenvectors of I in the complexification $\mathfrak{g}^{\mathbf{C}}$. Next, identify $\mathfrak{g}$ with the complex Lie algebra $(\mathfrak{g}, I)$. From Proposition 3.6 in [4] we can choose a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}^+$ such that $\{X_{r+1}, \ldots, X_n\}$ is a basis for $[\mathfrak{g}^+, \mathfrak{g}^+]$ and also such that the canonical coordinates of the second kind with respect to $\{X_1, \ldots, X_n\}$, denoted by $z_1, \ldots, z_n$, define a biholomorphic mapping $\Phi\colon G \to \mathbf{C}^n$ given by $\Phi(g) = (z_1(g), \ldots, z_n(g))$, where $z_1, \ldots, z_r$ are homomorphisms from G to $\mathbf{C}$, $r = \dim \mathfrak{g}^+/[\mathfrak{g}^+, \mathfrak{g}^+]$. Letting $H = (h_{jk}) \in M(r, \mathbf{C})$ be a hermitian matrix, we define a hermitian bihomomorphism $H\colon G \times G \to \mathbf{C}$ by

$$H(g_1, g_2) = \sum_{j,k=1}^{r} h_{jk}z_j(g)\bar{z}_j(g_2) \qquad \forall g_1, g_2 \in G. \tag{4.1}$$

Similarly, letting e_j ($j = 1, \ldots, r$) denote the standard basis of C^r and e_j^* ($j = 1, \ldots, r$) the corresponding dual basis, we obtain a hermitian form $\hat{H}$ on $\mathbf{C}^r$. Moreover, the map $\Phi_r(g) = (z_1(g), \ldots, z_r(g))$ is a holomorphic homomorphism of G to $\mathbf{C}^r$ with $\ker \Phi_r = [G, G]$. Φ_r then descends to a biholomorphism of $G/[G, G]$ onto $\mathbf{C}^r$. It is clear then that $\Phi_r^*\hat{H}(g_1, g_2) = H(g_1, g_2)$ and that the correspondence $\hat{H} \to H$ is injective.

We are interested in those hermitian bihomomorphisms H on G whose imaginary part A, difined by $A(g_1, g_2) = (H(g_1, g_2) - \overline{H(g_1, g_2)})/2i$, is integral valued on the lattice Γ. We note that A is integral valued on Γ if and only if the imaginary part of $\hat{H}$, $\hat{A}$, is integral valued on the lattice $\Phi_r(\Gamma)$ in $\mathbf{C}^r$. Note also that via Φ_r we can identify the complex torus $T = G/[G, G]/\pi(\Gamma)$ with $\mathbf{C}^r/\Phi_r(\Gamma)$ holomorphically and the lattice $\Lambda = \Gamma/\Gamma_1$ with $\Phi_r(\Gamma)$.

Let $\mathscr{L} \in \mathrm{Pic}(G/\Gamma)$. Then from Propositions 3.4 and 3.5 of [4] there exists a unique real right invariant 2-form $\alpha \in \Lambda^2\mathfrak{g}^*$ of type (1, 1) representing $c_1(\mathscr{L})$, and it is given by

$$\alpha = \frac{1}{2i} \sum_{j,k=1}^{r} h_{jk}dz_j\Lambda d\bar{z}_k \tag{4.2}$$

where (h_{jk}) is an $r \times r$ hermitian matrix and $r = \dim \mathfrak{g}^+/[\mathfrak{g}^+, \mathfrak{g}^+]$. Let $\hat{H}$ be the corresponding hermitian form on $\mathbf{C}^r$ as indicated above. Since $\hat{A}$ comes from an $\hat{H}$, it follows from the Appell-Humbert Theorem that $\hat{A}$ represents the Chern class of a holomorphic line bundle over T if and only if it is integral on the lattice $\Phi_r(\Gamma)$. Assuming that this is the case there exists $\mathscr{L}_1 \in \mathrm{Pic}(T)$ such that $c_1(\mathscr{L}_1)$ is represented by $\hat{A}$. Since $\pi^* c_1(\mathscr{L}_1) = c_1(\pi^*\mathscr{L}_1)$, it follows by an argument similar to the one in Proposition 3.1 showing the surjectivity of β that $\mathscr{L} \otimes (\pi^*\mathscr{L}_1)^{-1} \in \mathrm{Pic}^\circ(G/\Gamma)$. Hence, from Proposition 3.1 we obtain $\mathscr{L} = \mathscr{L}_\lambda \otimes \pi^*\mathscr{L}_1$ for some $\lambda \in \hat{\Gamma}$. Summarizing, we state the following theorem.

THEOREM 3. *Let $\mathscr{L} \in \mathrm{Pic}(G/\Gamma)$ and let α, as in* (4.2), *represent the Chern class of $\mathscr{L}$, $c_1(\mathscr{L})$. Then $\mathscr{L}$ can be expressed as*

$$\mathscr{L} = \mathscr{L}_\lambda \otimes \pi^*\mathscr{L}_1 \tag{4.3}$$

where $\lambda \in \hat{\Gamma}$ and $\mathscr{L}_1 \in \mathrm{Pic}(T)$ if and only if the imaginary part A of the hermitian bihomomorphism H defined by (4.1) *is integral on the lattice Γ.*

REMARK. Whether or not every $\mathscr{L} \in \mathrm{Pic}(G/\Gamma)$ can be written as in (4.3) is not known to the author.

5. An example. As mentioned in the introduction, $\Gamma_1 = \Gamma \cap [G, G] \supset [\Gamma, \Gamma]$ gives rise to $\Gamma_1/[\Gamma, \Gamma]$, a finite Abelian group of order possibly greater than one. Hence, it is possible to show the existence of a character λ on Γ which does not descend directly to a character on $\Lambda = \Gamma/\Gamma_1$. So in general, $\hat{\Gamma} \neq \hat{\Lambda}$ and hence $\mathrm{Pic}^\circ(G/\Gamma) \neq \mathrm{Pic}^\circ(T)$. In the language of exact sequences,

$$1 \to \Gamma_1 \xrightarrow{i} \Gamma \xrightarrow{\Phi_r} \Lambda \to 1$$

is exact, while

$$1 \to \hat{\Lambda} \xrightarrow{\Phi_r^*} \hat{\Gamma} \xrightarrow{i^*} \Gamma_1$$

is left exact. We provide an example showing that Φ_r^* is not surjective.

EXAMPLE. Let G be the simply connected complex nilpotent Lie group defined by

$$G = \left[\begin{bmatrix} 1 & z_{12} & z_{13} \\ 0 & 1 & z_{23} \\ 0 & 0 & 1 \end{bmatrix} \mid z_{ij} \in \mathbf{C},\ i < j \right]$$

and Γ be the lattice of G defined by

$$\Gamma = \left[\begin{bmatrix} 1 & 2x & z \\ 0 & 1 & 2y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbf{Z} \oplus i\mathbf{Z} \right].$$

One can show easily that $\Gamma_1/[\Gamma, \Gamma] \cong (\mathbf{Z} \oplus i\mathbf{Z})/4(\mathbf{Z} \oplus i\mathbf{Z})$. Next, define $\lambda\colon \Gamma \to \mathbf{C}_1^*$ by $\lambda(\gamma) = \varepsilon[(\operatorname{Re} c + \operatorname{Im} c)/4]$ where

$$\gamma = \begin{bmatrix} 1 & 2a & c \\ 0 & 1 & 2b \\ 0 & 0 & 1 \end{bmatrix}$$

and $c = \operatorname{Re} c + i \operatorname{Im} c$. Clearly, λ is a homomorphism. Further, if $c \in (\mathbf{Z} + i\mathbf{Z})\backslash 4(\mathbf{Z} + i\mathbf{Z})$, then $\lambda(\gamma) \neq 1$ where

$$\gamma = \begin{bmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, $\lambda|_{\Gamma_1} \neq 0$ and so $\Phi_r^*\colon \hat{\Lambda} \to \hat{\Gamma}$ is not surjective.

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