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A q-EXTENSION OF BAILEY'S BILINEAR GENERATING FUNCTION FOR THE JACOBI POLYNOMIALS

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ABSTRACT. This note presents a rather simple proof of an interesting q-extension of Bailey's bilinear generating function for the classical Jacobi polynomials. The proof given here uses only such elementary results as the q-analogues of Euler's transformation, Vandermonde's summation theorem, and binomial expansion.

1. Introduction. Several interesting proofs are given in the literature for Bailey's bilinear generating function for the classical Jacobi polynomials [**1**, p. 9, Eq. (2.1)]. One of the recent proofs is given by Stanton [**3**]; it uses the orthogonality property of Jacobi polynomials and a known quadratic transformation for a well-poised hypergeometric ${}_3F_2$ series. Indeed, as remarked by Stanton [**3**, p. 399], this technique applies *mutatis mutandis* to yield a q-extension of Bailey's result. The object of the present note is to give a rather simple proof of the q-extension, using only such elementary results as the q-analogues of Euler's transformation, Vandermonde's summation theorem, and binomial expansion.

2. Definitions and preliminaries. Put

$$(2.1)\quad (\lambda; q)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1-\lambda)(1-\lambda q)\ldots(1-\lambda q^{n-1}), & \forall n \in \{1, 2, 3, \ldots\}, \end{cases}$$

and let ${}_{p+1}\Phi_p$ denote the standard q-hypergeometric series with $p+1$ numerator and p denominator parameters. Then, in terms of the little q-Jacobi polynomials defined by

$$(2.2)\qquad p_n^{(\alpha,\beta)}(x; q) = \frac{(\alpha q; q)_n}{(q; q)_n}\, {}_2\Phi_1\left[\begin{matrix} q^{-n}, \ \alpha\beta q^{n+1}; \\ \alpha q; \end{matrix}\ q, qx\right],$$

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a q-extension of Bailey's result has the form (cf. [**3**, p. 400, Eq. (5)])

$$\sum_{n=0}^{\infty} \frac{(q;\,q)_n\,(\alpha\beta q;\,q)_n}{(\alpha q;\,q)_n\,(\beta q;\,q)_n}\,(\beta q)^n\,q^{n(n-1)/2}\,p_n^{(\alpha,\beta)}(x;\,q)\,p_n^{(\alpha,\beta)}(y;\,q)t^n$$

$$= \frac{(-\alpha\beta qt;\,q)_\infty}{(-t;\,q)_\infty}\,\Phi^{4:0;2}_{2:1;1}\left[\begin{matrix}\sqrt{\alpha\beta q},\,-\sqrt{\alpha\beta q},\,q\sqrt{\alpha\beta},\,-q\sqrt{\alpha\beta}: \\ -\alpha\beta qt,\,-q/t: \\ \text{—};\;\beta qx,\,\beta qy; \\ q;\;\beta q^2xy,\,q \\ \alpha q;\qquad \beta q;\end{matrix}\right] \tag{2.3}$$

$$= \frac{(-\alpha\beta qt;\,q)_\infty}{(-t;\,q)_\infty}\sum_{r,s=0}^{\infty}\frac{(\alpha\beta q;\,q)_{2r+2s}\,(\beta qx;\,q)_s\,(\beta qy;\,q)_s}{(-\alpha\beta qt;\,q)_{r+s}\,(-q/t;\,q)_{r+s}\,(\alpha q;\,q)_r\,(\beta q;q)_s}\cdot\frac{(\beta q^2xy)^r}{(q;\,q)_r}\,\frac{q^s}{(q;\,q)_s},$$

where, for convenience,

$$(\lambda;\,q)_\infty = \prod_{j=0}^{\infty}(1-\lambda q^j). \tag{2.4}$$

In our proof of the q-extension (2.3), detailed in the next section, we shall require the following elementary results in the theory of q-hypergeometric functions.

I. *A q-analogue of Euler's transformation* [**2**, p. 97, Eq. (3.3.2.3)]

$$ {}_2\Phi_1\left[\begin{matrix}a,\,b; \\ \quad q,\,z \\ c;\end{matrix}\right] = \frac{(abz/c;\,q)_\infty}{(z;\,q)_\infty}\,{}_2\Phi_1\left[\begin{matrix}c/a,\,c/b; \\ \quad q,\,abz/c \\ c;\end{matrix}\right]. \tag{2.5}$$

II. *A q-analogue of Vandermonde's theorem* [**2**, p. 97, Eq. (3.3.2.7)]

$$ {}_2\Phi_1\left[\begin{matrix}q^{-n},\,b; \\ \quad q,\,q \\ c;\end{matrix}\right] = b^n\,\frac{(c/b;\,q)_n}{(c;\,q)_n}, \qquad n = 0,\,1,\,2,\,\ldots. \tag{2.6}$$

III. *The q-binomial expansion* (*or Heine's theorem*) [**2**, p. 92, Eq. (3.2.2.11)]

$$ {}_1\Phi_0\left[\begin{matrix}a; \\ \quad q,\,z \\ \text{—};\end{matrix}\right] = \frac{(az;\,q)_\infty}{(z;\,q)_\infty}. \tag{2.7}$$

3. Proof of the q-extension (2.3). In view of (2.5), the definition (2.2) readily yields

$$p_n^{(\alpha,\beta)}(x;q)$$
$$= \frac{(\beta q;q)_n(\beta qx;q)_\infty}{(q;q)_n(qx;q)_\infty}\sum_{\ell=0}^{\infty} q^{\ell(\ell-2n+1)/2}\frac{(\alpha q;q)_{n+\ell}}{(\alpha q;q)_\ell(\beta q;q)_{n-\ell}}\frac{(-x)^\ell}{(q;q)_\ell}, \tag{3.1}$$

and, using (2.1) in (2.2), we also have

$$p_n^{(\alpha,\beta)}(y;q) = \frac{(\alpha q;q)_n}{(\alpha\beta q;q)_n}\sum_{m=0}^{n}\frac{q^{m(m-2n+1)/2}(\alpha\beta q;q)_{m+n}(-y)^m}{(q;q)_m(q;q)_{n-m}(\alpha q;q)_m}. \tag{3.2}$$

Denoting, for convenience, the first member of (2.3) by Ω, and substituting from (3.1) and (3.2), we find that

$$\begin{aligned}\Omega &= \frac{(\beta qx;q)_\infty}{(qx;q)_\infty}\sum_{n,\ell=0}^{\infty} q^{(1/2)n(n+1)+(1/2)\ell(\ell-2n+1)}(\beta t)^n\frac{(-x)^\ell}{(q;q)_\ell}\\ &\quad\cdot\frac{(\alpha q;q)_{n+\ell}}{(\alpha q;q)_\ell(\beta q;q)_{n-\ell}}\sum_{m=0}^{n}\frac{q^{m(m-2n+1)/2}(\alpha\beta q;q)_{m+n}(-y)^m}{(q;q)_m(q;q)_{n-m}(\alpha q;q)_m}\\ &= \frac{(\beta qx;q)_\infty}{(qx;q)_\infty}\sum_{\ell,m,n=0}^{\infty} q^{(1/2)\ell(\ell+1)+m+(1/2)n(n+1)-n\ell-\ell m}\\ &\quad\cdot\frac{(\alpha q;q)_{\ell+m+n}(\alpha\beta q;q)_{2m+n}}{(\alpha q;q)_\ell(\alpha q;q)_m(\beta q;q)_{m+n-\ell}}\frac{(-x)^\ell}{(q;q)_\ell}\frac{(-\beta yt)^m}{(q;q)_m}\frac{(\beta t)^n}{(q;q)_n}.\end{aligned} \tag{3.3}$$

From the q-summation theorem (2.6) it is easily verified that

$$\begin{aligned}&\frac{(\alpha q;q)_{\ell+m}}{(q;q)_\ell(q;q)_m(\alpha q;q)_\ell(\alpha q;q)_m}\\ &\quad= q^{\ell m}\sum_{r=0}^{\min(\ell,m)}\frac{q^{r(r-\ell-m)}}{(q;q)_r(q;q)_{\ell-r}(q;q)_{m-r}(\alpha q;q)_r}\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}&\frac{(\alpha q;q)_{\ell+m+n}}{(q;q)_n(\alpha q;q)_{\ell+m}(\beta q;q)_{m+n-\ell}} = (-\beta)^{-n}q^{(-1/2)n(n+1)-mn+n\ell}\\ &\quad\cdot\sum_{s=0}^{n}\frac{(-1)^s q^{(1/2)s(s-2n+1)}(\alpha\beta q^{2m+n+1};q)_s}{(q;q)_s(q;q)_{n-s}(\beta q;q)_{s+m-\ell}},\end{aligned} \tag{3.5}$$

which, together, yield

$$\begin{aligned}&\frac{(\alpha q;q)_{\ell+m+n}(\alpha\beta q;q)_{2m+n}}{(q;q)_\ell(q;q)_m(q;q)_n(\alpha q;q)_\ell(\alpha q;q)_m(\beta q;q)_{m+n-\ell}}\\ &= (-\beta)^{-n}q^{(-1/2)n(n+1)+\ell m-mn+n\ell}\sum_{r=0}^{\min(\ell,m)}\sum_{s=0}^{n}(-1)^s\frac{q^{r(r-\ell-m)+(1/2)s(s-2n+1)}}{(q;q)_r(q;q)_s}\\ &\quad\cdot\frac{(\alpha\beta q;q)_{s+2m+n}}{(q;q)_{\ell-r}(q;q)_{m-r}(q;q)_{n-s}(\alpha q;q)_r(\beta q;q)_{s+m-\ell}}.\end{aligned} \tag{3.6}$$

Substituting from (3.6) into (3.3), we obtain

$$\Omega = \frac{(\beta qx;\, q)_\infty}{(qx;\, q)_\infty} \sum_{\ell, m, n=0}^{\infty} q^{(1/2)\ell(\ell+1)-m(n-1)} (-x)^\ell (-\beta yt)^m (-t)^n$$

$$\cdot \sum_{r=0}^{\min(\ell, m)} \sum_{s=0}^{n} \frac{(-1)^s q^{r(r-\ell-m)+(1/2)s(s-2n+1)} (\alpha\beta q;\, q)_{s+2m+n}}{(q;q)_r (q;q)_s (q;q)_{\ell-r} (q;q)_{m-r} (q;q)_{n-s} (\alpha q;q)_r (\beta q;q)_{s+m-\ell}}$$

(3.7)

$$= \frac{(\beta qx;\, q)_\infty}{(qx;\, q)_\infty} \sum_{r, s, \ell, m, n=0}^{\infty} q^{(1/2)(\ell+r)(\ell+r+1)-(m+r)(n+s-1)-r(r+\ell+m)-(1/2)s(s+2n-1)}$$

$$\cdot \frac{(\alpha\beta q;\, q)_{2r+2s+2m+n}}{(\alpha q;q)_r (\beta q;q)_{s-m-\ell}} \frac{(\beta xyt)^r}{(q;q)_r} \frac{t^s}{(q;q)_s} \frac{(-x)^\ell}{(q;q)_\ell} \frac{(-\beta yt)^m}{(q;q)_m} \frac{(-t)^n}{(q;q)_n}.$$

Now replace the summation index s in (3.7) by $s - m$ and apply the definition (2.1). Each of the resulting ℓ-, m- and n-series can thus be separated, and from (3.7) we have

$$\Omega = \frac{(\beta qx;\, q)_\infty}{(qx;\, q)_\infty} \sum_{r, s=0}^{\infty} \frac{q^{(-1/2)(r+s)(r+s+1)} (\alpha\beta q;\, q)_{2r+2s}}{(\alpha q;q)_r (\beta q;\, q)_s}$$

(3.8)
$$\cdot \frac{(\beta q^2 xyt)^r}{(q;\, q)_r} \frac{(qt)^s}{(q;\, q)_s} {}_1\Phi_0\left[\begin{matrix} q^{-s}/\beta; & \\ & q,\ \beta q^{s+1}x \\ ; & \end{matrix}\right]$$

$$\cdot\ {}_1\Phi_0\left[\begin{matrix} q^{-s}; & \\ & q,\ \beta q^{s+1}y \\ ; & \end{matrix}\right] {}_1\Phi_0\left[\begin{matrix} \alpha\beta q^{2r+2s+1}; & \\ & q,\ -q^{-r-s}t \\ ; & \end{matrix}\right].$$

Applying Heine's theorem (2.7) to each of the ${}_1\Phi_0$'s occurring in (3.8) and noticing, in particular, that

$${}_1\Phi_0\left[\begin{matrix} \alpha\beta q^{2r+2s+1}; & \\ & q,\ -q^{-r-s}t \\ ; & \end{matrix}\right]$$

(3.9)
$$= \frac{(-\alpha\beta q^{r+s+1}t;\, q)_\infty}{(-q^{-r-s}t;\, q)_\infty}$$

$$= \frac{(-\alpha\beta qt;\, q)_\infty}{(-t;\, q)_\infty} \frac{t^{-r-s} q^{(1/2)(r+s)(r+s+1)}}{(-\alpha\beta qt;\, q)_{r+s} (-q/t;\, q)_{r+s}},$$

the last member of the q-extension (2.3) follows at once from (3.8); here we assume that the various parameters involved are so constrained that Equation (3.9) holds true.

This evidently completes our proof of the q-extension (2.3) in which the second member is obtained upon merely interpreting the double series as a q-hypergeometric function of two variables.

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