ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 13, Number 2, Spring 1983

SOME RATIONAL CONTINUA

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In this note there are presented some examples of rational continua. The first example is of a rational continuum X (of rim-type 2) and a confluent mapping of X onto a non-rational continuum. This answers in the negative Problem III which was posed by A. Lelek in [**6**, p. 57]. In the second example there is presented a rational continuum X of rim-type 2 and a confluent mapping of X onto a rational continuum of rim-type 3. These two examples give negative answers to the following question which was posed by B.B. Epps in his dissertation [**3**, p. 6]: If X is a rational continuum of finite rim-type and f: $X \to Y$ is a confluent map, is the rim-type of Y less than or equal to the rim-type of X? In the second example there is given a rational, uniquely arcwise connected continuum X which contains a dense ray (continuous one-to-one image of $[0, 1)$) which is of first category in X. This answers in the negative a question posed by J.B. Fugate in a talk given at the Auburn Topology Conference in March 1976 (see [**4**, Question 2]). The third and final example in this note is of a hereditarily locally connected continuum X which contains a dense ray which is of first category in X.

I wish to thank Professors A. Lelek and J.R. Martin for several very helpful conversations.

1. Definitions and preliminaries. Our notation follows that of Whyburn [**9**]. By a *continuum* is meant a compact, connected, metric space. The set of natural numbers is denote by **N**. A continuum X is *rational at a point* $x \in X$ if X has a neighbourhood basis at x of open sets with countable boundaries. A continuum is *rational* if it is rational at each of its points. A sequence of sets is said to form a *null sequence* if the diameters of the sets converge to zero. A continuous function f of a continuum X onto a continuum Y is *confluent* if for each continuum C in Y each component of $f^{-1}(C)$ maps onto C. Let Cl(A) and Bd(A) denote the closure and boundary, respectively, of a set A. By a *neighbourhood* we shall mean an open neighbourhood.

If A is a subset of a space X, let A' denote the derived set of A. Let

This work was supported in part by National Research Council (Canada) grant No. A5616.

Received by the editors on August 13, 1976.

$A^{(0)} = A$. If α is the successor of the ordinal number n, let $A^{(\alpha)} = (A^{(n)})'$. If α is a limit ordinal let

$$A^{(\alpha)} = \bigcap\{A^{(n)} | n < \alpha\}.$$

If C is a compact, countable subset of a metric space, then there exists a countable ordinal α such that $C^{(\alpha)} = \emptyset$. We denote the smallest such ordinal α by ttyp(C). If X is a continuum which is rational at x, then X has a countable neighbourhood basis at x of open sets with countable boundaries. We define the *rim-type of X at x by* $\text{rimt}_x(X) = \alpha$ where α is the smallest ordinal such that X has a neighbourhood basis at x of open sets $U_i)_{i \in N}$ such that $\text{ttyp}(\text{Bd}(U_i)) \leqq \alpha$ for each $i \in N$. Then $\text{rimt}_x(X)$ is a countable ordinal number.

If X is a rational continuum we denote the *rim-type of X* by

$$\text{rimt}(X) = \sup\{\text{rimt}_x(X) | x \in X\}.$$

It is well-known (see [5, p. 290]) that the rim-type of a rational continuum is an ordinal number that is strictly smaller than the first uncountable ordinal Ω. We shall need the following slightly stronger result.

LEMMA 1. *If X is a continuum which is rational at each point of a subset A of X, then there exists a countable ordinal α such that* $\text{rimt}_x(S) \leqq \alpha$ *for each $x \in A$.*

PROOF. Let $\mathscr{B}$ be a countable base for A of open sets in X with countable boundaries. Let

$$\alpha = \sup\{\text{ttyp}(\text{Bd}(U)) | U \in \mathscr{B}\}.$$

LEMMA 2. *Let $A_i)_{i \in N}$ be a null sequence of pairwise disjoint rational continua in a continuum X. If α and β are countable ordinal numbers such that* $\text{rimt}_x(X) \leqq \alpha$ *for each* $x \in X \backslash (A_0 \cup A_1 \cup \cdots)$ *and* $\text{rimt}(A_i) \leqq \beta$ *for each $i \in N$, then* $\text{rimt}(X) \leqq \alpha + \beta$.

PROOF. Let $x \in A_0$ and let U be a neighbourhood of x. Then

$$W = U \backslash \bigcup\{A_i | i \geqq 1 \text{ and } A_i \cap \text{Bd}(U) \neq \emptyset\}$$

is a neighbourhood of x since $A_i)_{i \in N}$ is a null sequence of closed sets. Since $\text{rimt}(A_0) \leqq \beta$, there exists a neighbourhood V of x in X such that $\text{Cl}(V) \subset W$ and $(\text{Bd}(V) \cap A_0)^{(\beta)} = \emptyset$.

Define an equivalence relation $\sim$ on X by setting $x \sim y$ if and only if $x = y$ or there exists $i \in N$ such that $x, y \in A_i$. Since the non-degenerate equivalence classes of $\sim$ form a null sequence of closed sets, it follows that $\sim$ is upper semi-continuous and the quotient space $X/\sim$ is a continuum. Let π be the natural projection of X onto the quotient space $X/\sim$. Notice that $\pi(W)$ is open in $X/\sim$.

Let $\mathscr{B}$ be a countable basis for $X/\sim$ of open sets whose boundaries miss the countable set $\pi(A_0 \cup A_1 \cup \cdots)$. We may suppose, since $X/\sim$ is a compact metric space, that the members of $\mathscr{B}$ form a null sequence. Let $\mathscr{C} \subset \mathscr{B}$ be a locally finite collection in $(X/\sim)\backslash\pi A_0)$ such that $\mathscr{C}$ is a cover for $\pi(\mathrm{Bd}(V))\backslash\pi(A_0)$ and such that, for each $C \in \mathscr{C}$, C meets $\pi(\mathrm{Bd}(V))$ and $\mathrm{Cl}(C)$ is contained in the open set $\pi(W)\backslash\pi(A_0)$. Let $\mathscr{C}' = \{\pi^{-1}(C)|C \in \mathscr{C}\}$. We may write $\mathscr{C}' = \{C_i|i \in N\}$. Then $\mathscr{C}'$ is a locally finite collection in $X\backslash A_0$ which covers $\mathrm{Bd}(V)\backslash A_0$ and, if $C_i \in \mathscr{C}'$, $\mathrm{Cl}(C_i) \subset W\backslash A_0$ and $\mathrm{Bd}(C_i) \subset X\backslash(A_0 \cup A_1 \cup \cdots)$.

Let $C_i \in \mathscr{C}'$. For each $y \in \mathrm{Bd}(C_i)$ let B_y be a neighbourhood of y with $(\mathrm{Bd}(B_y))^{(\alpha)} = \varnothing$, with diameter $B_y < 1/i$ and with $\mathrm{Cl}(B_y) \subset W\backslash A_0$. Since $\mathrm{Bd}(C_i)$ is compact, there exist $n \in N$ and $y_1, \ldots, y_n \in \mathrm{Bd}(C_i)$ such that $B_{y_1} \cup \cdots \cup B_{y_n}$ contains $\mathrm{Bd}(C_i)$. Then $D_i = C_i \cup B_{y_1} \cup \cdots \cup B_{y_n}$ is a neighbourhood of C_i with

$$\text{diameter } D_i \leqq \text{diameter } C_i + 2/i$$

and with $\mathrm{Cl}(D_i) \subset W\backslash A_0$. Also, $\mathrm{Bd}(D_i) \subset \mathrm{Bd}(B_{y_1}) \cup \cdots \cup \mathrm{Bd}(B_{y_n})$ so $(\mathrm{Bd}(D_i))^{(\alpha)} = \varnothing$. Let

$$P = V \backslash \bigcup\{\mathrm{Cl}(D_i)|i = 1, 2, \ldots\}.$$

Then P is an open neighbourhood of x and

$$\mathrm{Bd}(P) \subset (A_0 \cap \mathrm{Bd}(V)) \cup \bigcup\{\mathrm{Bd}(D_i)|i \in N\}$$

since the sets $D_1, D_2, \ldots$ form a null locally finite collection in $X\backslash A_0$. If $y \in \mathrm{Bd}(P)\backslash A_0$, then there exists a neighbourhood G of y and $n \in N$ such that $\mathrm{Bd}(P) \cap G \subset \mathrm{Bd}(D_1) \cup \cdots \cup \mathrm{Bd}(D_n)$. Hence $(\mathrm{Bd}(P))^{(\alpha)} \subset \mathrm{Bd}(V) \cap A_0$ and $(\mathrm{Bd}(P))^{(\alpha+\beta)} \subset (\mathrm{Bd}(V) \cap A_0)^{(\beta)} = \varnothing$. This completes the proof of the lemma.

COROLLARY 3. *Let X be a continuum and let $A_i)_{i\in N}$ be a null sequence of pairwise disjoint rational continua in X. Then X can not fail to be rational only at points of $A_0 \cup A_1 \cup \cdots$.*

PROOF. The corollary follows immediately from Lemma 1 and 2.

Lemma 2 and Corollary 3 fail if the continua $A_i)_{i\in N}$ do not form a null sequence. Lelek has given an example of an arclike Suslinian continuum which fails to be rational only at points in the union of a countable family of disjoint arcs. Another example relevant to this paper is the continuum Y given in Example 3.1 of [J. Grispolakis and E.D. Tymchatyn, *Confluent images of rational continua*, Houston J. Math. 5 (1979), 331–337].

A continuous mapping of a continuum X onto a locally connected continuum Y is said to be *pseudo-confluent* (see [7]) if for each arc A in Y some component of $f^{-1}(A)$ maps onto A. A confluent map is clearly pseudo-confluent.

The following proposition is related to a result in [7], Theorem 5.1]. It shows that Epps' question has a positive answer if the image space is locally connected.

PROPOSITION 4. *If $f: X \to Y$ is a pseudo-confluent mapping of a rational continuum X onto a locally connected continuum Y, then Y is rational and* $\operatorname{rimt}(T) \leqq \operatorname{rimt}(X)$.

PROOF. Let $\alpha = \operatorname{rimt}(X)$. Let $y, z \in Y$. Let A be a countable compact set in X such that A separates $f^{-1}(x)$ from $f^{-1}(y)$ and $\operatorname{ttyp}(A) \leqq \alpha$. Since f is pseudo-confluent and Y is locally connected, it follows (as in [7, Theorem 4.5]) that $f(A)$ separates x and y in Y. It is easy to check by transfinite induction that $(f(A))^{(n)} \subset f(A^{(n)})$ for each ordinal n. Hence $\operatorname{ttyp}(f(A)) \leqq \operatorname{ttyp}(A)$. Thus, $\operatorname{rimt}(Y) \leqq \operatorname{rimt}(X)$.

LEMMA 5. *Let f be a continuous mapping of a compact metric space X onto a compact metric space Y. Let $K = \{x \in Y \mid f^{-1}(x)$ is non-degenerate$\}$. If $\{f^{-1}(x) \mid x \in K\}$ forms a null sequence in X, then $f|_{X\setminus f^{-1}(K)}$ is an embedding of $X \setminus f^{-1}(K)$ into Y.*

PROOF. Let $x \in X \setminus f^{-1}(K)$ and let U be a neighbourhood of x. Then $f(X\setminus U)$ is compact and hence closed in Y. The set $X\setminus f^{-1}f(X\setminus U) \subset U$ is a neighbourhood of x since the sets $\{f^{-1}(y) \mid y \in K\}$ form a null sequence. Hence $f(X\setminus f^{-1}f(X\setminus U)) = Y\setminus f(X\setminus U) \subset f(U)$ is a neighbourhood of $f(x)$. Thus, $f|_{X\setminus f^{-1}(K)}$ is a homeomorphism.

2. Examples. We are now ready to present our first example. This is an example of a rational continuum Y (of rim type 2) and a confluent mapping f of y onto a non-rational continuum X.

EXAMPLE 1. Let S be the Sierpinski triangular curve (see Kuratowski [5, p. 276]). It is defined there as follows. Let T be the equilateral triangle in the plane with vertices $(0, 0)$, $(1, 1)$ and $(\sqrt{2}, 0)$. Partition T into four congruent triangles T_0, T_1, T_2, T_3. Let T_0, T_1, T_2 be the triangles which have a vertex in common with T. The triangles T_0, T_1 and T_2 are numbered clockwise and T_0 is the left-most triangle of the three. Let v_0, v_1, v_2 be the vertices of T_3 where v_0 is the left-most vertex of the three and the numbering is clockwise. In a similar way partition each of the triangles T_i for $i = 0, 1, 2$ into four congruent triagnles $T_{i,0}, T_{i,1}, T_{i,2}, T_{i,3}$, where $T_{i,3}$ is the triangle which has no vertices in common with T_i. Let $v_{i,0}, v_{i,1}$ and $v_{i,2}$ be the vertices of $T_{i,3}$. The vertices $v_{i,0}, v_{i,1}, v_{i,2}$ and the triangles $T_{i,0}, T_{i,1}, T_{i,2}$ are numbered clockwise starting with the left-most one.

Continue inductively in this manner. Let

$$S = \operatorname{Cl}(\bigcup_D \operatorname{Bd}(T_{\alpha_1, \ldots, \alpha_k})$$

where $D = \{(\alpha_1, \ldots, \alpha_k) | k = 1, 2, \ldots$ and $\alpha_1, \ldots, \alpha_k \in \{0, 1, 2\}\}$.

The local separating point of S are the vertices $v_{\alpha_1, \ldots, \alpha_k}$ where $(\alpha_1, \ldots, \alpha_k) \in D$.

Our example is obtained from the Sierpinski curve S as an inverse limit by successively exploding the local separating points of S to arcs.

Let $X_0 = Y_0 = S$ and let f_0: $Y_0 \to X_0$ be the identity map. Let $Y_1 = \bigcup_{i=0}^2 (A_i \cup S_i \cup B_i)$ be a plane continuum and π_1: $Y_1 \to Y_0$ a continuous map such that for each $i = 1, 2, 0$, π_1 carries S_i homeomorphically onto $T_i \backslash \{v_0, v_1, v_2\}$, $\pi_1^{-1}(v_i) = A_i \cup B_i$, $\mathrm{Cl}(S_i) = S_i \cup A_i \cup B_{(i+2) \bmod 3}$, $\mathrm{Cl}(S_i)$ has three arc components, and A_i and B_i are line segments of the same length such that $A_i \cap B_i = \{m_i\}$ where m_i is a common endpoint of A_i and B_i. Suppose also that if K is a ray in S_i such that $v_j \in \mathrm{Cl}(\pi_1(K))$, then $A_i \subset \mathrm{Cl}(K)$ when $i = j$, and $B_j \subset \mathrm{Cl}(K)$ when $j \equiv (i + 2) \bmod 3$. We identify the points of $Y_0 \backslash \{v_0, v_1, v_2\}$ with their preimages in Y_1. Let

$$T^1_{\alpha_1, \ldots, \alpha_k} = \pi_1^{-1}(T_{\alpha_1, \ldots, \alpha_k}) \cap \mathrm{Cl}\,(S_{\alpha_1})$$

for $(\alpha_1, \ldots, \alpha_k) \in D$.

Define an equivalence relation $\sim_1$ on Y_1 by setting $x \sim_1 y$ if and only if $x = y$ or $x, y \in A_i \cup B_i$ for some i and the distance from x to m_i equals the distance from y to m_i. Then $\sim_1$ is an upper semi-continuous relation on Y_1. Let X_1 be the quotient space $Y_1/\sim_1$ and let f_1: $Y_1 \to X_1$ be the natural projection. Let ϕ_1: $X_1 \to X_0$ be such that $\phi_1 \circ f_1 = f_0 \circ \pi_1$. See Figure 1.

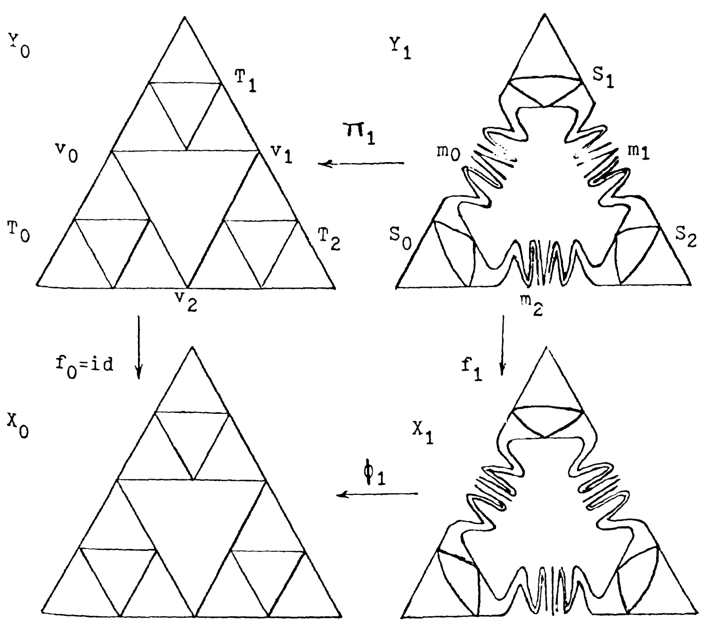

Figure 1.

The space Y_1 was obtained from Y_0 by replacing by arcs $A_i \cup B_i$ each of the three local separating points v_i of $S = Y_0$ which were obtained at the first stage of construction of S. The space X_1 was obtained from Y_1 by folding in half each of the arcs $A_i \cup B_i$ and thus eliminating the three local separating points m_i, $i = 0, 1, 2$, in Y_1. Both Y_1 and X_1 have six arc components.

Let

$$Y_2 = \bigcup\{S_{\alpha_1,\alpha_2} \cup A_{\alpha_1,\alpha_2} \cup B_{\alpha_1,\alpha_2} | \alpha_1, \alpha_2 = 0, 1, 2\}$$

be a plane continuum and let π_2: $Y_2 \to Y_1$ be a continuous map such that π_2 carries $\bigcup\{S_{\alpha_1,\alpha_2} | \alpha_1, \alpha_2 = 0, 1, 2\}$ homeomorphically onto $Y_1 \backslash \{v_{\alpha_1,\alpha_2} | \alpha_1, \alpha_2 = 0, 1, 2\}$, and for each $\alpha_1, \alpha_2 = 0, 1, 2$,

$$\pi_2(S_{\alpha_1,\alpha_2}) = T^1_{\alpha_1,\alpha_2} \backslash \{v_{\alpha_1,\alpha_2} | \alpha_1, \alpha_2 = 0, 1, 2\},$$
$$\pi_2^{-1}(v_{\alpha_1,\alpha_2}) = A_{\alpha_1,\alpha_2} \cup B_{\alpha_1,\alpha_2},$$
$$\mathrm{Cl}(S_{\alpha_1,\alpha_2}) = S_{\alpha_0,\alpha_2} \cup A_{\alpha_1,\alpha_2} \cup B_{\alpha_1,(\alpha_2+2) \bmod 3},$$

the number of arc components of $\mathrm{Cl}(S_{\alpha_1,\alpha_2})$ is two more than the number of arc components of $T^1_{\alpha_1,\alpha_2}$, A_{α_1,α_2} and B_{α_1,α_2} are line segments of the same length such that $A_{\alpha_1,\alpha_2} \cap B_{\alpha_1,\alpha_2} = \{m_{\alpha_1,\alpha_2}\}$ where m_{α_1,α_2} is a common endpoint of A_{α_1,α_2} and B_{α_1,α_2}. Suppose also that if K is a ray in S_{α_1,α_2} such that $v_{\alpha_1,j} \in \mathrm{Cl}(\pi_2(K))$, then $A_{\alpha_1,\alpha_2} \subset \mathrm{Cl}(K)$ when $j = \alpha_2$ and $B_{\alpha_1,j} \subset \mathrm{Cl}(K)$ when $j = (\alpha_2 + 2) \bmod 3$. We identify the points of $Y_1 \backslash \{v_{\alpha_1,\alpha_2} | \alpha_1, \alpha_2 \in \{0, 1, 2\}\}$ with their preimages in Y_2. Let

$$T^2_{\alpha_1,\ldots,\alpha_k} = \pi_2^{-1}(T^1_{\alpha_1,\ldots,\alpha_k}) \cap \mathrm{Cl}(S_{\alpha_1,\alpha_2})$$

for $(\alpha_1, \ldots, \alpha_k) \in D$ and $k \geqq 2$. Let $T^2_{\alpha_1} = \pi_2^{-1}(T^1_{\alpha_1})$ for $\alpha_1 = 0, 1, 2$.

Define an equivalence relation $\sim_2$ on Y_2 by setting $x \sim_2 y$ in Y_2 if and only if $x = y$ or $x, y \in A_{\alpha_1,\alpha_2} \cup B_{\alpha_1,\alpha_2}$ for some $\alpha_1, \alpha_2 \in \{0, 1, 2\}$ and the distance from x to m_{α_1,α_2} is the same as the distance from y to m_{α_1,α_2}. Then $\sim_2$ is an upper semi-continuous relation on Y_2. Let X_2 be the quotient space $Y_2/\sim_2$ and let f_2: $Y_2 \to X_2$ be the natural projection. Let ϕ_2: $X_2 \to X_1$ be such that $\phi_2 \circ f_2 = f_1 \circ \pi_2$.

The space Y_1 was obtained from Y_2 by replacing by arcs $A_{\alpha_1,\alpha_2} \cup B_{\alpha_1,\alpha_2}$ each of the nine local separating points of Y_1 which correspond to the vertices of the triangles $T_{i,3}$ $i = 0, 1, 2$, which were introduced at the second stage of construction of S. The only point of $A_{\alpha_1,\alpha_2} \cup B_{\alpha_1,\alpha_2}$ which is a local separating point of Y_2 is m_{α_1,α_2}. The space X_2 was obtained from Y_2 by folding in half each of the arcs $A_{\alpha_1,\alpha_2} \cup B_{\alpha_1,\alpha_2}$ so that $f_2(A_{\alpha_1,\alpha_2}) = f_2(B_{\alpha_1,\alpha_2})$ contains no local separating points of X_2.

We can continue this process inductively to define for each $n = 1, 2, \cdots$, space Y_n and X_n and maps π_n and ϕ_n such that the rectangles in the following diagram commute

$$\begin{array}{ccccccccc} Y_0 & \xleftarrow{\pi_0} & \cdots & \xleftarrow{\pi_{n-1}} & Y_{n-1} & \xleftarrow{\pi_n} & Y_n & \leftarrow & \cdots \\ \downarrow f_0 & & \cdots & & \downarrow f_{n-1} & & \downarrow f_n & & \cdots \\ X_0 & \xleftarrow{\phi_0} & \cdots & \xleftarrow{\phi_{n-1}} & X_{n-1} & \xleftarrow{\phi_n} & X_n & \leftarrow & \cdots . \end{array}$$

Let Y be the inverse limit of the sequence (Y_n, π_n) and let X be the inverse limit of the sequence (X_n, ϕ_n). Let $\pi: Y \to S$, $\phi: X \to S$ and $f: Y \to X$ be the natural maps induced by the above diagram. Then Y is clearly rational at each point of

$$Y \setminus \bigcup \{\pi^{-1}(v_{\alpha_1,\dots,\alpha_k}) | (\alpha_1, \dots, \alpha_k) \in D\} .$$

In fact at each of these points Y has a neighbourhood basis of open sets with boundaries consisting of at most four points (the boundary points are the points in Y which correspond to points in $\{m_{\alpha_1,\dots\alpha_k} | (\alpha_1, \dots, \alpha_k) \in D\}$ in Y_k). The sets $\pi^{-1}(v_{\alpha_1,\dots,\alpha_k}))$ where $(\alpha_1, \dots, \alpha_k) \in D$ form a null sequence of pairwise disjoint arcs in Y. By Lemma 2, Y is rational and $\operatorname{rimt}(Y) \leq 2$. It is easy to see that Y does not have a basis of open sets with finite boundaries at the point corresponding to m_0. Hence $\operatorname{rimt}(Y) = 2$. The continuum X is not rational since X contains no local separating points (see [**9**, III.9.43]).

It remains to prove only that f: $Y \to X$ is confluent. Notice that f is at most two-to-one on Y and f is one-to-one off of the inverse image under π of the local separating points of S. Let K be a continuum in X and suppose K meets $f(A_{\alpha_1,\dots,\alpha_k})$. If $K \subset f(A_{\alpha_1,\dots,\alpha_k})$, then $f^{-1}(K)$ has at most two components and both of these are mapped onto K by f. If $K \not\subset f(A_{\alpha_1,\dots,\alpha_k})$, then $K \supset f(A_{\alpha_1,\dots,\alpha_k})$ by the construction of X and Y. Thus

$$A_{\alpha_1,\dots,\alpha_k} \cup B_{\alpha_1,\dots,\alpha_k} \subset f^{-1}(K) .$$

Now $\pi^{-1} : S \to Y$ is upper semi-continuous. If $K \not\subset f(A_{\alpha_1,\dots,\alpha_k})$, then π^{-1} restricted to $\phi(K)$ is monotone. Hence $\pi^{-1}\phi(K) = f^{-1}(K)$ is connected. In each case each component of $f^{-1}(K)$ maps onto K and f is confluent.

A continuum is said to be *decomposable* if it can be written as the union of two proper subcontinua. A continuum is said to be *hereditarily decomposable* if each subcontinuum is decomposable. Since every rational continuum contains a countable set whose complement is zero-dimensional and every indecomposable continuum has an uncountable family of pairwise disjoint, non-degenerate continua (see [**5**, p. 212, Theorem 7]), every rational continuum is hereditarily decomposable. A continuum X is said to be uniquely *arcwise connected* if for each $x \neq y$ in X there is a unique arc in X with endpoints x and y.

The next example is of a rational uniquely arcwise connected continuum which contains a dense ray which is of first category. This answers Question 2 of Fugate [4].

EXAMPLE 2. Let X, S and $\phi: X \to S$ be as in Example 1. Let $x * y$ in X if and only if $x = y$ or there exists $k \in N$ and $\alpha_1, \ldots, \alpha_{k-1} \in \{0, 1, 2\}$ and $\alpha_k \in \{0, 1\}$ such that $x, y \in \phi^{-1}(v_{\alpha_1,\ldots,\alpha_k})$. Then $*$ is an equivalence relation on X. The equivalence classes of $*$ that are non-degenerate form a null sequence of arcs. Hence, $*$ is upper semi-continuous. Let Z be the quotient space $X/*$ and let $\theta: X \to Z$ be the natural projection.

If $(x, 0) \in S$ where $0 < x < \sqrt{2}$ and $(x, 0)$ is not a local separating point of S, then there exists a sequence $u_n)$ of local separating points of S where $u_n = v_{\alpha_1,\ldots,\alpha_{k(n)}}, \alpha_1 \ldots, \alpha_{k(n)-1} \in \{0, 1, 2\}$, $\alpha_{k(n)} \in \{0, 1\}$, the sequence $u_n)$ converges to $(x, 0)$ and $\{(x, 0)\} \cup \bigcup_{n=1}^{\infty}\{u_n\}$ separates $[0, x) \times \{0\}$ from $(x, \sqrt{2}] \times \{0\}$ in S. Also,

$$\theta \circ \phi^{-1}(\{(x, 0)\} \cup \bigcup_{n=1}^{\infty} \{u_n\})$$

separates $\theta \circ \phi^{-1}([0, x) \times \{0\})$ from $\theta \circ \phi^{-1}((x, \sqrt{2} \times \{0\})$ in Z. Notice that $\theta \circ \phi^{-1}(\bigcup_{n=1}^{\infty}\{u_n\})$ is a sequence in Z which converges to the point $\theta \circ \phi^{-1}((x, 0)$. It is now easy to show by a similar argument that if $(x, y) \in S$ such that $z = \theta \circ \phi^{-1}((x, y))$ is a single point, then Z has a neighbourhood basis at z of open sets whose boundaries have at most three limit points. Thus, $\mathrm{rimit}_z(Z) \leqq 2$. It is easy to see that no finite set separates Z between $\theta \circ \phi^{-1}((y, 0))$ and $\theta \circ \phi^{-1}((x, 0))$ for all x and y such that $0 \leqq y < x \leqq \sqrt{2}$. Hence $\mathrm{rimt}_{(0,0)}(Z) = 2$. By Lemma 3, $\mathrm{rimt}(Z) \leqq 3$ since the set of points z in Z such that $\mathrm{rimt}_z(Z) > 2$ is contained in the union of a null sequence of pairwise disjoint arcs. If $z \in \theta \circ \phi^{-1}(v_2)$ and U is a small neighbourhood of z, then the boundary of U disconnects $\theta \circ \phi^{-1}([0, \sqrt{2}])$ into infinitely many components. It follows from the above that ttyp $(\mathrm{Bd}(U)) \geqq 3$. Thus $\mathrm{rimt}(Z) = 3$.

Let Y and $f: Y \to X$ be as in Example 1. The map $\theta \circ f: Y \to Z$ is a confluent map (since it is a composition of confluent maps) which carries a continuum of rim-type 2 onto a continuum of rim-type 3.

Let $W = S \backslash \{v_{\alpha_1,\ldots,\alpha_k,2} | k \in N$ and $\alpha_1, \ldots, \alpha_k = 0, 1, 2\}$. Then W is a uniquely arcwise connected set. By Lemma 5, $\theta \circ \phi^{-1}(W)$ is homeomorphic to W. It is now easy to see that the arc components of Z are $\theta \circ \phi^{-1}(W)$ and the null sequence of pairwise disjoint arcs $\theta \circ \phi^{-1}(v_{\alpha_1,\ldots,\alpha_k,2})$ where $k \in N$ and $\alpha_1, \ldots, \alpha_k \in \{0, 1, 2\}$.

We may suppose Z lies in a hyperplane in E^4 since it is one-dimensional. Adjoin to Z a null sequence of pairwise disjoint arcs $D_i)_{i \in N}$ as follows.

(1) $D_i \cap Z$ consists of exactly two points.

(2) D_0 is a semi-circle in E^4 such that D_0 meets Z in $\theta \circ \phi^{-1}((1, 1))$ and one of the endpoints of $\theta \circ \phi^{-1}(v_2)$.

(3) $R = (D_0 \cup D_1 \cup \cdots) \cup \bigcup\{\theta \circ \phi^{-1}(v_{\alpha_1,\ldots,\alpha_k,2}) | k \in N, \alpha_1, \ldots, \alpha_k \in \{0, 1, 2\}\}$ is a ray.

(4) If n is the smallest integer such that D_n meets $\theta \circ \phi^{-1}(v_{\alpha,\ldots,\alpha_k,2})$, then D_{n+1} also meets $\theta \circ \phi^{-1}(v_{\alpha_1,\ldots,\alpha_k,2})$.

(5) If $\beta_1, \ldots, \beta_{k-1} \in \{0, 1, 2\}$ and $j \in N$ such that $\theta \circ \phi^{-1}(v_{\alpha_1,\ldots,\alpha_{k+j},2})$ meets D_n for some $\alpha_1, \ldots, \alpha_{k+j} \in \{0, 1, 2\}$, then there exists $m \in N$ such that $m < n$ and D_m meets $\theta \circ \phi^{-1}(v_{\beta_1,\ldots,\beta_{k-1},2})$.

(6) If D_n meets $\theta \circ \phi^{-1}(v_{\alpha_1,\ldots,\alpha_k,2})$ and $\theta \circ \phi^{-1}(v_{\beta_1,\ldots,\beta_j,2})$, then $T_{\alpha_1,\ldots,k}$ meets $T_{\beta_1,\ldots,\beta_k}$.

It is easy to find a null family of pairwise disjoint arcs D_i) satisfying conditions (1)–(6). Then $Z' = Z \cup \bigcup D_i$ is an arcwise connected continuum. It is also not very difficult to see that Z' is uniquely arcwise connected.

Define $x \# y$ in Z' if and only if $x = y$ or there exists $i \in N$ such that $x, y \in D_i$. Then $\#$ is an equivalence relation on Z' since the sets D_i are pairwise disjoint. Since the non-degenerate equivalence classes of $\#$ are closed and form a null sequence, $\#$ is upper semi-continuous. Thus $Z'/\# = Z/\#$ is a continuum. The image of R in $Z'/\#$ is a ray which is dense and of first category in $Z'/\#$.

It is well known that a continuum is rational if and only if it contains a countable set with zero-dimensional complement. If C is a countable set in Z with zero-dimensional complement, then the image of C together with the image of $D_0 \cup D_1 \cup \cdots$ is a countable set in $Z'/\#$ with zero-dimensional complement in $Z'/\#$ by Lemma 5. Thus $Z'/\#$ is a rational continuum.

We next give an example of a hereditarily locally connected continuum X which contains a dense ray which is of first category in X. Note that such an example cannot be uniquely arcwise connected for a uniquely arcwise connected, locally connected continuum is a dendrite.

EXAMPLE 3. Let $[0, 1]$ denote a unit segment on the z-axis in Euclidean three-space. Let $C_1, C_2, \ldots,$ be a sequence of Cantor sets in $[0, 1]$ such that for each $n = 1, 2, \ldots,$

(1) the components of $[0, 1]\backslash C_n$ have diameter less than $1/n$,

(2) if n is even $C_n \cap C_{n-1} = \{b_n\}$ where $b_n = \sup C_{n-1} = \sup C_n$,

(3) if $n > 1$ is odd, $C_n \cap C_{n-1} = \{a_n\}$ where $a_n = \inf C_{n-1} = \inf C_n$, and

(4) $C_n \cap (C_1 \cup \cdots \cup C_{n-2}) = \varnothing$.

If C is a Cantor set in $[0, 1]$, x and y two points of C are said to be *consecutive endpoints* of C if x and y are the two endpoints of the closure of a component of $[0, 1]\backslash C$.

For each natural number n let P_n be the plane in Euclidean three-space which contains the z-axis and the point $(1, n, 0)$. If x and y are consecutive endpoints of C_n, let $\overline{xy}$ be a semi-circle in P_n with endpoints x and y. For each n let $A_n = C_n \cap \bigcup\{\overline{xy} | x$ and y are consecutive endpoints of $C_n\}$. Then each A_n is an arc in P_n.

Let $X = [0, 1] \cup A_1 \cup A_2 \cup \cdots$. Then X is obtained by attaching to the arc $[0, 1]$ a null sequence of disjoint arcs each of which meets $[0, 1]$. By [**8**, p. 94] X is a hereditarily locally connected continuum. Also $R = A_1 \cup A_2 \cup \cdots$ is a dense ray in X.

Let $x \sim y$ in X if and only if $x = y$ or $x, y \in \overline{z_1 z_1}$ for some z_1 and z_2 consecutive endpoints of C_n for some $n = 1, 2, \ldots$. Then $\sim$ is an upper semi-continuous equivalence relation on X. The quotient space $X/\sim$ is hereditarily locally connected since the projection map is monotone and monotone mappings preserve hereditarily locally connected continua (see [**6**, p. 58]). The image of the ray R in $X/\sim$ under the natural projection mapping is a ray which is dense and of first category in $X/\sim$.

It is easy to modify Example 3 to obtain a hereditarily locally connected continuum with countably infinitely many disjoint dense rays.

QUESTION 1. Does there exist a hereditarily decomposable continuum which contains uncountably many disjoint dense rays?

QUESTION 2. If X is a locally connected continnum, is it true that X is finitely Suslinian if and only if the closure of every ray in X is an arc, a simple closed curve, or a simple closed curve with an arc adjoined by its endpoint? (A continuum X is said to be *finitely Suslinian* if every sequence of disjoint continua in X is a null sequence). The necessity can be proved along the following lines. Let R be a ray in a finitely Suslinian continuum X such that R is not compact. Let $x \in \mathrm{Cl}(R)\backslash R$. By Whyburn [**10**, p. 334] $R \cup \{x\}$ is arcwise connected. If 0 is the endpoint of R, it follows from Sierpinski's theorem that $R \cup \{x\}$ is the only arc in $R \cup \{x\}$ from 0 to x. Hence $R \cup \{x\}$ is an arc.

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