ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 13, Number 3, Summer 1983

KRULL DIMENSION OF DIFFERENTIAL OPERATOR RINGS II: THE INFINITE CASE

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In a recent paper [2], Goodearl and Warfield have considered the problem of computing the Krull dimension of the differential operator ring $R[\theta; \delta]$, when R is a commutative Noetherian ring with a derivation δ. They have given a reasonably complete description in the case that K.dim(R) is finite, but have only obtained partial results in the infinite case. Here we obtain a description of the infinite case that parallels the results of Goodearl and Warfield in the finite case. The notations and definitions of [2] will be used here and the reader is recommended to have a copy of that paper at hand since the proofs in this paper rely heavily on the methods of [2].

Throughout the paper, R will be a commutative Noetherian ring and δ a derivation on R. The differential operator ring $R[\theta; \delta]$ will be denoted by T.

The major result of [2] shows that if R has finite Krull dimension n then K.dim$(T) = n$ except when there is a maximal ideal M of height n with $\delta(M) \subseteqq M$ or char$(R/M) > 0$, in which case K.dim$(T) = n + 1$. Example 4.7 of [2] shows that the maximal ideals of R do not control the Krull dimension of T in the case that K.dim(R) is infinite. It will be shown here that if K.dim$(R) = \eta + n$, where η is a limit ordinal and n a natural number, then it is the prime ideals M such that K.dim$(R/M) = \eta$ that control the Krull dimension of T. For this reason we begin with a careful analysis of the limit ordinal case.

I would like to thank Ken Goodearl for his helpful comments.

THEOREM 1. *Let x be a non-zero divisor in R. Let R_x and R_C denote the localisations at the denominator sets $\{x^n | n = 0, 1, 2, \ldots\}$ and $C = \{1 - xr \mid r \in R\}$ of R.*

(i) *$\{x^n\}$ and C are denominator sets in T, and δ extends to the localisations of R by the quotient rule; so that there are natural isomorphisms $T_x \cong R_x[\theta; \delta]$ and $T_C \cong R_C[\theta; \delta]$.*

(ii) *The diagonal map from T to $T_x \oplus T_C$ is a faithfully flat embedding.*

This research was partially supported by a National Science Foundation grant.
Received by the editors on February 22, 1982.

(iii) $\text{K.dim}(R_C)$ *is not a limit ordinal.*

PROOF. (i) This is immediate from [**2**, Proposition 1.1].

(ii) The map is flat; so suppose that K is a maximal right ideal of T with $K_x = T_x$ and $K_C = T_C$. Then $1.x^n \in K$, for some n, and $1.(1 - xr) \in K$, for some $r \in R$. Suppose that n has been chosen to be as small as possible. If $n > 0$ then $x^{n-1} = x^n.r + (1 - xr)x^{n-1} \in K$, a contradiction. Thus $n = 0$ and $1 \in K$, therefore $K = T$. Hence the map is faithfully flat.

(iii) If x is a unit of R then R_c is the zero ring, in which case $\text{K.dim}(R_C) = -1$. Otherwise, the image of x in R_C is a nonzero divisor in R_C, and an easy calculation shows that it is in the Jacobson radical of R_C. Thus, by [**4**, Theorem 2.4], $\text{K.dim}(R_C)$ is not a limit ordinal.

COROLLARY 2. *Let* $\text{K.dim}(R) = \alpha$ *be a limit ordinal and* $\delta\text{-K.dim}(R) = \beta < \alpha$. *Suppose that* M *is a finitely generated* T*-module such that* M_x *is a finitely generated* R_x*-module. Then* $\text{K.dim}_T(M) < \alpha$.

PROOF. $\text{K.dim}(R_C) \leqq \text{K.dim}(R) = \alpha$. However, by Theorem 1, $\text{K.dim}(R_C) \neq \alpha$; so $\text{K.dim}(R_C) < \alpha$. Now $T_C \cong R_C[\theta;\, \delta]$, so that $\text{K.dim}(T_C) \leqq \text{K.dim}(R_C) + 1 < \alpha$, since α is a limit ordinal.

Because the map $T \to T_x \oplus T_C$ is faithfully flat, the map $N \to N_x \oplus N_C$ of the lattice of T-submodules of M preserves strict inclusions. Hence

$$\text{K.dim}_T(M) \leqq \text{K.dim}_{T_x \oplus T_C}(M_x \oplus M_C) = \max\{\text{K.dim}_{T_x}(M_x), \text{K.dim}_{T_C}(M_C)\}.$$

Now

$$\text{K.dim}_{T_C}(M_C) \leqq \text{K.dim}(T_C) < \alpha,$$

and, by [**2**, Theorem 1.6]

$$\text{K.dim}_{T_x}(M_x) \leqq \delta\text{-K.dim}(R_x) \leqq \delta\text{-K.dim}(R) = \beta < \alpha.$$

Therefore $\text{K.dim}(M) < \alpha$.

Using the above result, we are able to generalize [**2**, Proposition 4.2] to arbitrary ordinals.

THEOREM 3. *Let* R *be a commutative Noetherian differential ring and let* P *be a prime ideal of* R *such that* $\alpha = \text{K.dim}(R/P) = \eta + n$, *where* η *is a limit ordinal and* n *is a natural number. If* $\delta\text{-K.dim}(R) < \eta$ *then* $\text{K.dim}_T(T/PT) = \text{K.dim}(R/P)$.

NOTE. Proposition 4.2 of [**2**] is the above in the case that $\delta\text{-K.dim}(R)$ is finite. The proof given here follows the proof of [**2**, Proposition 4.2] closely, but uses Corollary 2 above to deal with the case of arbitrary limit ordinals.

PROOF. By [**2**, Proposition 1.2], $\text{K.dim}(T/PT) \geqq \alpha$. The reverse inequality is proved by showing that $K.\dim_T(A) < \alpha$, for any proper T-module factor A of T/PT. The proof of this by induction on α, beginning at $\alpha = \eta$. The inductive step is proved exactly as in [**2**, Proposition 4.2] with statements 'K.dim( ) is finite' replaced by 'K.dim( ) $< \eta$'; so we present only the case $\alpha = \eta$.

Without loss of generality, assume that no T-submodule of A has Krull dimension less than η. If $A = 0$ we are finished. Otherwise, by [**2**, Proposition 2.3], we may assume that A is annihilated by a power of P, that there exists $x \in R\backslash P$ such that A_x is a finitely generated R_x-module and that x is a non zero divisor modulo $\mathrm{ann}_R(A)$. Now, $\mathrm{ann}_R(A)$ is a δ-ideal of R, since A is a T-module, and, applying Corollary 2 to A viewed as a $T/\mathrm{ann}_T(A)$ – module,

$$K.\dim_T(A) = K.\dim_{T/\mathrm{ann}_T(A)}(A) < \eta.$$

COROLLARY 4. *If η is a limit ordinal such that δ-K.dim$(R) < \eta$ while* K.dim$(R) \geqq \eta$ *then* K.dim$(R[\theta;\,\delta])$ = K.dim(R).

PROOF. Set $T = R[\theta;\,\delta]$. Let $P_1, \ldots, P_n$ be the minimal prime ideals of R. If K.dim$(R/P_i) < \eta$, for some i, then K.dim$(T/P_iT) < \eta$ [**2**, Proposition 1.2], so K.dim$(T/P_iT) <$ K.dim(R). If K.dim$(R/P_i) \geqq \eta$, then K.dim(T/P_iT) = K.dim$(R/P_i) \leqq$ K.dim(R), by Theorem 3.

Therefore, K.dim$(T) = \max\{K.\dim_T(T/P_iT)\} \leqq$ K.dim(R). The reverse inequality is [**2**, Proposition 1.2].

Specializing this to the limit ordinal case gives the following result.

COROLLARY 5. *If η is a limit ordinal and* K.dim$(R) = \eta$, *then* K.dim$(R[\theta;\,\delta]) = \eta$ *unless* δ-K.dim$(R) = \eta$ *in which case* K.dim$(R[\theta;\,\delta])$ $= \eta + 1$.

PROOF. If δ-K.dim$(R) = \eta$ then K.dim$(R[\theta;\,\delta]) = \eta + 1$ by [**2**, Proposition 1.3].

Goodearl and Warfield have conjectured that when K.dim(R) is infinite then K.dim$(R[\theta;\,\delta]) = \max\{\delta\text{-K.dim}(R) + 1,\ \text{K.dim}(R)\}$, and Corollary 5 shows that this is the case if R has limit ordinal Krull dimension. However, the conjecture is not true in general, as the following example shows.

EXAMPLE 6. Let A be a commutative Noetherian Q-algebra with Krull dimension ω, the first limit ordinal, for example [**5**, p. 203]. Let K be the field of fractions of A and $\tilde{K}$ the algebraic closure of K. Let x, y be commuting indeterminates over $\tilde{K}$ and let $R = A[x, y]$, $R_1 = K[x, y]$ and $R_2 = \tilde{K}[x, y]$; so that $R \subseteqq R_1 \subseteqq R_2$ and R_2 is an integral extension of R_1. Let δ be the derivation on R_2 (and so also by restriction on R_1 and R) given by

$$\delta = 2y\frac{\partial}{\partial x} + (y^2 + x)\frac{\partial}{\partial y}.$$

Set $T = R[\theta;\, \delta]$. Note that $\text{K.dim}(R) = \omega + 2$; so $\text{K.dim}(T) \leqq \omega + 3$. We show that $\delta\text{-K.dim}(R) = \omega + 1$ and $\text{K.dim}(T) = \omega + 3$, so that $\text{K.dim}(T) \neq \max\{\text{K.dim}(R), \delta\text{-K.dim}(R) + 1\}$.

Now $xR + yR$ is a δ-prime ideal of R and $R/xR + yR \cong A$, so $\delta\text{-K.dim}(R/xR + yR) = \omega = \text{K.dim}(R/xR + yR)$. Hence, $\text{K.dim}(T/xT + yT) = \text{K.dim}((R/xR + yR)[\theta;\, \delta]) = \omega + 1$, by Corollary 5. Now, for any prime P of R, T/PT is a critical T-module [**2**, Lemma 2.1]; so the proper chain of prime ideals $0 \leqq xR \leqq xR + yR$ forces $\omega + 1 = \text{K.dim}(T/xT + yT) < \text{K.dim}(T/xT) < \text{K.dim}(T)$. Thus $\text{K.dim}(T) \geqq \omega + 3$; so $\text{K.dim}(T) = \omega + 3$.

Now $\delta\text{-K.dim}(R) \geqq \omega + 1$ since $\delta\text{-K.dim}(R/xR + yR) = \omega$. Suppose that $\delta\text{-K.dim}(R) \geqq \omega + 2$. Then there exists δ-prime ideals $0 < P < Q$ of R with $\delta\text{-K.dim}(R/Q) = \omega$ and $\delta\text{-K.dim}(R/P) > \omega$. Note that $\text{K.dim}(R/Q) \geqq \delta\text{-K.dim}(R/Q) = \omega$. Hence $A \cap Q = 0$, for otherwise R/Q is a homomorphic image of $(A/A \cap Q)[x, y]$ and so has finite Krull dimension. Thus in R_1 there is a proper chain of δ-prime ideals $0 < PR_1 < QR_1$. Since R_2 is integral over R_1 there is a proper chain of prime ideals $0 < \tilde{P} < \tilde{Q}$, such that $\tilde{P}$ is minimal over PR_2 and $\tilde{Q}$ is minimal over QR_2. By [**6**, Theorem 1] $\tilde{P}$ and $\tilde{Q}$ are δ-prime ideals of R_2, and hence $\delta\text{-K.dim}(R_2) \geqq 2$. However, by [**2**, Example 2.15], $\delta\text{-K.dim}(R_2) = 1$. Hence $\delta\text{-K.dim}(R) = \omega + 1$.

In order to find a formula for the Krull dimension of $R[\theta;\, \delta]$ in the general case, it is necessary to look at arbitrary prime factor rings R/P with Krull dimension a limit ordinal. To retain a small amount of clarity, the cases of characteristic zero and characteristic non zero are presented separately.

LEMMA 7. *Let P be a prime ideal of R such that* $\text{char}(R/P) = 0$ *and that* $\text{K.dim}(R/P) = \eta$ *is a limit ordinal.*

(i) *If* $\delta(P) \subseteqq P$ *and* $\delta\text{-K.dim}(R/P) = \eta$, *then* $\text{K.dim}(T/PT) = \eta + 1$.

(ii) *If* $\delta(P) \subseteqq P$ *and* $\delta\text{-K.dim}(R/P) < \eta$, *then* $\text{K.dim}(T/PT) = \eta$.

(iii) *If* $\delta(P) \subsetneqq P$, *then* $\text{K.dim}(T/PT) = \eta$.

PROOF. (i) and (ii) are just Corollary 5 applied to the ring R/P. (iii) $\text{K.dim}(T/PT) \geqq \eta$, by [**2**, Proposition 1.2]. An easy adaptation of the argument due to Hart [**3**, Lemma 2.4] gives the reverse inequality.

For any ideal I of R, set

$$(I\colon \delta) = \{r \in R \mid \delta^n(r) \in I, \text{ for all } n = 0, 1, 2, \ldots\}.$$

Then $(I\colon \delta)$ is the largest δ-ideal contained in I.

LEMMA 8. *Let P be a prime ideal of R such that* $\mathrm{char}(R/P) > 0$ *and that* $\mathrm{K.dim}(R/P) = \eta$. *Then* $\mathrm{K.dim}(T/PT) = \eta + 1$.

PROOF. By [1, Lemma 13], if Q is a prime ideal containing P then $Q/(Q{:}\ \delta)$ is nilpotent. It follows that the map $Q \to (Q{:}\ \delta)$ is an order isomorphism from the set of primes of R containing P to the set of δ-primes of $R/(P{:}\ \delta)$. Hence δ-$\mathrm{K.dim}(R/(P{:}\ \delta)) = \eta$. Therefore, by [2, Proposition 1.3], $\mathrm{K.dim}(T/(P{:}\ \delta)T) = \eta + 1$. Since $P/(P{:}\ \delta)$ is nilpotent, we may choose a series of ideals $(P{:}\ \delta) = A_0 \leqq A_1 \leqq \cdots \leqq A_n = R$ such that each factor is either isomorphic to R/P or a prime homomorphic image of R/P. Thus there are right ideals $(P{:}\ \delta)T \leqq A_1T \leqq \cdots \leqq A_nT = T$ such that each factor is isomorphic to a homomorphic image of T/PT. Hence $\eta + 1 = \mathrm{K.dim}(T/(P{:}\ \delta)T) = \max(\mathrm{K.dim}(A_{i+1}T/A_iT)) \leqq \mathrm{K.dim}(T/PT) \leqq \mathrm{K.dim}(R/P) + 1 = \eta + 1$; so $\mathrm{K.dim}(T/PT) = \eta + 1$.

In order to make it easier to compare our general result with that of Goodearl and Warfield, we shall say that, given a limit ordinal η, a prime ideal P of R is *η-maximal* if $\mathrm{K.dim}(R/P) = \eta$. All that remains to be done is to rephrase Proposition 2.7 and Theorem 2.9 of [2] in terms of η-maximal ideals and to check that the proofs go through.

PROPOSITION 9. *Let η be a limit ordinal and let P be a prime ideal such that* $\mathrm{K.dim}(R/P) = \eta + n$, *for some natural number $n \geqq 1$. Set*

$$\eta + m = \max\{\mathrm{K.dim}(T/QT) \mid Q \textit{ prime in } R \textit{ and } P < Q\}.$$

Then $\mathrm{K.dim}(T/PT) = \eta + m + 1$.

PROOF. As in [2, Proposition 2.7].

THEOREM 10. *Let I be an ideal of R and let η be a limit ordinal such that* $\mathrm{K.dim}(R/I) = \eta + n$, *for some natural number n. Let*

$$\mathcal{M} = \{M \lhd R \mid M \textit{ is } \eta\textit{-maximal and } I \subseteqq M \textit{ and either}$$
$$\text{(i) } \delta(M) \subseteqq M \textit{ and } \delta\text{-}\mathrm{K.dim}(R/M) = \eta, \textit{ or}$$
$$\text{(ii) } \mathrm{char}(R/M) > 0\}.$$

Set $m = \max\{\mathrm{height}(M/I) \mid M \in \mathcal{M}\}$, *with* $m = -1$ *if* $\mathcal{M} = \emptyset$. *Then* $\mathrm{K.dim}(T/IT) = \max\{\eta + (m + 1), \mathrm{K.dim}(R/I)\}$.

PROOF. As in [2, Theorem 2.9], using Lemmas 7 and 8 in place of [2, Lemma 2.8].

COROLLARY 11. *Let* $\mathrm{K.dim}(R) = \eta + n$, *for some limit ordinal η and natural number n. Set* $\mathcal{M} = \{M \lhd R \mid M$ *is η-maximal and either* (i) $\delta(M) \subseteqq M$ *and* δ-$\mathrm{K.dim}(R/M) = \eta$ *or* (ii) $\mathrm{char}(R/M) > 0\}$ *and set* $m =$

$\max\{\text{height}(M) | M \in \mathcal{M}\}$, *with* $m = -1$ *if* $\mathcal{M} = \emptyset$. *Then* $\text{K.dim}(R[\theta; \delta]) = \max\{\eta + (m+1), \text{K.dim } R\}$.

If R is an algebra over a field of finite characteristic then it is easy to see, from Lemma 8, that $\text{K.dim}(R[\theta; \delta]) = \text{K.dim}(R) + 1$. In the case that R is a Q-algebra we can give the following slight improvement to Corollary 4.

THEOREM 12. *Let R be a Q-algebra with* δ-$\text{K.dim}(R) \leqq \eta$, *for some limit ordinal* η. *Suppose that* $\text{K.dim}(R) > \eta$. *Then* $\text{K.dim}(R[\theta; \delta]) = \text{K.dim}(R)$.

PROOF. If $\text{K.dim} R \geqq \eta + \omega$, then Corollary 4 applies. Otherwise, suppose that $\text{K.dim}(R) = \eta + n$, for some natural number $n \geqq 1$. Consider the set M defined in Theorem 10. If $\mathcal{M} = \emptyset$ then $\text{K.dim}(R[\theta; \delta]) = \max\{\eta + (-1 + 1), \text{K.dim}(R)\} = \text{K.dim}(R)$. Otherwise, let $M \in \mathcal{M}$. Then, since $\text{char}(R/M) = 0$, δ-$\text{K.dim}(R/M) = \eta$. Now minimal prime ideals of R are δ-primes; so, since δ-$\text{K.dim}(R) \leqq \eta$, M must be a minimal prime. Thus $m = \max\{\text{height}(M) | M \in \mathcal{M}\} = 0$ and $\text{K.dim}(R[\theta; \delta]) = \max\{\eta + 1, \text{K.dim}(R)\} = \text{K.dim}(R)$.

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