ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 13, Number 3, Summer 1983

CONVEX POLYTOPES AND RETRACTIONS OF ABELIAN GROUPS

WALTER S. SIZER

Introduction. For any group G, let $F(G)$ denote the semigroup of finite non-empty subsets of G. A semigroup homomorphism σ: $F(G) \to G$ satisfying $\sigma(\{g\}) = g$ for all g in G is called a *retraction* of G. The notion of a group admitting a retraction generalizes the notion of a lattice-ordered group because in any lattice-ordered group the mapping $A \to \wedge A$ is a retraction (cf., [**1**]). This example of a retraction induced by a lattice order has the property that the effect of the mapping on $F(G)$ is determined uniquely by its effect on two element subsets. This is not so for all retractions, and [**1**, example 6.1], gives an instance where two distinct retractions agree on all two element subsets. The question of whether distinct retractions can agree on sets of cardinality less than or equal to n for arbitrary n is dealt with in this paper.

Also, in looking at a retraction σ on a group G, the notion which corresponds to that of an l-subgroup is the notion of a σ-subgroup—a subgroup H of G such that σ restricted to $F(G)$ is a retraction of H. In this paper we also deal with the question of whether a subgroup H of G with the property that all sets in $F(H)$ of cardinality less than n get mapped by σ to H must necessarily be a σ-subgroup.

Our approach considers only retractions of divisible abelian groups and builds on observations made in [**3**] and [**4**]. In the process of studying retractions we get a correspondence between retractions and homomorphisms from a semigroup of convex polytopes in Q^n to Q^n, so some of our results are essentially geometric in nature.

I. Retractions and convex polytopes. Throughout, G will be a torsion free divisible abelian group, hence a rational vector speace. For convenience we take G to be of finite rank.

If σ is any retraction of G, and A, B, C are sets satisfying $A + C = B + C$, then $\sigma(A) = \sigma(B)$. Hence for A, B in $F(G)$, we define $A \sim B$ if there is a C in $F(G)$ with $A + C = B + C$. The following proposition is then easy to verify.

Received by the editors on January 13, 1981, and in revised form on May 17, 1982.

Proposition 1. *The relation* $\sim$ *is a cancellative congruence.*

Proof. Omitted.

If for the moment we denote the equivalence class of A under $\sim$ by $[A]$, we have observed that σ takes on the same value on all sets in $[A]$. Hence we can factor any retraction of G through $F(G)/\sim$.

The following proposition gives an alternate way of conceptualizing $\sim$.

Proposition 2. $A \sim B$ *if and only if the convex hull of* A (*in the rational vector space* G) *equals the convex hull of* B.

Proof. If: This is proved in [3] in the proof of lemma 11. Only if: Suppose $A \sim B$; then $A \sim A \cup B$, so it suffices to show that if $A \sim A \cup \{x\}$, then x is in the convex hull of A. Suppose $A + C = (A \cup \{x\}) + C$. Let $c_0 \in C$. From the above equation we get successively relations

$$x + c_0 = a_1 + c_1,\ a_1 \in A,\ c_1 \in C;$$
$$x + c_1 = a_2 + c_2,\ a_2 \in A,\ c_2 \in C;$$
$$x + c_2 = a_3 + c_3.\ a_3 \in A,\ c_3 \in C;$$
$$\vdots$$
$$x + c_{n-1} = a_n + c_n,\ a_n, \in A,\ c_n \in C;$$
$$\vdots$$

Since C is finite, for some pair of integers $i, j,\ i \neq j,\ c_i = c_j$. Without loss of generality we take $j = n,\ i = 0$. Summing the first n equations above and setting c_n equal to c_0, we get $nx + \sum c_i = \sum a_i + \sum c_i$, or $nx = a_1 + \cdots + a_n$. Then $x = (1/n)\ a_1 + \cdots + (1/n)a_n$, so x is in the convex hull of A.

The set of convex polytopes in G forms a semigroup S_0 under the addition $P + Q = \{p + q \mid p \in P,\ q \in Q\}$. Using this notation we get the following result.

Proposition 3. $F(G)/\sim\ \cong S_0$.
(This was noted in [4] for the case of $G = Q^2$).

Proof. That the mapping $[A] \to$ convex hull of A is 1–1 and onto follows from proposition 2. That this mapping is a homomorphism follows from the fact that {extreme points of $A + B$} $\subseteqq$ {extreme point of A} + {extreme points of B}, itself a consequence of the equation min $(A + B)$ = min (A) + min (B) for any total order on G.

Our observations so far are summed up as follows.

COROLLARY. *There is a* 1-1 *correspondence between retractions of* G *and semigroup homomorphisms* $\sigma^*: S_0 \to G$ *satisfying* $\sigma^*(\{g\}) = g$.

We can eliminate the added condition on σ^* in the corollary by passing to a subsemigroup of S_0. Let P be a positive cone of a G for some total order, and let $S = \{Q \in S_0 | Q \subseteqq P, \min Q = 0\}$. S is a subsemigroup, and any Q_0 in S_0 can be written uniquely as $p + Q$, p in G, Q in S (namely $p = \min Q_0$, $Q = Q_0 - \min Q_0$). Any homomorphism $\sigma^*: S_0 \to G$ satisfying $\sigma^*(\{g\}) = g$ restricts to a homomorphism $\sigma^{**}: S \to G$, and any homomorphism $\sigma^{**}: S \to G$ extends uniquely to a homomorphism $\sigma^*: S_0 \to G$ satisfying $\sigma^*(\{g\}) = g$ for all g in G by the rule $\sigma^*(Q_0) = \min Q_0 + \sigma^{**}(Q_0 - \min Q_0)$. Thus we get the following result.

COROLLARY. *There is a* 1-1 *correspondence between retractions of* G *and semigroup homomorphisms* $\sigma: S \to G$.

In what follows we shall make use of the fact that S, as subsemigroup of the cancellative semigroup S_0, is cancellative. In the next section we give a way of describing semigroup homomorphisms from S to G.

II. Algebra of polytopes. The cancellative, commutative semigroup S can be embedded in a group $Q(S)$ of quotients by adjoining formal inverses, and $Q(S)$ will have a vector space structure over the rational numbers which agrees with the multiplication by non-negative rational numbers defined on S.

We say a set $X = \{x_a | a \text{ in } A\}$ of polytopes in S is *independent* if it is independent in the vector space $Q(S)$, and that X *forms a basis* of S if it is a basis in $Q(S)$. We get this simple result.

PROPOSITION 4. *Let* Y *be a basis in* S. *An arbitrary map* $\sigma: Y \to G$ *extends unqiuely to a homomorphism* $\sigma^*: S \to G$.

PROOF. Clear.

Thus to define a homomorphism $S \to G$—and hence a retraction $F(G) \to G$—it suffices to define an arbitrary map from a basis in S to G. Bases in S are not easy to construct, however. One is given, without verification, at the close of [**4**] for the case $G = Q^2$. We shall limit our inquiries here to properties of bases in S.

A k-simplex in S is a polytope with $k + 1$ vertices which span a k-dimensional affine subspace of G.

Let $P \in S$, and let H be a hyperplane in G such that (1) P lies in one of the closed half-spaces determined by H, and (2) $P \cap H \neq \emptyset$. Then $F = P \cap H$ is a face of P. The hyperplane H can be described as $\{x \text{ in } G | \langle x, v \rangle = c\}$ for suitable v in G and c rational. By replacing v by $-v$ and

c by $-c$ if necessary, we can assume that $c = \max\{\langle y, v\rangle | y \text{ in } P\}$; in this case, $F = F(P, v)$ is called the face of P with outer normal v. Going backwards, given v in G, $v \neq 0$, $H = \{x \text{ in } G | \langle x, v\rangle = \max\{\langle y, v\rangle | u \text{ in } P\}\}$ is a hyperplane and $H \cap P = F(P, v)$, (see **[2]**). We denote by $F^*(P, v)$ the translate of $F(P, v)$ in S. By [2, p. 317, theorem 1], if $P + Q = R$ for P, Q, R in S, then $F^*(P, v) + F^*(Q, v) = F^*(R, v)$ for all v in $G - \{0\}$. We are now able to prove our next result.

THEOREM 5. *Let X be a k-simplex in S. Let $X_1, \ldots, X_r$ be polytopes in S of dimension at most k-1. Then no relation $X + \sum a_i X_i = 0$ holds for a_i rational.*

PROOF. We will show that no relation $X + \sum r_i X_i = \sum s_i X_i$ holds for r_i, s_i non-negative. We use induction on k. If $k = 1$ the result is trivially valid. Suppose the theorem holds for k-1, and we will show it is true for k also. Let v be the outer normal of a k-1 dimensional face of X. Then (1) $F^*(X, v) + \sum r_i F^*(X_i, v) = \sum s_i F^*(X_i, v)$. $F^*(X, v)$ is k-1 dimensional. $F^*(X_i, v)$ is k-1 dimensional only if $X_i \perp v$, in which case $F^*(X_i, v) = F^*(X_i, -v)$. But we also have (2) $F^*(X, -v) + \sum r_i F^*(X_i, -v) = \sum s_i F^*(X_i, -v)$, and the first term here is zero-dimensional. Combining (1) and (2), we get

$$\begin{aligned} &F^*(X, v) + \textstyle\sum r_i F^*(X_i, v) + \sum s_i F^*(X_i, -v) \\ &\quad = F^*(X, -v) + \textstyle\sum r_i F^*(X_i, -v) + \sum s_i F^*(X_i, v). \end{aligned} \tag{3}$$

In (3) the k-1 dimensional summands of the form $r_i F^*(X_i, v)$ and $s_i F^*(X_i, -v)$ on the left are balanced by terms $r_i F^*(X_i - v)$ and $s_i F^*(X_i, v)$ on the right, and since S is cancellative we cancel them. On the left we have remaining a single term of dimension k-1, namely $F^*(X, v)$, and on the right no terms of dimension k-1. This is not possible by our induction hypothesis, so the theorem is valid.

COROLLARY. *If G is n-dimensional, any basis in S contains a polytope with at least $n + 1$ vertices.*

PROOF. By theorem 5, a basis cannot consist solely of polytopes of dimension less than n. A polytope of dimension n contains at least $n + 1$ vertices.

Our next objective is to show that there exist bases in S which consist only of simplices. First we need the following Lemma.

LEMMA 6. *Let $P \in S$ be a polytope. Let H be a hyperplane which cuts P. Let P_0, P_1 be the polytopes formed by intersecting P with the closed half-spaces determined by H, and let $F = P \cap H$. Suppose $0 \in P_0$, and set $P_1' = P_1 - \min P_1$, $F' = F - \min F$. Then $P + F' = P_0 + P_1'$.*

PROOF. First we show that min P_1 = min F. Let a = min P_1, b = min$F$. Then $b \geqq a$, 0, so for every rational number r in [0, 1], $b \geqq ra + (1 - r)0$. Since 0 is in one closed half-space determined by H and a is in the other, some $ra + (1 - r)0$ is in H, so is in F. But then by the minimality of b, $ra \geqq b$, so $ra = b$. But also $ra \leqq a$, so $ra = a$, and $a = b$.

Hence, to show $P + F' = P_0 + P_1'$, we need only show that $P + F = P_0 + P_1$.

$P + F \subseteqq P_0 + P_1$: Let $p \in P$, $f \in F$. Then $p \in P_i$, $f \in P_{1-i}$, so $p + f \in P_0 + P_1$.

$P_0 + P_1 \subseteqq P + F$: Let $a \in P_0$, $b \in P_1$. For some rational number r in [0, 1], $ra + (1 - r)b \in H$. Then $a + b = (ra + (1 - r)b) + ((1 - r)a + rb)$ is in $F + P$.

THEOREM 7. *There is a basis in S consisting only of simplices.*

PROOF. We need a result of Tverberg [5], namely that any polytope can be dissected into simplices by first cutting it with a hyperplane, then cutting one of the pieces with a hyperplane, and continuing this process for a finite number of steps.

Let X be an independent set of polytopes in S maximal subject to containing only simplices. We want to show that, for any polytope $P \in S$, $P + \sum r_i X_i = \sum s_i X_i$, for r_i, s_i non-negative rational numbers, X_i in X. We use induction on k, the dimension of P. If $k = 1$, P is a simplex, so by the maximality of X, either P is in X or P can be expressed as a linear combination of elements of X, and we can get the desired relationship. Assume then that any polytope of dimension at most k-1 satisfies an equation of the desired form.

Using Tverberg's result and lemma 6, we get equations

$$P + F_1' = P_0' + P_1',$$
$$P + F_1' + F_2' = P_0'' + P_1'' + P_2'',$$

where F_2' is the face we get cutting P_i' with a hyperplane, and P_i'' and P_2'' are the resulting pieces, translated to lie in S. Eventually we get

$$(*) \qquad P + \sum F_j' = \sum P_j^{(k)},$$

where the $P_j^{(k)}$ are simplices and the F_j' have dimension at most k-1. By induction, we get relations $F_j' + \sum r_{ij} X_i = \sum s_{ij} X_i$, and by maximality of X we get equations $P_j^{(k)} + \sum a_{ij} X_i = \sum b_{ij} X_i$. Then, adding $\sum r_{ij} X_i + \sum a_{ij} X_i$ to both sides of (*), we get

$$P + \sum F_j' + \sum r_{ij} X_i + \sum a_{ij} X_i = \sum P_j^{(k)} + \sum a_{ij} X_i + \sum r_{ij} X_i,$$

or $P + \sum s_{ij} X_i + \sum a_{ij} X_i = \sum b_{ij} X_i + \sum r_{ij} X_i$, as desired.

III. Aplications to retractions. With some care in constructing bases X in S and defining maps $X \to G$ we can answer some previously unsettled questions about retractions.

THEOREM 8. *Let G be a torsion free abelian group of rank n. Two retractions of G are the same if and only if they agree on all sets of cardinality at most $n + 1$, and agreement on all sets of cardinality at most $k < n + 1$ is not sufficient to guarantee agreement on sets of cardinality $n + 1$.*

PROOF. By [3] we can take $G = Q^n$.

Only if: clear.

If: If two retractions agree on all sets of cardinality at most $n + 1$, the corresponding homomorphisms $S \to G$ agree on all sets with at most $n + 1$ vertices. Hence they agree on all simplices, so they agree on a basis and hence are identical. Thus the original retractions were the same.

To show that agreement on sets of cardinality at most $k < n + 1$ is not sufficient for retractions to be the same we proceed as follows. In constructing a basis Y we first get a maximal independent set X of polytopes of dimension less than n. The maximality of X will imply that the image under any semigroup homomorphism $S \to G$ of any polytope of dimension at most $n - 1$ will be determined by its effect on X. The corollary to theorem 5 tells us that X is not a basis, so it can be extended to a basis Y. Clearly, two distinct mappings of Y into G can agree on X, so two distinct retractions on G can agree on all sets of cardinality less than or equal to n.

Regarding σ-subgroups of retractable groups, we get the following result.

THEOREM 9. *For any n there is a retractable group G with retraction σ, and a subgroup H of G, such that σ maps all subsets of $F(H)$ of cardinality less than n to H, yet H is not a σ-subgroup of G.*

PROOF. We take $G = Q^n$, and H to be any n-1 dimensional subspace. This time we first take an independent set X of polytopes of dimension less than n-1 maximal subject to having all vertices in H. We extend X to a maximal independent set X' of polytopes with all vertices in H, then extend X' to a basis Y in S. The maximality of X implies that if P is a polytope with vertices in H and dimension less than n-1, then the image of P under any homomorphism $S \to G$ is determined by the corresponding images of X. Also, the corollary to theorem 5 applied to H shows that $X \neq X'$. Thus in defining a map $Y \to G$ we can assign values in H to elements of X, and yet assign a value outside of H to some element of X' not in X. The corresponding retraction would have the desired property.

Acknowledgement. The author would like to thank Professor F.D.

Pedersen of Southern Illinois University for his interest in this work and for making helpful suggestion during its progress.

REFERENCES

1. R. D. Byrd, J. T. Lloyd, R. A. Mena, and J. R. Teller, *Retractable Groups*, Act. Math. Hung. **29** (1977), 219–233.

2. Branko Grunbaum, *Convex Polytopes*, Interscience Publishers, New York, N.Y., 1967.

3. F. D. Pedersen and Walter S. Sizer, *Subsemigroups of the Finite Complexes of a Group*, Semigroup Forum **20** (1980), 285–292.

4. ———, *Certain Semigroups of Complexes of an Abelian Group*, Proceedings of the Conference on Semigroups in honor of Alfred H. Clifford, Tulane University Lecture Notes, 215–221.

5. Helge Tverberg, *How to Cut a Convex Polytope into Simplices*, Geometriae Dedicata **3** (1974), 239–240.

MOORHEAD STATE UNIVERSITY, MOORHEAD, MN 56560