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LIMIT BEHAVIOR OF SOBOLEV TOTAL FLUX BOUNDARY CONTROL PROBLEMS

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ABSTRACT. We consider control of certain nonlocal boundary value problems of Sobolev equations. Regularity results are obtained for the optimal controls. Further, convergence results are obtained for the solution of the boundary value problems as well as the control problem as the equations formally approach the diffusion equation.

1. Introduction. Let D be a nonempty bounded domain in $\mathbf{R}^n$, $n = 2$ or 3, with smooth boundary Γ. Let $a \in D$ and $B(a, \rho)$ be the ball centered at a with radius $\rho > 0$ and boundary Γ_ρ such that $B(a, \rho) \subset D$. Let $\Omega = D - B(a, \rho)$ so that $\partial\Omega = \Gamma_\rho \cup \Gamma$. Finally, let $Q = \Omega \times (0, T)$ with $\Sigma = \Gamma \times (0, T)$ and $\Sigma_\rho = \Gamma_\rho \times (0, T)$. We study control problems governed by the following nonlocal boundary value problem

$$(1)\qquad (1 - \varepsilon\Delta)y_t^{(\varepsilon)} - \Delta y^{(\varepsilon)} = 0 \text{ in } Q$$

$$(2)\qquad y^{(\varepsilon)}(x, 0) = 0 \text{ in } \Omega$$

$$(3)\qquad y^{(\varepsilon)}(x, t) = 0 \text{ on } \Sigma$$

$$(4)\qquad y^{(\varepsilon)}(x, t) = C_\varepsilon(t) \text{ on } \Sigma_\rho$$

where $C_\varepsilon(t)$ is an unknown function of t only

$$(5)\qquad \int_{\Gamma_\rho} \frac{\partial}{\partial n}(y^{(\varepsilon)} + \varepsilon y_t^{(\varepsilon)})d\sigma = v(t) \text{ a.e. in } [0, T]$$

where $v \in L^2(0, T)$.

In equation (1) the Laplacian may be replaced by another operator A that is second order symmetric uniformly strongly elliptic in Ω. Equation (1) with $\varepsilon > 0$ arises in the modelling of many physical phenomena **[2]**. Of particular interest is the flow of a fluid through fissured media **[1]**. In this case a system of fractures is assumed that separates blocks of a porous media thereby creating a material having two porosities. The solution y of equation (1) at a point represents an average pressure of the

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fluid in the fissures in a neighborhood of that point. The term $-\varepsilon\Delta y_t$ is related to the contributions of fluid into the fissures from seepage from the blocks [1]. The boundary conditions of this problem attempt to model a fluid boundary condition in which the pressure is known in the cavity $B(a, \rho)$ only to be independent of position on the boundary Γ_ρ and with the total flux through Γ_ρ known but not known locally. That is, the total quantity of fluid flowing through Γ_ρ can be measured, although not at each point on Γ_ρ.

These boundary value problems have been treated in [3, 6]. There limiting behavior of the solutions is studied as the radius ρ approaches 0. It is found that for fixed $\varepsilon \geqq 0$ the solution of (1)–(5) converges to the solution of

$$
\begin{aligned}
(1 - \varepsilon\Delta)y_t^{(\varepsilon)} - \Delta y^{(\varepsilon)} &= v(t)\delta(x - a) \text{ in } Q \\
y^{(\varepsilon)}(x, 0) &= 0 \text{ in } \Omega \\
y^{(\varepsilon)}(x, t) &= 0 \text{ on } \Sigma
\end{aligned} \tag{6}
$$

as $\rho \to 0$.

In this note we consider control problems given by

$$
\begin{aligned}
&\text{minimize } J_\varepsilon^{(1)}(v) = \int_0^T \int_\Omega (y^{(\varepsilon)}(x, t; v) - z(x, t))^2 dx\, dt + \int_0^T v^2(t) dt \\
&\text{subject to } v \in L^2(0, T)
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
&\text{minimize } J_\varepsilon^{(1)}(v) = \int_\Omega (y^{(\varepsilon)}(x, T; v) - z(x))^2 dx + \int_0^T v^2(t) dt \\
&\text{subject to } v \in L^2(0, T).
\end{aligned} \tag{8}
$$

Again for problems in which $\varepsilon \geqq 0$ is fixed, it can be shown [3, 6] that the optimal controls of these problems converge as $\rho \to 0$ to the solution of the control problem with (6) for underlying equations. In the case of (6) limiting behavior of optimal controls is studied as $\varepsilon \to 0$, [7].

Here we study the convergence properties of problem (1)–(5) with $\rho > 0$ fixed as $\varepsilon \to 0$ along with convergence of the optimal controls $u_\varepsilon^{(1)}$ and $u_\varepsilon^{(2)}$. In §2 we provide a priori estimates of the solutions of (1)–(5) and of certain adjoint problems. From these estimastes we are able to establish convergence properties of the solution of these problems. In §3 we give the optimality equations for the solution of the control problems (7) and (8). These conditions along with the estimates of the previous section enable us to deduce regularity properties of the optimal controls $u_\varepsilon^{(1)}$ and $u_\varepsilon^{(2)}$ for $\varepsilon \geqq 0$. Finally, in §4 we apply the above results to obtain convergence properties.

2. Estimates and convergence properties of solutions. We provide estimates that will be useful in determining convergence and regularity properties. Our first result gives estimates on $y^{(\varepsilon)}$.

LEMMA 1. *Let $\varepsilon \geqq 0$. Then*

$$\int_{\Omega} ((y^{(\varepsilon)}(x, t))^2 + \varepsilon|\nabla y^{(\varepsilon)}(x, t)|^2)dx + \int_0^t \int_{\Omega} |\nabla y^{(\varepsilon)}(x, s)|^2 \, dx \, ds \leqq K \int_0^t v^2(s) ds$$

where K is independent of ε.

PROOF. Multiplying (1) by $y^{(\varepsilon)}$ and integrating by parts, we have

$$\int_{\Omega} ((y^{(\varepsilon)}(x, t)^2 + \varepsilon|\nabla y(x, t)|^2)dx + 2\int_0^t \int_{\Omega} |\nabla y^{(\varepsilon)}(x, s)|^2 \, dx \, ds \leqq \frac{1}{K} \int_0^t c_\varepsilon^2(s) ds + K \int_0^t v^2(s) ds.$$

Since $y^{(\varepsilon)}|_{\Sigma} = 0$, we see that

$$\int_0^t \| y^{(\varepsilon)}(\cdot, s)\|_{H^1(\Omega)}^2 \, dx \leqq K \int_0^t \int_{\Omega} |\nabla y^{(\varepsilon)}|^2 \, dx \, ds.$$

Further, the trace map from $H^1(\Omega)$ into $L^2(\Gamma_\rho)$ is continuous. Hence, we have

$$\int_0^t c^2(s) ds \leqq K \int_0^t \int_{\Omega} |\nabla y^{(\varepsilon)}(x, s)|^2 \, dx \, ds,$$

and we obtain (9).

Similarly, we may obtain estimates for the solution of the following problem.

(10) $$(1 - \varepsilon\Delta)\varphi_t^{(\varepsilon)} - \Delta\varphi^{(\varepsilon)} = \psi \text{ in } Q$$

(11) $$\varphi^{(\varepsilon)}(x, 0) = \theta(x) \text{ in } \Omega$$

(12) $$\varphi^{(\varepsilon)}(x, t) = 0 \text{ on } \Sigma$$

(13) $$\varphi^{(\varepsilon)}(x, t) = d_\varepsilon(t) \text{ on } \Sigma_\rho$$

where d_ε is an unknown function of t only

(14) $$\int_{\Gamma_0} \frac{d\varphi^{(\varepsilon)}}{dn}(x, t) d\sigma = 0 \text{ a.e. in } [0, T].$$

LEMMA 2. *Let $\varepsilon \geqq 0$ and $\theta = 0$. Then*

$$\int_\Omega (\varphi^{(\varepsilon)}(x, t))^2 dx + c\int_0^t\int_\Omega |\nabla\varphi^{(\varepsilon)}(x, t)|^2 dx\, ds + \varepsilon\int_\Omega |\nabla\varphi^{(\varepsilon)}(x, t)|^2 dx \leqq \int_0^t\int_\Omega \psi^2(x, s)dx\, ds, \tag{15}$$

$$\int_0^t\int_\Omega (\varphi_t^{(\varepsilon)}(x, s))^2 dx\, ds + \varepsilon\int_0^t\int_\Omega |\nabla\varphi_t^{(\varepsilon)}(x, s)|^2 dx\, ds + \int_\Omega |\nabla\varphi^{(\varepsilon)}(x, t)|^2 dx \leqq \int_0^t\int_\Omega \psi^2(x, s)dx\, ds \tag{16}$$

and

$$\int_\Omega |\nabla\varphi^{(\varepsilon)}(x, t)|^2 dx + \int_0^t\int_\Omega (\Delta\varphi^{(\varepsilon)}(x, s))^2 dx\, ds + \varepsilon\int_\Omega (\Delta\varphi(x, t))^2 dx \leqq \int_0^t\int_\Omega \psi^2(x, s)dx\, ds. \tag{17}$$

We introduce the subspace H of $H^1(\Omega)$ defined by taking the closure in $H^1(\Omega)$ of the set

$$\{\alpha \in C^1(\Omega) : \alpha = \text{constant on } \Gamma_\rho \text{ and } \alpha = 0 \text{ on } \Gamma\}. \tag{18}$$

LEMMA 3. *Let* $\varepsilon \geqq 0$, $\theta = 0$, *and* $\psi \in L^2(0, T; H)$. *Then*

$$\int_0^t\int_\Omega |\nabla\varphi_t^{(\varepsilon)}|dx\, ds + \varepsilon\int_0^t\int_\Omega (\Delta\varphi_t^{(\varepsilon)})^2 dx\, ds + \int_\Omega (\Delta\varphi^{(\varepsilon)}(x, t))^2 dx \leqq \int_0^t\int_\Omega |\nabla\psi|^2 dx\, ds. \tag{19}$$

The preceding results give the following corollary.

COROLLARY 4. *Let* $\varepsilon \geqq 0$, $\theta = 0$, *and* $\psi \in L^2(Q)$. *The following estimates hold independently of* ε.

$$\|\varphi^{(\varepsilon)}\|_{H^1(Q} \leqq C\|\psi\|_{L^2(Q)}$$
$$\|\varphi^{(\varepsilon)}\|_{L^2(O, T; H^2(\Omega))} \leqq C\|\psi\|_{L^2(Q)}.$$

If $\psi \in L^2(0, T; H)$, *then for any* $\varepsilon \geqq 0$,

$$\|\varphi_t^{(\varepsilon)}\|_{L^2(O, T; H^1(\Omega))} \leqq C\|\psi\|_{L^2(O, T; H^1(\Omega))}.$$

On the other hand, we may establish the following lemma.

LEMMA 5. *Let* $\varepsilon \geqq 0$, $\psi = 0$, *and* $\theta \in H^1(\Omega)$. *Then for almost every* $t \in [0, T]$,

$$\int_\Omega \{(\varphi^{(\varepsilon)}(x, t))^2 + \varepsilon|\nabla\varphi^{(\varepsilon)}(x, t)|^2\}dx + C\int_0^t\int_\Omega |\nabla\varphi^{(\varepsilon)}(x, s)|^2 dx\, ds \leqq \int_\Omega (\theta^2(x) + \varepsilon|\nabla\theta(x)|^2 dx \tag{20}$$

$$\begin{aligned} &\int_0^t \int_\Omega ((\varphi_t^{(\varepsilon)}(x, s))^2 + \varepsilon |\nabla \varphi_t^{(\varepsilon)}(x, s)|^2 dx\, ds \\ &+ \int_\Omega |\nabla \varphi^{(\varepsilon)}(x, t)|^2 dx \leqq \int_\Omega |\nabla \theta(x)|^2 dx. \end{aligned} \tag{21}$$

We now consider convergence results for equations (10)–(14).

PROPOSITION 6. *Let $\varepsilon \to 0$, $\psi_\varepsilon \to \psi$ weakly in $L^2(0, T; H)$, and $\theta = 0$. Then $\varphi^{(\varepsilon)} \to \varphi^{(0)}$ strongly in $L^2(Q)$.*

PROOF. Let (ε_i) be a sequence such that $\varepsilon_i \to 0$ as $i \to \infty$. Then by Corollary 4 there is a subsequence, again denoted by (ε_i), such that $\varphi^{(\varepsilon_i)} \to \varphi$ weakly in $H^1(Q)$ and strongly in $L^2(Q)$. Define the set $V = \{\theta \in C^1(Q)\colon \theta(\cdot, 0) = 0 \text{ in } \Omega, \theta|_\Gamma = 0 \text{ and } \theta|_{\Gamma_\rho} = \text{function of } t \text{ only}\}$. Let V_0 be the closure in $H^1(Q)$ of the set V. Hence, V_0 must be weakly closed, and the limit function φ belongs to V_0.

We now demonstrate that φ is indeed the solution of (10)–(14) with $\varepsilon = 0$. To this end let $\alpha \in C_0^\infty(Q)$. Then we see by integrating that

$$\int_Q \varphi^{(\varepsilon_i)}[-(\alpha_t - \varepsilon_i \Delta \alpha_t - \Delta\alpha)] dx\, dt = \int_Q \psi_{\varepsilon_i} \alpha\, dx\, dt.$$

As $\varepsilon_i \to 0$, we have then $\int_Q \varphi[-\alpha_t - \Delta\alpha] dx\, dt = \int_Q \psi\alpha\, dx\, dt$. Accordingly, for all $\alpha \in C_0^\infty(Q)$, $\int_Q(\varphi_t - \Delta\varphi - \psi)\alpha\, dx\, dt = 0$ holds, and $\varphi_t - \Delta\varphi = \psi$ in the sense of distributions. However, $\varphi \in H^1(Q)$ so that $\varphi_t \in L^2(Q)$ and $\Delta\varphi \in L^2(0, T; H^{-1}(\Omega))$. With $\psi \in L^2(Q)$, we conclude that in fact $\Delta\varphi \in L^2(Q)$, and the equation holds in $L^2(Q)$. Furthermore, since φ belongs to V_0, we see that equations (11), (12) and (13) hold as well.

It remains to show that φ also satisfies equation (14). Since the trace map is continuous and onto from $H^2(\Omega)$ to $H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$, [5], we let $\alpha \in H^2(\Omega)$ have the property

$$\alpha|_{\Gamma_\rho} = 1 \text{ and } \alpha|_\Gamma = \left.\frac{\partial\alpha}{\partial n}\right|_{\partial\Omega} = 0. \tag{22}$$

Setting $w^{(\varepsilon)} = \varphi^{(\varepsilon)} - \varphi$ we see that $w_t^{(\varepsilon)} - \Delta w^{(\varepsilon)} = -\varepsilon\Delta\varphi_t^{(\varepsilon)} + \psi_\varepsilon - \psi$. Multiplying by α, we have

$$\begin{aligned} &\int_0^t \int_\Omega w_t^{(\varepsilon)}\, dx\, ds - \int_0^t \int_{\Gamma_\rho} \frac{\partial w^{(\varepsilon)}}{\partial n}\, d\sigma\, ds - \int_0^t \int_\Omega w^{(\varepsilon)} \Delta\alpha\, dx\, ds \\ &= -\varepsilon \int_0^t \int_\Omega \nabla\varphi_t^{(\varepsilon)} \cdot \nabla\alpha\, dx\, ds + \int_0^t \int_\Omega (\psi_\varepsilon - \psi)\alpha\, dx\, ds. \end{aligned} \tag{23}$$

Since $\psi_{\varepsilon_i} \to \psi$ weakly in $L^2(0, T; H)$, it is a uniformly bounded sequence in $L^2(0, T; H)$. As a consequence of Corollary 4, we see that

$$\int_0^t \|\nabla\varphi_t^{(\varepsilon)}\|_{L^2(\Omega)}\, ds \leqq M$$

for all $\varepsilon \geqq 0$. Hence, the first term on the left goes to zero as $\varepsilon_i \to 0$. As all the other terms go to zero, we see that

$$\int_0^t \int_{\Gamma_\rho} \frac{\partial w^{(\varepsilon_i)}}{\partial n} \, d\sigma \, ds \to 0$$

as $\varepsilon_i \to 0$ as well. But we note that

$$\begin{aligned}\int_0^t \int_{\Gamma_\rho} \frac{dw^{(\varepsilon)}}{dn} \, d\sigma \, ds &= \int_0^t \left[\left(\int_{\Gamma_\rho} \frac{d\varphi^{(\varepsilon)}}{dn} \, d\sigma \right) - \left(\int_{\Gamma_\rho} \frac{d\varphi}{dn} \, d\sigma \right) \right] ds \\ &= - \int_0^t \int_{\Gamma_\rho} \frac{d\varphi}{dn} \, d\sigma \, ds.\end{aligned}$$

Thus, we have for almost all $t \in [0, T]$,

$$\int_0^t \int_{\Gamma_\rho} \frac{\partial \varphi}{dn} \, d\sigma \, ds = 0,$$

and conclude that

$$\int_{\Gamma_\rho} \frac{d\varphi}{dn} (x, t) d\sigma = 0$$

for almost all $t \in [0, T)$.

The limit φ is a solution of equations (10)–(14) for $\varepsilon = 0$ and $\theta = 0$. By uniqueness then $\varphi^{(0)} = \varphi$. Furthermore, we see the above arguments holds for any sequence $\varepsilon_i \to 0$ so that $\varphi^{(\varepsilon)} \to \varphi^{(0)}$ strongly in $L^2(Q)$.

PROPOSITION 7. *Let $\psi_\varepsilon \to \psi$ weakly in $L^2(0, T; H)$ as $\varepsilon \to 0$. Then $d_\varepsilon \to d_0$ strongly in $L^2(0, T)$.*

PROOF. Let $\alpha \in H^2(\Omega)$ have the properties

$$\alpha|_{\Gamma_\rho} = \alpha|_\Gamma = \left.\frac{\partial \alpha}{\partial n}\right|_\Gamma = 0 \text{ and } \left.\frac{\partial \alpha}{\partial n}\right|_{\Gamma_\rho} = 1. \tag{24}$$

Then with $w^{(\varepsilon)} = \varphi^{(\varepsilon)} - \varphi^{(0)}$ we see that

$$\begin{aligned}\int_0^t \int_\Omega w_t^{(\varepsilon)} \alpha \, dx \, ds + \int_0^t \int_{\Gamma_\rho} w^{(\varepsilon)} d\sigma \, ds - \int_0^t \int_\Omega w^{(\varepsilon)} \Delta\alpha \, dx \, ds \\ = - \varepsilon \int_0^t \int_\Omega \Delta\varphi^{(\varepsilon)} \alpha \, dx \, ds + \int_0^t \int_\Omega (\psi_\varepsilon - \psi) \alpha \, dx \, ds.\end{aligned}$$

Certainly, from the preceding proposition, as $\varepsilon \to 0$, we have

$$\int_0^t \int_{\Gamma_\rho} w^{(\varepsilon)} d\sigma \, ds \to 0.$$

Thus, $\int_0^t (d^{(\varepsilon)}(s) - d^{(0)}(s)) ds \to 0$ as $\varepsilon \to 0$. But this implies along with estimates in Corollary 4 that $d^{(\varepsilon)} \to d^{(0)}$ weakly in $L^2(0, T)$. In fact, from Corollary 4 we have strong convergence in $L^2(0, T)$.

In an analogous manner we may use Lemma 5 to obtain other convergence result for $\psi = 0$.

PROPOSITION 8. *Let* $\psi = 0$ *and* $\theta_\varepsilon \to \theta$ *weakly in* $H^1(\Omega)$ *as* $\varepsilon \to 0$. *Then* $\varphi^{(\varepsilon)} \to \varphi^{(0)}$ *strongly in* $L^2(Q)$ *and* $d_\varepsilon \to d_0$ *weakly in* $L^2(0, T)$.

By superposition we obtain the following result.

COROLLARY 9. *Let* $\psi_\varepsilon \to \psi$ *weakly in* $L^2(0, T; H)$ *and* $\theta_\varepsilon \to \theta$ *weakly in* $H^1(\Omega)$ *as* $\varepsilon \to 0$. *Then* $\varphi^{(\varepsilon)} \to \varphi^{(0)}$ *strongly in* $L^2(Q)$ *and* $d_\varepsilon \to d_0$ *weakly in* $L^2(0, T)$.

Finally, we give results for the convergence behavior of $y^{(\varepsilon)}$ in (1)–(5).

PROPOSITION 10. *Let* $v_\varepsilon \to v$ *weakly in* $L^2(0, T)$ *as* $\varepsilon \to 0$. *Then* $y^{(\varepsilon)}(v_\varepsilon) \to y^{(0)}(v)$ *weakly in* $L^2(Q)$ *as* $\varepsilon \to 0$.

PROOF. We note as a consequence of Lemma 1 that

$$\|y^{(\varepsilon)}(v_\varepsilon)\|_{L^\infty(0, T; H^0(\Omega))} \leqq \|v_\varepsilon\|_{L^2(0,T)} \tag{25}$$

so that there exists a sequence $\varepsilon_i \to 0$ such that

$$y^{(\varepsilon_i)}(v_{\varepsilon_i}) \to y$$

weak* in $L^\infty(0, T; H^0(\Omega))$, and thus weakly in $L^2(Q)$.

Introduce the adjoint equations which are (10)–(14) with time reversed.

$$-(1 - \varepsilon\Delta)\varphi_t^{(\varepsilon)} - \Delta\varphi^{(\varepsilon)} = \psi \text{ in } Q \tag{10'}$$

$$\varphi^{(\varepsilon)}(x, T) = \alpha_\varepsilon(x) \text{ in } \Omega \tag{11'}$$

$$\varphi^{(\varepsilon)}(x, t) = 0 \text{ on } \Sigma \tag{12'}$$

$$\varphi^{(\varepsilon)}(x, t) = d_\varepsilon(t) \text{ on } \Sigma_\rho \tag{13'}$$

where d_ε is an unknown function of t only, and

$$\int_{\Gamma_\rho} \frac{d\varphi^{(\varepsilon)}}{dn}(x, t)d = 0 \text{ a.e. in } [0, T]. \tag{14'}$$

By setting $\alpha_\varepsilon = 0$ and taking $\psi \in L^2(0, T; H)$ we may obtain the useful Green formula

$$\int_0^T \int_\Omega y^{(\varepsilon)}(v_\varepsilon)\psi dx\, ds = \int_0^T v_\varepsilon(t)d_\varepsilon(t)dt$$

for each $\varepsilon \geqq 0$. Now as $\varepsilon \to 0$ we see that, since, by Proposition 7, $d_\varepsilon \to d_0$ strongly in $L^2(0, T)$,

$$\int_0^T \int_\Omega y\psi dx\, dt = \int_0^T v(t)d(t)dt$$

for any $\psi \in L^2(0, T; H)$. This formula in fact implies that $y = y^{(0)}(v)$. Hence, we see that

$$y^{(\varepsilon_i)}(v_{\varepsilon_i}) \to y^{(0)}(v)$$

weakly in $L^2(Q)$. Again, the argument holds for any sequence $\varepsilon_i \to 0$ so we have the result.

PROPOSITION 11. *Let* $v_\varepsilon \to v$ *weakly in* $L^2(0, T)$ *as* $\varepsilon \to 0$. *Then* $y^{(\varepsilon)}(\cdot, T; v_\varepsilon) \to y^{(0)}(\cdot, T; v)$ *weakly in* $L^2(\Omega)$.

PROOF. Again we use a Green formula linking (1)–(5) to an adjoint equation (10′)–(14′). Set $\psi = 0$ in (10′) and take $\alpha^{(\varepsilon)}$ to be the solution of

$$\begin{gathered}(1 - \varepsilon\Delta)\alpha^{(\varepsilon)} = \beta \text{ in } \Omega \\ \alpha^{(\varepsilon)}|_\Gamma = 0 \\ \alpha^{(\varepsilon)}|_{\Gamma_\rho} = C_\varepsilon \text{ unknown constant} \\ \int_{\Gamma_\rho} \frac{d\varphi^{(\varepsilon)}}{dn} d\sigma = 0\end{gathered} \tag{26}$$

where $\beta \in L^2(\Omega)$. Note that

$$\int_\Omega (\alpha^{(\varepsilon)}(x))^2 dx + \varepsilon \int_\Omega |\nabla \alpha^{(\varepsilon)}(x)|^2 dx \leqq C \int_\Omega \beta^2(x) dx. \tag{27}$$

Furthermore, for $\beta \in H_0^1(\Omega)$, we have

$$\int_\Omega |\nabla \alpha^{(\varepsilon)}(x)|^2 dx + \varepsilon \int_\Omega |\Delta \alpha^{(\varepsilon)}(x)|^2 dx \leqq \int_\Omega |\nabla \beta(x)|^2 dx \tag{28}$$

and for $\beta \in H_0^2(\Omega)$,

$$\int_\Omega (\Delta \alpha^{(\varepsilon)}(x))^2 dx + \varepsilon \int_\Omega |\nabla \Delta \alpha^{(\varepsilon)}(x)|^2 dx \leqq \int_\Omega |\Delta \beta(x)|^2 dx. \tag{29}$$

By multiplying (10′) by $-\Delta\varphi_t$ and integrating, we have the estimate

$$\begin{aligned}\int_0^t \int_\Omega |\nabla \varphi_t^{(\varepsilon)}(x, s)|^2 dx\, ds + \int_\Omega (\Delta \varphi^{(\varepsilon)}(x, t))^2 dx \\ \leqq \int_\Omega (\Delta \alpha^{(\varepsilon)}(x))^2 dx \\ \leqq \int_\Omega (\Delta \beta(x))^2 dx.\end{aligned} \tag{30}$$

In this case we may obtain

$$\begin{aligned}\|d_\varepsilon\|_{H^1(0,T)} &\leqq K \|\varphi^{(\varepsilon)}\|_{H^1(0,T; H^1(\Omega))} \\ &\leqq K \|\beta\|_{H^2(\Omega)}\end{aligned}$$

where the constant is independent of $\varepsilon \geqq 0$. Hence, in contrast to Proposition 8, we see that there exists a sequence such that $d_{\varepsilon_i} \to d_0$ strongly in $L^2(0, T)$.

The Green formula is given by

$$\int_\Omega y^{(\varepsilon_i)}(x, T; v_{\varepsilon_i})\beta(x)dx = \int_0^T d_{\varepsilon_i}(t)v_{\varepsilon_i}(t)dt.$$

in the limit we see that

$$\int_\Omega \tilde{y}(x)\beta(x)dx = \int_0^T d_0(t)v(t)dt \tag{31}$$

since

$$y^{(\varepsilon_i)}(x, T; v_{\varepsilon_i}) \to \tilde{y}$$

weakly in $L^2(\Omega)$. However, equation (31) implies $\tilde{y} = y^{(0)}(\cdot, T; v)$. Again, from uniqueness, the above argument may be carried through for any sequence $\varepsilon_i \to 0$. Hence, the proposition is proved.

3. Control problems. In this section we consider the control problems (7) and (8) with underlying equations (1)–(5) in which v serves as a control variable. Our problems seek to find functions $u_\varepsilon^{(1)}$ and $u_\varepsilon^{(2)}$ in $L^2(0, T)$ that minimize (7) and (8) over $L^2(0, T)$, respectively. We apply the estimates obtained in the previous section to deduce regularity properties of the optimal control. The existence and uniqueness of these solutions is straightforward. Hence, we have the following proposition, c.f. [**4**].

PROPOSITION 12. *For each $\varepsilon \geqq 0$ there exist unique solutions $u_\varepsilon^{(1)}$ and $u_\varepsilon^{(2)}$ to problems* (7) *and* (8), *respectively.*

Further, in the usual manner, we may obtain optimality equations as the Euler equations of the various problems.

PROPOSITION 13. *Let $\varepsilon \geqq 0$. Then the equations characterizing $u_\varepsilon^{(i)}$ are given by*

$$(1 - \varepsilon\Delta)y_t^{(\varepsilon)} - \Delta y^{(\varepsilon)} = 0 \text{ in } Q, \tag{32}$$

$$y^{(\varepsilon)}(x, 0) = 0 \text{ in } \Omega, \tag{33}$$

$$y^{(\varepsilon)}(x, t) = 0 \text{ on } \Sigma, \tag{34}$$

$$y^{(\varepsilon)}(x, t) = c_\varepsilon(t) \text{ on } \Sigma_\rho \tag{35}$$

where $c_\varepsilon(t)$ is an unknown function of t only, and

$$\int_{\Gamma_\rho} \frac{\partial}{\partial n}(y^{(\varepsilon)} + \varepsilon y_t^{(\varepsilon)}d = u_\varepsilon^{(i)}(t) \text{ a.e. in } [0, T] \tag{36}$$

for $i = 1, 2$. Coupled with the adjoint equations

(37) $$-(1 - \varepsilon\Delta)p_t^{(\varepsilon)} - \Delta p^{(\varepsilon)} = y^{(\varepsilon)}(u_\varepsilon^{(1)}) - z \text{ in } Q,$$

(38) $$p^{(\varepsilon)}(\cdot, T) = 0 \text{ in } \Omega,$$

(39) $$p^{(\varepsilon)}(x, t) = 0 \text{ on } \Sigma,$$

(40) $$p^{(\varepsilon)}(x, t) = d_\varepsilon^{(1)}(t) \text{ on } \Sigma_\rho$$

where $d_\varepsilon^{(1)}(t)$ *is an unknown function of* t *only,*

(41) $$\int_{\Gamma_\rho} \frac{dp^{(\varepsilon)}}{dn}\, d\sigma = 0 \text{ a.e. in } [0, T],$$

(42) $$u_\varepsilon^{(1)}(t) = -d^{(1)}(t) \text{ a.e. in } [0, T],$$

and

(43) $$-(1 - \varepsilon\Delta)q_t^{(\varepsilon)} - \Delta q^{(\varepsilon)} = 0 \text{ in } Q,$$

(44) $$q^{(\varepsilon)}(x, T) = \alpha_\varepsilon(x) \text{ in } \Omega,$$

(45) $$q^{(\varepsilon)}(x, t) = 0 \text{ on } \Sigma,$$

(46) $$q^{(\varepsilon)}(x, t) = d_\varepsilon^{(2)}(t) \text{ on } \Sigma_\rho$$

where $d_\varepsilon^{(2)}(t)$ *is an unknown function of* t *only,*

(47) $$\int_{\Gamma_\rho} \frac{dq^{(\varepsilon)}}{dn}\, d\sigma = 0 \text{ a.e. in } [0, T],$$

(48) $$u_\varepsilon^{(2)}(t) = -d_\varepsilon^{(2)}(t) \text{ a.e. in } [0, T]$$

where α_ε *satisfies*

(49) $$\begin{gathered}(1 - \varepsilon\Delta)\alpha_\varepsilon(\cdot) = y^{(\varepsilon)}(\cdot, T; u_\varepsilon^{(2)}) - z(\cdot) \text{ in } \Omega,\\ \alpha_\varepsilon|_\Gamma = 0,\\ \alpha_\varepsilon|_{\Gamma_\rho} = c\end{gathered}$$

where c *is an unknown constant independent of* $x \in \Gamma_\rho$, *and*

$$\int_{\Gamma_\rho} \frac{d\alpha_\varepsilon}{dn}\, d\sigma = 0.$$

THEOREM 14. *Let* $z \in L^2(Q)$ *in* (7). *If* $\varepsilon > 0$, *then* $u_\varepsilon^{(1)}$ *belongs to* $H^1(0, T)$. *If* $\varepsilon = 0$, *then* $u_0^{(1)}$ *belongs to* $L^\infty(0, T)$. *If* $z \in L^2(0, T; H)$, *then for* $\varepsilon = 0$ *it follows that* $u_0 \in H^1(0, T)$.

PROOF. Since $u_\varepsilon^{(1)}(t) = -d_\varepsilon^{(1)}(t)$ on Γ_ρ, we may write for any $\varepsilon \geqq 0$,

$$\int_0^T (u_\varepsilon^{(1)}(t))^2 dt = \int_0^T (d_\varepsilon^{(1)}(t))^2 dt$$
$$= \int_0^T \frac{1}{\omega_n \rho^{n-1}} \int_{\Gamma_\rho} (p^{(\varepsilon)}(x, t))^2 dx\, dt.$$

By trace properties [5],

$$\frac{1}{\omega_n \rho^{n-1}} \int_0^T \int_{\Gamma_\rho} (p^{(\varepsilon)}(x, t))^2 dx\, dt \leqq \frac{1}{\omega_n \rho^{n-1}} \int_0^T \| p^{(\varepsilon)}(\cdot, t)\|_{H^1(\Omega)}^2 dt.$$

Similarly, we have

$$\int_0^T \left(\frac{d}{dt} u_\varepsilon^{(1)}(t)\right)^2 dt \leqq \frac{1}{\omega_n \rho^{n-1}} \int_0^T \| p_t^{(\varepsilon)}(\cdot, t)\|_{H^1(\Omega)}^2 dt.$$

From inequalities (15) and (16) we see that for $\varepsilon > 0$,

$$\int_0^T \| p^{(\varepsilon)}(\cdot, t)\|_{H^1(\Omega)}^2 + \int_0^T \| p_t^{(\varepsilon)}(\cdot, t)\|_{H^1(\Omega)}^2$$
$$\leqq C_\varepsilon \int_0^T \int_\Omega (y^{(\varepsilon)}(u_\varepsilon^{(1)}) - z)^2 dx\, dt.$$

However, if $\varepsilon = 0$, then from (15) it follows that

$$\int_0^T \| p^{(0)}(\cdot, t)\|_{H^1(\Omega)}^2 \leqq \int_0^T \int_\Omega (y^{(0)}(u_0^{(1)}) - z)^2 dx\, dt.$$

In fact, from (17), for each $t \leqq T$,

$$\| p^{(0)}(\cdot, t)\|_{H^1(\Omega)}^2 \leqq \int_0^T \int_\Omega (y^{(0)}(u_0^{(1)}) - z)^2 dx\, dt;$$

on the other hand, if $z \in L^2(0, T; H)$, then $y^{(\varepsilon)}(u_\varepsilon^{(1)}) - z$ belongs to $L^2(0, T; H)$ for any $\varepsilon \geqq 0$, and inequality (19) holds with ψ replaced by $y^{(\varepsilon)}(u_\varepsilon^{(1)}) - z$ and $\varphi^{(\varepsilon)}$ replaced by $p^{(\varepsilon)}$.

THEOREM 15. *Let $z \in L^2(\Omega)$ in* (8). *If $\varepsilon > 0$, then $u_\varepsilon^{(2)}$ belongs to $L^\infty(0, T)$. For $\varepsilon = 0$, $u_0^{(2)}$ belongs to $L^2(0, T)$. If $\varepsilon > 0$ and $z \in H$, then $u_\varepsilon^{(2)} \in H^1(0, T)$, and for $\varepsilon = 0$, $u_0^{(2)} \in L^\infty(0, T)$.*

PROOF. These results follow from the inequalities of Lemma 5 and (26)–(28) with β replaced by $y^{(\varepsilon)}(\cdot, T; u_\varepsilon) - z(\cdot)$ and $\varphi^{(\varepsilon)}$ replaced by $q^{(\varepsilon)}$.

4. Convergence behavior of optimal controls. We consider the family of optimal controls $\{u_\varepsilon^{(1)}: \varepsilon \geqq 0\}$ for problem (7). First we note that from Lemma 1 with v in $L^2(0, T)$ fixed,

$$\| y^{(\varepsilon)}(v)\|_{L^2(0, T; H^1(\Omega))} \leqq C\| v\|_{L^2(0, T)}$$

for any $\varepsilon \geqq 0$. Accordingly, we see from

$$J_\varepsilon^{(1)}(u_\varepsilon^{(1)}) \leqq J_\varepsilon^{(1)}(v) \leqq K(\|v\|_{L^2(0,T)}^2 + \|z\|_{L^2(Q)}^2)$$

that the sets $\{y^{(\varepsilon)}(u_\varepsilon^{(1)}): \varepsilon \geqq 0\}$ and $\{u_\varepsilon^{(1)}: \varepsilon \geqq 0\}$ are bounded in $L^2(Q)$ and $L^2(0, T)$, repsectively, independently of $\varepsilon \geqq 0$. Hence, there exists a sequence $\varepsilon_i \to 0$ as $i \to \infty$ such that

$$y^{(\varepsilon_i)}(u_{\varepsilon_i}^{(1)}) \to \tilde{y}$$

weakly in $L^2(Q)$ and

$$u_{\varepsilon_i}^{(1)} \to u$$

weakly in $L^2(0, T)$. As a consequence of Proposition 10, however, we see that

$$y^{(\varepsilon_i)}(u_{\varepsilon_i}^{(1)}) \to y^{(0)}(u)$$

weakly in $L^2(Q)$ as $i \to \infty$. Actually, by Lemma 1 there is a subsequence again $\varepsilon_i \to 0$ such that

$$y^{(\varepsilon_i)}(u_{\varepsilon_i}^{(1)}) \to y^{(0)}(u)$$

weakly in $L^2(0, T; H^1(\Omega))$. The next step is to show that u and $y^{(0)}(u)$ satisfy the characterizing equations (32)–(42). By uniqueness then we conclude $u = u_0^{(1)}$. But this step follows from Propositions 6 and 7. Indeed, we see that if $z \in L^2(0, T: H)$, then

$$u_{\varepsilon_i}^{(1)} \to u_0^{(1)}$$

strongly in $L^2(0, T)$. From Proposition 7,

$$p^{(\varepsilon_i)}(u_{\varepsilon_i}^{(1)}) \to p^{(0)}(u_0^{(1)})$$

strongly in $L^2(0, T; H)$. Finally, from the uniqueness of $u_0^{(1)}$, the above holds for any sequence $\varepsilon_i \to 0$.

THEOREM 16. *Let $\varepsilon \to 0$ and z belong to $L^2(0, T; H)$. Then $u_\varepsilon^{(1)} \to u_0^{(1)}$ strongly in $L^2(0, T)$ and $y^{(\varepsilon)}(u_\varepsilon^{(1)}) \to y^{(0)}(u_0^{(1)})$ weakly in $L^2(0, T; H)$.*

For problem (8) we may, as was done above, show that the sets $\{y^{(\varepsilon)}(\cdot, T; u_\varepsilon^{(2)}): \varepsilon \geqq 0\}$ and $\{u_\varepsilon^{(2)}: \varepsilon \geqq 0\}$ are bounded in $L^2(\Omega)$ and $L^2(0, T)$, respectively, independently of $\varepsilon \geqq 0$. Hence, from Proposition 11 there is a sequence such that

$$u_{\varepsilon_i}^{(2)} \to u$$

weakly in $L^2(0, T)$ and

$$y^{(\varepsilon_i)}(\cdot, T; u_{\varepsilon_i}^{(2)}) \to y^{(0)}(\cdot, T; u)$$

weakly in $L^2(\Omega)$. Note from Lemma 5 and inequality (27) that

$$\int_\Omega \{(q^{(\varepsilon)}(x, t))^2 + \varepsilon|\nabla q^{(\varepsilon)}(x, t)|^2\}dx + \int_0^t \int_\Omega |\nabla q^{(\varepsilon)}(x, s)|^2 dx\, ds$$
$$\leq \int_\Omega (y^{(\varepsilon)}(x, T; u_{\varepsilon_i}^{(2)}) - z(x))^2 dx$$

Hence, we see there is a subsequence $q^{(\varepsilon_i)}$ such that $q^{(\varepsilon_i)} \to q$ weakly in $L^2(0, T; H^1(\Omega))$, $q^{(\varepsilon_i)} \to q$ weak* in $L^\infty(0, T; L^2(\Omega))$ and accordingly,

$$d_{\varepsilon_i}^{(2)} \to d$$

weakly in $L^2(0, T)$. Now since

$$u_{\varepsilon_i}^{(2)} = -d_{\varepsilon_i}^{(2)},$$

we see that $u^{(2)} = -d$. Furthermore, it is easy to show that $q^{(0)} = q$ in $L^2(Q)$. For the boundary conditions consider $w^{(\varepsilon)} = q^{(\varepsilon)} - q^{(0)}$ with $\alpha \in C^\infty(\Omega)$ such that $\alpha|_\Gamma = \alpha|_{\Gamma_\rho} = d\alpha/dn|_{\Gamma_\rho} = 0$ and $d\alpha/dn|_{\Gamma_\rho} = 1$, we obtain

$$\int_0^t \int_\Omega w^{(\varepsilon)}\alpha_t + \int_0^t \int_{\Gamma_\rho} w^{(\varepsilon)} d\sigma - \int_0^t \int_\Omega w^{(\varepsilon)}\Delta\alpha = \varepsilon \int_0^t \int_\Omega w^{(\varepsilon)}\Delta\alpha_t.$$

As $\varepsilon \to 0$, we observe that for each $t \in [0, T]$, $\lim_{\varepsilon\to 0}\int_0^t [d^{(\varepsilon)}(s) - d^{(0)}(s)]ds = 0$. Hence, we have for each $t \in [0, T]$, $\int_0^t [d(s) - d^{(0)}(s)]ds = 0$. Therefore, $d = d^{(0)}$ a.e. in $[0, T]$ so that $u = -d^{(0)} = u_0^{(2)}$. By uniqueness the above argument holds for any sequence $\varepsilon_j \to 0$.

REMARK 17. Note that Proposition 8 is not used above since Lemma 1 only gives estimates of $|\nabla y^{(\varepsilon)}|$ in $L^2(0, T; L^2(\Omega))$ and not in $L^\infty(0, T; L^2(\Omega))$. Hence, increasing regularity of z (as in problem (7)) does not give better results. There is typically less regularity in problems of the type (8).

THEOREM 18. *Let $\varepsilon \to 0$. Then $u_\varepsilon^{(2)} \to u_0^{(2)}$ weakly in $L^2(0, T)$, $y^{(\varepsilon)}(u_\varepsilon^{(2)}) \to y^{(0)}(u_0^{(2)})$ weakly in $L^2(Q)$ and $y^{(\varepsilon)}(\cdot, T; u_\varepsilon^{(2)}) \to y^{(0)}(\cdot, T; u_0^{(2)})$ weakly in $L^2(\Omega)$.*

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