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DISTRIBUTIVE, MODULAR AND SEPARATING ELEMENTS IN LATTICES

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Given the importance of distributive lattices as a class, it was a natural step to consider distributivity of elements in an arbitrary lattice L. For instance an element d is called *distributive* if $d \vee (x \wedge y) = (d \vee x) \wedge (d \vee y)$ for all $x, y \in L$, and *separating* if $d \vee x = d \vee y$ and $d \wedge x = d \wedge y$ to gether imply $x = y$. An important early result was that in a modular lattice any distributive (or dually distributive) element is in fact *neutral*, that is, distributive, dually distributive and separating. A deeper result is that the same is true in weakly modular lattices [**3**, §III. 2].

In this paper the above result is extended in other directions, notably in M-symmetric and "Θ- modular" lattice. To do this we introduce some notions which might be considered as "modularity" of elements in a fashion similar to that for "distributivity" above. For instance an element d of L is *left* [*right*] *modular* if $d\ M\ a$ $[a\ M\ d]$ for all $a \in L$, and *weakly separating* if $d \vee x = d \vee y$, $d \wedge x = d \wedge y$ and $x \leqq y$ together imply $x = y$. Such elements do indeed arise (in a nontrivial manner) in congruence lattices, for example.

The first main result proved is that in an M-symmetric lattice, any element which is both distributive and dually distributive is neutral. Given the lack of duality inherent in M-symmetry, this is perhaps the strongest result that might be expected. On the other hand it is shown that in an M-symmetric algebraic lattice satisfying DCC, any dually distributive element is neutral. Counterexamples show that these results cannot be extended.

In the final section the concept of "Θ-modularity", introduced (in a rather special context) by Spitznagel [**7**], is considered and its relationship with the earlier concepts is demonstrated. Roughly speaking, given an equivalence Θ on the lattice L, L is Θ-modular if each of its elements weakly separates each Θ-class. (Thus a modular lattice is Θ-modular for any Θ). The case in which we are most interested corresponds to the equivalence $\Theta_d = \{(a, b) \in L \times L : a \vee d = b \vee d\}$, when d is an arbitrary element of L. Our main result here is that if d is distributive (so that Θ_d is in fact a congruence on L) and if L is Θ_d-modular, then d must be neutral.

In a sequel [**5**] these results will be used in a discussion of congruence

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lattices (of semigroups). The relevent point is that if A is any (universal) algebra, with congruence lattice $C(A)$, and if $\rho \in C(A)$, then ρ is neutral in $C(A)$ if and only if the map

$$\tau \to (\tau \cap \rho, (\tau \vee \rho)/\rho)$$

is an isomorphism of $C(A)$ upon a subdirect product of the principal ideal $(\rho]$ of $C(A)$ and the lattice $C(A/\rho)$ of congruences on the quotient algebra A/ρ.

1. Definitions and elementary results. We adhere generally to the notation and terminology of Grätzer [**3**]. In the following, L is an arbitrary lattice. Recall that an element d of L is (i) *distributive* if $d \vee (x \wedge y) = (d \vee x) \wedge (d \vee y)$ for all $x, y \in L$; (ii) *standard* if d is distributive and *separates* L, in the sense that $d \wedge x = d \wedge y$ and $d \vee x = d \vee y$ together imply $x = y$; and (iii) *neutral* if d is distributive, dually distributive and separating.

(In fact standard and neutral elements are defined a little differently in [**3**]—the forms (ii) and (iii) just given are equivalent to the definitions in [**3**] by [**3**, Theorems III, 2.3. and III. 2.4]). We shall have occasion to use the following alternative characterization of standard elements.

RESULT 1.1. [**3**, Theorem III. 2.3]. *An element d of L is standard if and only if $a \wedge (d \vee b) = (a \wedge d) \vee (a \wedge b)$ for all $a, b \in L$.*

The modularity relation M on L is defined by aMb if $(x \vee a) \wedge b = x \vee (a \wedge b)$ for all $x \leqq b$ or, equivalently, if $(x \vee a) \wedge b = x$ for all $x \in [a \wedge b, b]$. A lattice L is *M-symmetric* if aMb implies bMa for all $a, b \in L$.

We call d *left modular* if dMa for all $a \in L$, and *right modular* if aMd for all $a \in L$. *A* useful observation [**6**, Lemma 1.2] is that left modularity is self-dual. The element d is *weakly separating* if $d \wedge x = d \wedge y, d \vee x = d \vee y$ and $x \leqq y$ together imply $x = y$.

One of the elementary relationships between these concepts we will demonstrate below is that a weakly separating element is left modular. A ready source of such elements is then provided by the following result, whose proof is easily obtained by modifying the proof, by Jonsson, that a lattice with a "type two" representation (as in the next proposition, but for all $a, d \in L$) is modular. (See [**3**; Theorem IV. 4.8].)

PROPOSITION 1.2. *Let L be a lattice and $d \in L$. Suppose L has a representation ϕ: $L \to$ Part(X) such that $(a \vee d)\phi = a\phi \circ d\phi \circ a\phi$ for all $a \in L$. Then d is weakly separating.*

Here Part(X) is the partition lattice of the set X and the symbol $\circ$ denotes composition: if $\alpha, \beta \in$ Part(X), then $(a, b) \in \alpha \circ \beta$ if $a \equiv c(\alpha)$ and

$c \equiv b(\beta)$ for some $c \in X$. Clearly if A is an algebra and $\rho \in C(A)$ has the property that for all $\tau \in C(A)$, $\rho \vee \tau = \rho \circ \tau \circ \rho$, then ρ weakly separates $C(A)$. We now demonstrate some elementary relationships among these concepts.

LEMMA 1.3. (i) *Any weakly separating element is left modular.*
(ii) *Any dually distributive element is right modular.*
(iii) *Any distributive, left modular element is separating.*

PROOF. (i) Suppose d weakly separates L and let $a, b \in L$, $b \leqq a$. It is easily veriified that $b \vee (d \wedge a) \leqq (b \vee d) \wedge a$, that $d \wedge (b \vee (d \wedge a)) = d \wedge ((b \vee d) \wedge a)$ and that $d \vee (b \vee (d \wedge a)) \wedge d = d \vee ((b \vee d) \wedge a)$, so $b \vee (d \wedge a) = (b \vee d) \wedge a$ and dMa.
(ii) Immediate from the definition.
(iii) Suppose d is distributive and left modular, and that

$$d \wedge a = d \wedge b,\ d \vee a = d \vee b \text{ for some } a, b \in L. \text{ Now}$$

$$\begin{aligned} a \wedge b &= (a \wedge b \vee (d \wedge a), \text{ (using } d \wedge a = d \wedge b), \\ &= ((a \wedge b) \vee d) \wedge a, \text{ (since } a \wedge b \leqq a \text{ and } dMa), \\ &= ((a \vee d) \wedge (b \vee d)) \wedge a, \text{ (using distributivity)} \\ &= (a \vee d) \wedge a, \text{ (using } a \vee d = b \vee d) \\ &= a. \end{aligned}$$

Thus $a \leqq b$, and similarly $b \leqq a$, as required.

Noting again the self-duality of left modularity we obtain the following corollary.

COROLLARY 1.4. *If d is distributive or dually distributive, the following are equivalent*:
(i) *d is weakly separating,*
(ii) *d is left modular, and*
(iii) *d is separating.*

2. Distributive elements and M-symmetry.

THEOREM 2.1. *Let L be an M-symmetric lattice. Any dually distributive element separates L. Hence any element which is both distributive and dually distributive is neutral.*

PROOF. If d is dually distributive, then by Lemma 1.3 (ii), d is right modular whence, by M-symmetry, left modular. The result now follows from Corollary 1.4.

EXAMPLE 2.2. It is interesting to note that the theorem fails in "semi-modular" lattices. (Again following [3], L is *semimodular* if $a \prec b$ implies

$a \vee c \prec b \vee c$ or $a \vee c = b \vee c$, for all $c \in L$ or, equivalently, if $x \succ x \wedge y$ implies $x \vee y \succ y$ for all x, y. Here $\prec$ denotes the covering [relation in L). For example the lattice L_1 of Figure 1(a) (where the intervals $[a, 1]$ and $[0, d]$ are isomorphic dense bounded chains, b is the only other element and the meets and join are as shown) is easily verified to be semimodular, but the element d (as shown) is dually distributive yet not separating.

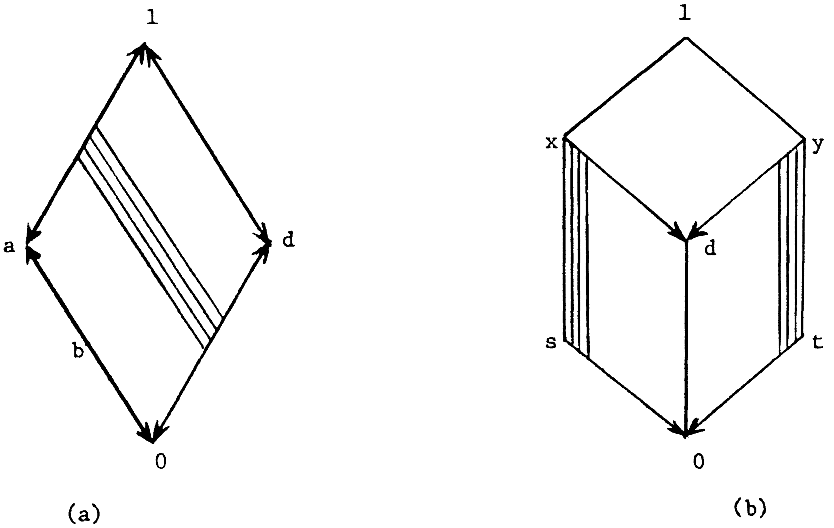

Figure 1

EXAMPLE 2.3. As remarked in the introduction, we cannot really expect such a strong result as "distributive implies neutral" in M-symmetric lattices in general. In fact the lattice L_2 of figure 1(b) shows that even "standard implies neutral" is not true. (See however the theorem below). In that diagram each interval $(0, s]$, $(0, t]$, $(d, x]$ and $(d, y]$ is isomorphic with the chain of natural numbers ordered by $1 > 2 > 3 \ldots$, the remaining meets and joins again being as shown. It is easily seen that L_2 is M-symmetric and that the element d (as shown) is standard. But d is not dually distributive, for $d \wedge (s \vee t) = d$ whilst $(d \wedge s) \vee (d \wedge t) = 0$.

The lattice L_2 satisfies ACC but not DCC. A similar example may be constructed satisfying DCC but not ACC. However, if the lattice is assumed to be algebraic we have the following result. (A lattice L is *algebraic* if it is complete and every element is a join of compact elements).

THEOREM 2.4. *In an algebraic, M-symmetric lattice with* DCC, *any standard element is neutral.*

PROOF. Let d be a standard element of L and suppose d is not dually distributive: thus there exist $x, y \in L$ such that

$$(1) \qquad d \wedge (x \vee y) > (d \wedge x) \vee (d \wedge y) = b, \text{ say.}$$

Since d is separating it is left modular (by Lemma 1.3 (i)), whence, by M-symmetry, right modular. In particular xMd and yMd, and since $b \in [d \wedge x, d]$ and $[d \wedge y, d]$, we have

$$(2) \qquad b = d \wedge (x \vee b) = d \wedge (y \vee b).$$

Thus

$$\begin{aligned} d \wedge \{(x \vee b) \vee (y \vee b)\} &\geqq d \wedge (x \vee y) > b \\ &= \{d \wedge (x \vee b)\} \vee \{d \wedge (y \vee b)\}, \end{aligned}$$

so that in (1) we may assume, without loss of generality, that $x, y \geqq b$. In fact since $d > b$ already, the inequality (1) is valid for some x, y, d in $[b)$, the principal dual ideal generated by b. Moreover the hypotheses of the theorem remain valid in $[b)$ so we may assume from now on that $b = 0$. Thus for some x and y, $d \wedge (x \vee y) > 0$ whilst $d \wedge x = d \wedge y = 0$.

Using DCC we may choose z in L minimally with respect to the following property: there is an element p of L for which $d \wedge (z \vee p) > 0$ whilst $d \wedge z = d \wedge p = 0$; let p be such an element.

On the other hand by the maximum principle there is an element m in $[p, z \vee p]$ maximal such that $d \wedge m = 0$. (For if $\{u_i\}$ is a chain in $[p, z \vee p]$ with each $d \wedge u_i = 0$, then by join continuity (true in any algebraic lattice) $d \wedge (\bigvee u_i) = \bigvee (d \wedge u_i) = 0$). Note that $m < z \vee p$, for otherwise $d \wedge (z \wedge p) = 0$. Thus $z \nleqq m$.

We now show that z covers $z \wedge m$. Let $w \in [z \wedge m, z)$. From the minimality of z (and noting that $d \wedge w = d \wedge m = 0$) it follows that $d \wedge (w \vee m) = 0$. But $w \vee m \in [m, p \vee z]$, so from the maximality of m we deduce that $w \vee m = m$, so that $w = m \wedge z$. Hence $z \succ z \wedge m$.

From M-symmetry (which implies semimodularity) it is now immediate that $z \vee p \succ m$. But if we put $c = d \wedge (z \vee p)$, then $c \vee m \in [m, z \vee p] = [m, z \vee m]$; thus $c \vee m = m$ or $c \vee m = z \vee m$. As in equation (2), using mMd, $d \wedge m = 0$ and $c \leqq d$ we obtain $c = d \wedge (c \vee m)$. Thus if $c \vee m = m$, then $c = d \wedge m = 0$, a contradiction. Otheriwse $d \vee m \geqq c \vee m = z \vee m \geqq z$, in which case

$$\begin{aligned} z = z \wedge (d \vee m) &= (z \wedge d) \vee (z \wedge m) \text{ (using Result 1.1)} \\ &= z \wedge m, \text{ contradicting } z \nleqq m. \end{aligned}$$

Hence d is dually distributive and neutral.

REMARK. In algebraic lattices satisfying DCC, M-symmetry and semimodularity are equivalent [**4**].

COROLLARY 2.5. [**1**; Theorem 1.9]. *In a semimodular lattice L without infinite chains, every standard element is neutral.*

PROOF. By the above remark L is M-symmetric, and since L satisfies ACC, it is algebraic, so the hypotheses of theorem are satisfied.

3. Θ-modularity. In [**7**] Spitznagel introduced the following generalization of modularity: if Θ is an equivalence on the lattice L, call L Θ-modular if for any $a, b \in L$, $a \Theta b$, $a \leqq b$ and $a \wedge x = b \wedge x$, $a \vee x = b \vee x$ together imply $a = b$. (In fact Spitznagel required Θ to be a congruence on L, and it is that situation which will occur most often, but we prefer the slightly more general definition), Note that if L is Θ-modular any sublattice of L which is contained in a single Θ-class is modular.

If $d \in L$, then the relation $\Theta_d = \{(a, b) \in L \times L: a \vee d = b \vee d\}$ is an equivalence on L. In fact from [**3**; Theorems III. 2.2, III. 2.5] it follows that d is distributive if and only if Θ_d is a congruence, (in which case Θ_d is the congruence generated by the principal ideal $(d]$ of L). Our particular interest will be in Θ_d-modularity. We will also briefly consider Θ^d-modularity, where Θ^d is defined dually to Θ_d. We first establish some elementary relationships between Θ_d-modularity and the concepts of the previous sections. Throughout, L is an arbitrary lattice and $d \in L$.

LEMMA 3.1. *If L is Θ_d-modular, then*
(i) *the principal ideal $(d]$ is modular,*
(ii) *d is weakly separating, and*
(iii) *d is left modular.*

PROOF. (i) Clearly $(d]$ is contained in a Θ_d-class.
(ii) If $a \leqq b$, $a \wedge d = b \wedge d$ and $a \vee d = b \vee d$, then obviously $a \Theta_d b$, so $a = b$.
(iii) Immediate from Lemma 1.3.(i).

A partial converse is provided by the following lemma.

LEMMA 3.2. *If d is dually standard, (that is, dually distributive and separating) and if $(d]$ is modular, then L is Θ_d-modular.*

PROOF. Let $a, b \in L$, $a \leqq b$, with $a \Theta_d b$, that is, $a \vee d = b \vee d$. Suppose $a \wedge x = b \wedge x$ and $a \vee x = b \vee x$, for some $x \in L$. Since d is dually distributive, $(d \wedge a) \vee (d \wedge x) = d \wedge (a \vee x) = d \wedge (b \vee x) = (d \wedge b) \vee (d \wedge x)$. Moreover $(d \wedge a) \wedge (d \wedge x) = (d \wedge b) \wedge (d \wedge x)$, since $a \wedge x = b \wedge x$; but $d \wedge a \leqq d \wedge b$ and so modularity of $(d]$ yields $d \wedge a = d \wedge b$. Since d separates L, $a = b$, as required.

Combining these results with Corollary 1.4 yields the next corollary.

COROLLARY 3.3. *Suppose d is dually distributive. Then the following are equivalent:*

(i) *L is Θ_d-modular,*
(ii) *d is (weakly) separating and $(d]$ is modular, and*
(iii) *d is left modular and $(d]$ is modular.*

In the M-symmetric case a simpler result holds, by applying Theorem 2.1.

COROLLARY 3.4. *Suppose d is dually distributive. If L is M-symmetric, then L is Θ_d-modular if and only if $(d]$ is modular.*

Clearly the dual of this result would appear to require M^*-symmetry (the dual of M-symmetry) as a hypothesis. However a similar result can be obtained for Θ^d-modularity in an M-symmetric lattice by strengthening the hypotheses.

COROLLARY 3.5. *Suppose d is both distributive and dually distributive. If L is M-symmetric, then L is Θ^d-modular if and only if $[d)$ is modular.*

PROOF. By Theorem 2.1, d is separating. The result now follows from the duals of Lemmas 3.1 and 3.2.

Our main result on Θ-modularity (which uses only Lemma 3.1 among the foregoing results) is that alluded to in the introduction. This theorem is a common abstraction and generalization of Proposition 3.13 of [7] and Theorem 3.7 of [2], each of which was set in a rather special situation. Our proof, in addition, represents a major simplification of the proofs of those two theorems.

THEOREM 3.6. *If d is distributive and L is Θ_d-modular, then d is neutral.*

PROOF. By Lemma 3.1, d is left modular whence, by Lemma 1.3(iii), separating. Rather than proving d is dually distributive directly we will use the dual of Result 1.1 to show d is dually standard. So let $a, b \in L$. Clearly $a \vee (d \wedge b) \leqq (a \vee d) \wedge (a \vee b)$. Further, $d \vee ((a \vee d) \wedge (a \vee b)) \leqq d \vee (a \vee d) = a \vee d = d \vee (a \vee (d \wedge b))$ so $d \vee (a \vee (d \wedge b)) = d \vee ((a \vee d) \wedge (a \vee b))$, that is, $a \vee (d \wedge b)\Theta_d(a \vee d) \wedge (a \vee b)$. Similarly $b \vee (a \vee (d \wedge b)) = b \vee ((a \vee d) \wedge (a \vee b))$. But

$$\begin{aligned} b \wedge (a \vee (d \wedge b)) &\geqq (b \wedge a) \vee (b \wedge (d \wedge b)) \\ &= (b \wedge a) \vee (b \wedge d) \\ &= b \wedge (a \vee d), \quad \text{using Result 1.1 itself, since } d \text{ is standard,} \\ &= b \wedge ((a \vee d) \wedge (a \vee b)), \end{aligned}$$

and since the reverse inequality is clearly satisfied, equality holds. From Θ_d-modularity, $a \vee (d \wedge b) = (a \vee d) \wedge (a \vee b)$ and the theorem follows from the dual of Result 1.1.

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