ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 13, Number 3, Summer 1983

MARKOV OPERATORS AND INVARIANT BAIRE FUNCTIONS

ROBERT E. ATALLA

ABSTRACT. Let X be a compact T_2-space and T a Markov operator on $C(X)$, i.e., $T \geqq 0$ and $T1 = 1$. We deal with Baire functions which are almost everywhere T-invariant, and focus on conditions under which such Baire functions belong to the smallest system of Baire functions containing the continuous invariant functions, and closed under almost everywhere convergence. Our main result is that this occurs precisely when both T and T^* are strongly ergodic. We also show that almost everywhere invariant Baire functions separate the invariant probabilities, although everywhere invariant Baire functions need not.

1. Introduction. We define $F(T) = \{f \text{ in } C(X): Tf = f\}$, $F(T^*) = \{m \text{ in } C(X)^*: T^*m = m\}$, and $P(T^*) = F(T^*) \cap$ probabilities. T is called *strongly ergodic* if for each f in $C(X)$, $T_n f$ converges in the Banach space $C(X)$, where $T_n = (1/n)(I + \cdots + T^{n-1})$. Recall that T is s.e. if and only if $F(T)$ separates $F(T^*)$ if and only if $F(T)$ separates $P(T^*)$ [7, Theorems 2.2 and 2.7].

Let B be the set of Baire functions, so $B = \bigcup\{B_a: a < \omega_1\}$, where ω_1 is the first uncountable ordinal and B_a the a-th Baire class. We define $B(F(T))$ to be the smallest set of bounded functions containing $F(T)$, and closed under bounded pointwise sequential convergence. If f and g are in B, we say $f = g$ $P(T^*)$-ae if $f = g$ m-ae for all m in $P(T^*)$. Let $B(F(T))^a$ be the smallest set of bounded Baire functions containing $F(T)$, and closed under $P(T^*)$-ae convergence.

It will be convenient to extend the operator T to an operator (again called T) on the Baire functions by letting $Tf(x) = \int fd(T^*\delta_x)$, where δ_x is the Dirac measure at x. An easy transfinite induction over the Baire classes shows that if f is in B, then so is Tf. The same argument shows that if f is in B and m in $F(T^*)$, then $\int fdm = \int Tfdm$.

We shall need the following known result.

THEOREM. [1, Proposition 2.1]. *The following are equivalent.*

(i) *Both T and T^* are strongly ergodic.*

(ii) $F(T^{**}) = \sigma(C(X)^{**}, C(X)^*)$*-closure of* $F(T)$.

(iii) *Norm-closure*$(I - T)^*(C(X)^*) =$ *weak-*closure*$(I - T)^*(C(X)^*)$.

AMS subject classification (1980): Primary 47A35, Secondary 26A21.
Received by the editors on March 16, 1982.

Theorem 1 of §2 gives a criterion for joint strong ergodicity of T and T^* in terms of $P(T^*)$-ae invariant Baire functions. It has one advantage over the criteria (ii) and (iii) given above, namely it refers directly to the space X, rather than to the dual and second dual spaces. In Theorem 2 we show that $P(T^*)$ is always separated by the $P(T^*)$-ae invariant Baire functions. This may be contrasted to Sine's theorem that $F(T)$ separates $P(T^*)$ if and only if T is strongly ergodic. We then give an example to show that everywhere invariant Baire functions need not separate $P(T^*)$.

The material of this paper is influenced by an exchange of letters with Robert Sine which took place a few years ago. In particular he suggested that one should expect only ae-results in working with invariant Baire functions.

2. Main results.

THEOREM 1. *The following are equivalent.*

(a) *Both T and T^* are strongly ergodic.*

(b) *If f is in B and $Tf = f$ $P(T^*)$-ae, then there exists g in $B(F(T))$ such that $f = g$ $P(T^*)$-ae.*

(c) *If f is in B and $Tf = f$ $P(T^*)$-ae, then f is in $B(F(T))^a$.*

PROOF. (c) implies (a). If (a) fails, then by the theorem cited in the introduction there exists F in $F(T^{**})\backslash$*-closure$(F(T))$, and then by Hahn-Banach there exists in $C(X)^*$ an m such that $F(m) = 1$ and $0 = \int f dm$ for all f in $F(T)$. Let $p = \sum_{n=0}^{\infty} 2^{-n}(T^*)^n|m|$. By the representation of $C(X)^{**}$ ([**6**, page 481] or [**4**]), for each q in $C(X)^*$, there exists a Baire function F_q such that $F(q) = \int F_q dq$. Further, $r \ll q$ implies $F_r = F_q$ r-ae, and hence $F(r) = \int F_q dr$. In our case we have $F_m = F_p$ m-ae, so $1 = \int F_p dm$. The advantage of p over m is that we can prove $TF_p = F_p$ p-ae (and hence m-ae). For this it suffices to show that $\int TF_p dq = \int F_p dq$ for all q such that $q \ll p$. Now p has the property that if $q \ll p$, then $T^*q \ll p$. Also, by an easy transfinite induction, if g is a Baire function and q a measure, then $\int Tg dq = \int g dT^*q$. Hence if $q \ll p$, then $\int TF_p dq = \int F_p dT^*q = F(T^*q) = T^{**}F(q) = F(q) = \int F_p dq$.

Let f be the sub-invariant majorant of F_p, which is defined as follows: $f_0 = F_p$, $f_{n+1} = \max(F_p, Tf_n)$, and $f(x) = \lim f_n(x)$, an increasing limit [**3**, page 19]. Then $Tf \leqq f$, and hence $f = Tf$ $P(T^*)$-ae. By an easy induction, $f_n = F_p$ m-ae for all n. (Note that for $n \geqq 1$,

$$\begin{aligned}\int |F_p - T^n F_p| dm &\leqq \int |F_p - T^{n-1}F_p| dm + \int |T^{n-1}F_p - T^n F_p| dm \\ &\leqq \int |F_p - T^{n-1}F_p| dm + \int T^{n-1}|F_p - TF_p| dm \\ &= \int |F_p - T^{n-1}F_p| dm + \int |F_p - TF_p| dm.)\end{aligned}$$

Hence $f = F_p$ m-ae, whence $\int f dm = \int F_p dm = 1$, so f is not in $B(F(T))^a$.

(a) implies (b). Since T is strongly ergodic, there exists a projection P such that $T_n f \to Pf$ uniformly for all f in $C(X)$, and range$(P) = F(T)$. Hence for all f in $C(X)$, $TPf = Pf$. An easy transfinite induction over the Baire classes shows that this holds for all f in B. Let f in B satisfy $Tf = f$ $P(T^*)$-ae. For (b) it suffices to prove (1) $f = Pf$ $P(T^*)$-ae and (2) Pf is in $B(F(T))$.

For (1), since T^* is strongly ergodic, T^{**} is weak-*ergodic, i.e., for all F in $C(X)^{**}$ and m in $C(X)^*$, $(T^{**})_n F(m) \to P^{**}F(m)$. Let $F = F_f$, i.e., $F(m) = \int f dm$ for all m in $C(X)^*$, and let $m = \delta_x$. Then we have $T_n f(x) \to Pf(x)$ for all x. But for all m in $P(T^*)$, $Tf = f$ m-ae, and an easy inducton as in "(c) implies (a)" gives $T^n f = f$ m-ae for all $n \geqq 1$. Hence $f(x) = Pf(x)$ m-ae for all m in $P(T^*)$.

(2) follows from an easy transfinite induction. (Suppose for all x, $g_n(x) \to g(x)$. Then $Pg_n(x) \to Pg(x)$. By inductive assumption, Pg_n is in $B(F(T))$, and hence Pg is in $B(F(T))$.)

(b) implies (c) obviously.

REMARKS. (a) Under stronger hypotheses, stronger results on the relation between invariant Baire functions and invariant continuous functions can be obtained. For instance it follows from [**5**, Lemma 3] that if T is irreducible and weakly almost periodic, then a $P(T^*)$-ae invariant Baire function is equal $P(T^*)$-ae to a continuous invariant function.

(b) Let F_1 be the Baire functions with $Tf = f$ $P(T^*)$-ae, let F_2 be the Baire functions satisfying the conclusion of (c) in Theorem 1, and let F_3 be the Baire functions satisying the conclusion (b). Then $F_3 \subset F_2 \subset F_1$, and if T and T^* are both strongly ergodic, we have equality. It is an open question whether it is true in general that $F_2 = F_3$.

(c) If $C(X)$ is a Grothendieck space (i.e., weak-*sequential convergence in $C(X)^*$ is equivalent with weak sequential convergence), then strong ergodicity of T implies that of T^* (see, e.g., [**2**, page 79]). In this case Theorem 1 is a criterion for strong ergodicity of T itself. (Another criterion for G-spaces is given in [**2**, Theorem 2.2].)

THEOREM 2. *The* $P(T^*)$-ae *invariant Baire functions separate* $P(T^*)$.

PROOF. Let m and p be in $P(T^*)$ with $m \neq p$, and let g in $C(X)$ satisfy $\int g dm \neq \int g dp$. Now T defines a contraction in $L^2(q)$, where $q = 2^{-1}(m + p)$, since $(Tf)^2 \leqq T(f^2)$, and hence $\|Tf\|^2 = \int (Tf)^2 dq \leqq \int T(f^2) dq = \int f^2 dq = \|f\|^2$. By the L^2-ergodic theorem, there exists a Baire function h with $Th = h$ q-ae and $\int |T_n g - h|^2 dq \to 0$. By standard measure theory there exists a subsequence $\{T_{n(k)} g\}$ which converges to h q-ae, and we may assume h is bounded. Also, $\int h dm = \lim \int T_{n(k)} g dm =$

$\int gdm$, and likewise $\int hdp = \int gdp$. Hence $\int hdm \neq \int hdp$. Finally, let f be the sub-invariant majorant of h. Then $Tf = f$ $P(T^*)$-ae, and by the arguments used in the proof of Theorem 1, $f = h$ q-ae, so $\int fdm \neq \int fdp$.

EXAMPLE. This example will show that Baire functions which are everywhere invariant instead of just $P(T^*)$-ae invariant may fail to separate $P(T^*)$. Let Y be the set of ordinals less than or equal to the first uncountable ordinal ω_1, and let $X = Y \times \{1, 2\}$. (For convenience the smallest ordinal is taken to be 1 rather than 0.) Every ordinal x can be written $x = l + n$, where l is a limit ordinal and n a non-negative integer. We define Markov T on $C(X)$ as follows: if x is a limit ordinal, $Tf(x, e) = f(x, e)$ for $e = 1$ or 2; and otherwise if $x = l + n$,

$$Tf(x, 1) = (1 - n^{-1})f(l + n, 1) + n^{-1}f(l + n, 2),$$

$$Tf(x, 2) = n^{-1}f(l + n, 1) + (1 - n^{-1})f(l + n, 2).$$

If f is any function on X with $Tf = f$, then $f(l + n, 1) = f(l + n, 2)$ whenever $x = l + n$, $n \geqq 1$. If, in addition, f is a Baire function, then it is constant in some neighborhood of $(\omega_1, 1)$, and also in some neighborhood of $(\omega_1, 2)$. But then f cannot separate the invariant probabilities which represent the Dirac measures at these two points.

Note that extreme elements of $P(T^*)$ are just the Dirac measures at (x, e), where x is a limit ordinal and $e = 1$ or 2. Hence every Baire function satisfies $Tf = f$ $P(T^*)$-ae. On the other hand if g is in $F(T)$, then $g(x, 1) = g(x, 2)$ for every limit ordinal x. Clearly, ae-invariant Baire functions are pretty far from $F(T)$.

REFERENCES

1. R. Atalla, *Markov operators and quasi-stonian spaces*, Proc. Amer. Math. Soc. **83** (1981), 613–618.

2. ———, *On the ergodic theory of contractions*, Revista Colombiana de Mat. **10** (1976), 75–81.

3. S. Foguel, *The ergodic theory of Markov processes*, Van Nostrand Mathematical Studies #21, 1969.

4. H. Gordon, *The maximal ideal space of a ring of measurable functions*, Amer. J. Math. **88** (1966), 827–843.

5. B. Jamison and R. Sine, *Irreducible almost periodic Markov operators*, J. Math. and Mech. **18** (1969), 1043–1057.

6. Z. Semadeni, *Banach spaces of continuous functions* I, PWN, Warsaw, 1971.

7. R. Sine, *Geometric theory of a single Markov operator*, Pac. J. Math. **27** (1968), 155–166.

DEPARTMENT OF MATHEMATICS, OHIO UNIVERSITY, ATHENS, OH 45701