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A PAIR OF BIORTHOGONAL SETS OF POLYNOMIALS

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ABSTRACT. The two sets of polynomials $\{J_n^{(\alpha,\beta)}(x)\}$ and $\{K_n^{(\alpha,\beta)}(x)\}$ where $J_n^{(\alpha,\beta)}(x)$ is of degree n in x^k and $K_n^{(\alpha,\beta)}(x)$ is of degree n in x $(n = 0, 1, 2, \ldots)$ are constructed so that they are biorthogonal on $(0, 1)$ with respect to discrete distribution $d\Omega(\alpha, \beta; x)$ which has jumps $[\alpha q]_\infty[\beta q]_i(\alpha q)^i/[\alpha\beta q^2]_\infty[q]_i$ at $x = q^i$. When $k = 1$ these reduce to the little q-Jacobi polynomials. Various other properties are also given.

1. Introduction. Let $\alpha(x)$ be a "distribution" on $[a, b]$ (finite or infinite), that is, $\alpha(x)$ is bounded, incrasing on (a, b), with infinitely many points of increase, and such that $\int_a^b x^n\, d\alpha(x) < \infty$ for all $n \geqq 0$.

The set of polynomials $\{P_n(x)\}$, and the set of polynomials $\{Q_n(x)\}$, deg $Q_n(x) = n$ for $n = 0, 1, 2, \ldots$ are said to be biorthogonal on (a, b) with respect to $d\alpha(x)$ if

$$\int_a^b P_n(x)Q_m(x)d\alpha(x) = h_n\delta_{n,m} \tag{1.1}$$

with $h_n \neq 0$ and δ_{nm} the familiar Kronecker delta. In this paper we shall take $P_n(x)$ to be of degree n in x^k where k is fixed.

Didon [7] and Deruyts [6] considered this concept in some detail. For example, given the set $\{P_n(x)\}$ the set $\{Q_n(x)\}$ is uniquely determined and conversely.

This concept has been reconsidered in [11], [12]. It is shown that (1.1) is equivalent to (1.2) and (1.3),

$$\int_a^b x^i P_n(x)d\alpha(x) \begin{cases} = 0 & 0 \leqq i < n, \\ \neq 0 & i = n \end{cases} \tag{1.2}$$

and

$$\int_a^b x^{ik} Q_n(x)d\alpha(x) \begin{cases} = 0 & 0 \leqq i < n, \\ \neq 0 & i = n. \end{cases} \tag{1.3}$$

Thus if $k = 1$, $\{P_n(x)\}$ and $\{Q_n(x)\}$ collapse to the set of orthogonal polynomials associated with $\alpha(x)$ on (a, b). Both Didon and Deruyts gave as an example the case $d\alpha(x) = x^{\alpha-1}(1 - x)^{\beta-1}\, dx$ on $(0, 1)$. More

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recently Chai [13] suggested a polynomial of degree n in x^k which is orthogonal to $x^i (i = 0, 1, \ldots, n-1)$ on $(0, 1)$ with respect to $d\alpha(x) = x^{\alpha-1}(1-x)^{\beta-1}dx$. This case was considered earlier in [6] and [7].

Lately considerable interest has been shown in the little q-Jacobi polynomials $P_n(x; \alpha, \beta|q)$ [1, 2, 3, 10, 15] defined by

$$P_n(x; \alpha, \beta|q) = {}_2\phi_1\begin{bmatrix} q^{-n}, & \alpha\beta q^{n+1}; & q, & qx \\ & \alpha q & & \end{bmatrix} \tag{1.4}$$

whose orthogonality relation is then

$$\int_0^1 P_n(x; \alpha, \beta|q)P_m(x; \alpha, \beta|q)\, d\Omega(\alpha, \beta; x) = h_n\delta_{nm} \tag{1.5}$$

where $\Omega(\alpha, \beta; x)$ is a step function with jumps, at $x = q^i$,

$$\begin{aligned} d\Omega(\alpha, \beta; x) &= \frac{[\alpha q]_\infty[\beta q]_i}{[\alpha\beta q^2]_\infty[q]_i}(\alpha q)^i, \quad i = 0, 1, 2, \ldots, \\ h_n &= \frac{[\beta q]_n[q]_n[\alpha\beta q]_{2n}}{[\alpha q]_n[\alpha\beta q]_n[\alpha\beta q^2]_{2n}}(\alpha q)^n . \end{aligned} \tag{1.6}$$

In this paper we shall employ the following notation: For $|q| < 1$ we put $(a; q)_\infty = \prod_{j=0}^\infty (1 - aq^j)$ and for arbitrary complex number n, $(a; q)_n = (a; q)_\infty/(aq^n; q)_\infty$, so that in particular if n is a non-negative integer $(a; q)_n = (1-a)(1-aq)\cdots(1-q^{n-1})$ in which case the restriction $|q| < 1$ is no longer necessary. For writing economy we shall write $[a]_n$ to mean $(a; q)_n$. If the base is not q but, say, p then we shall mention it explicitly as $(a; p)_n$.

The Heine or q-series

$${}_r\phi_{r+k-1}\begin{bmatrix} a_1, a_2, \ldots, a_r; q, x \\ b_1, b_2, \ldots, b_{r-k+1} \end{bmatrix} = \sum_{j=0}^\infty \frac{[a_1]_j[a_2]_j\cdots[a_r]_j(-1)^{kj}q^{(1/2)kj(j-1)}x^j}{[q]_j[b_1]_j\cdots[b_{r-k+1}]_j} .$$

The q-difference operator (with base q) is $D_qf(x) = (f(x) - f(qx)/x$ and its n-th iterate is then

$$D_q^nf(x) = x^{-n}\sum_{j=0}^n ([q^{-n}]_j/[q]_j)q^jf(xq^j) \qquad (n = 0, 1, 2, \ldots). \tag{1.7}$$

Its fractional extension is then

$$D_q^\mu f(x) = x^{-\mu}\sum_{j=0}^\infty ([q^{-\mu}]_j/[q]_j)q^j f(xq^j) . \tag{1.8}$$

The q-binomial theorem is [14]

$$(1.9)\qquad \sum_{j=0}^{\infty} ([a]_j/[q]_j)z^j = [az]_\infty/[z]_\infty .$$

Gauss' theorem for the sum of ${}_2\phi_1$ series is given by [14]

$$(1.10)\qquad {}_2\phi_1\left[\begin{matrix} a, b; q, c/ab \\ c \end{matrix}\right] = [c/a]_\infty[c/b]_\infty/[c]_\infty[c/ab]_\infty$$

and the q-Vandermonde theorem is [14]

$$(1.11)\qquad {}_2\phi_1\left[\begin{matrix} q^{-n}, b; q, q \\ c \end{matrix}\right] = ([c/b]_n/[c]_n)b^n .$$

We shall find it useful to know the "moments" for the distribution (1.6). Using the q-binomial theorem (1.9) we get

$$(1.12)\quad \mu_j = \int_0^1 x^j \, d\Omega(\alpha, \beta; x) = [\alpha q]_\infty[\alpha\beta q^{2+j}]_\infty/[\alpha q^{1+j}]_\infty[\alpha\beta q^2]_\infty .$$

Jackson [8] gave the following q-analog of Taylor's formula for polynomials of degree $\leqq n$.

$$(1.13)\qquad f(x) = \sum_{r=0}^{n} x^r([1/x]_r/[q]_r)[D_q^r f(x)]_{x=1} .$$

2. A Biorthogonal System of Polynomials. Let us define for $n = 0, 1, 2, \ldots,$

$$(2.1)\qquad \mathscr{J}_n^{(\alpha,\beta)}(x; k|q) = \sum_{j=0}^{n} \frac{(q^{-nk}; q^k)_j[\alpha\beta q^{n+1}]_{kj}}{(q^k; q^k)_j[\alpha q]_{kj}} (qx)^{kj}$$

and

$$(2.2)\qquad \begin{aligned} \mathscr{K}_n^{(\alpha,\beta)}(x; k|q) = {} & \sum_{r=0}^{n} [\beta qx]_{n-r}(qx)^r/[q]_r[\beta q]_{n-r} \\ & \sum_{j=0}^{r} [q^{-r}]_j(\alpha q^{1+j}; q^k)_n q^{j(r-n)}/[q]_j . \end{aligned}$$

We shall prove that $\{\mathscr{J}_n^{(\alpha,\beta)}(x, k|q)\}$ and $\{\mathscr{K}_n^{(\alpha,\beta)}(x, k|q)\}$ are biorthogonal to each other. More specifically we assert that

$$(2.3)\qquad \int_0^1 \mathscr{J}_n^{(\alpha,\beta)}(x; k|q)\mathscr{K}_m^{(\alpha,\beta)}(x; k|q)d\Omega(\alpha, \beta; x) = h_n(\alpha, \beta, k)\delta_{nm}$$

where $\Omega(\alpha, \beta; x)$ is the distribution function given in (1.6) and

$$(2.4)\qquad h_n(\alpha, \beta, k) = (q^k; q^k)_n(q\alpha)^n/[\alpha\beta q^2]_{n-1}(1 - \alpha\beta q^{kn+n+1}) .$$

Before proving (2.3) we remark that when $k = 1$ the polynomials $\mathscr{J}_n^{(\alpha,\beta)}(x; 1|q)$ reduce to the little q-Jacobi polynomial (1.4). This is obvious. Less obvious is that $\mathscr{K}_n^{(\alpha,\beta)}(x; 1|q)$ also reduces to a constant multiple of the little q-Jacobi polynomial. To see this we put $k = 1$ in (2.2) and evaluate the inside sum by Gauss' summation formula (1.10) to get

$$\mathscr{K}_n^{(\alpha,\beta)}(x;1|q) = \frac{[\alpha q]_n[\beta q x]_n}{[\beta q]_n}\,{}_3\phi_2\left[\begin{matrix} q^{-n}, q^{-n}/\beta, 0; q, q \\ \alpha q, q^{-n}/\beta x \end{matrix}\right]$$
$$= \frac{[\alpha q]_n[\beta q x]_n}{[\beta q]_n}\,{}_2\phi_2\left[\begin{matrix} p^{-n}, \beta p^{-n}; p, x/\alpha p^{1+n} \\ p/\alpha, \beta x p^{-n} \end{matrix}\right],$$

where $p = 1/q$. Now the ${}_2\phi_2$ in the second equality can be transformed using Heine's transformation **[14]**

$${}_2\phi_2\left[\begin{matrix} A, B; p, y\,c/B \\ c, yA \end{matrix}\right] = ([y;p]_\infty/[yA;p]_\infty)\,{}_2\phi_1\left[\begin{matrix} c/B, A; p, y \\ c \end{matrix}\right]$$

to get, after changing the base back to q,

(2.5) $$\mathscr{K}_n^{(\alpha,\beta)}(x;1|q) = ([\alpha q]_n/[\beta q]_n)P_n(x;\alpha,\beta|q).$$

Thus we see that when $k = 1$ the orthogonality relation (2.3) reduces to (1.5).

Now to prove (2.3) it is sufficient to prove

(2.6a) $$I_{n,m} = \int_0^1 x^m \mathscr{J}_n^{(\alpha,\beta)}(x;k|q)\,d\Omega(\alpha,\beta;x)\begin{cases} = 0, & 0 \leqq m < n \\ \neq 0, & m = n \end{cases},$$

and

(2.6b) $$I'_{n,m} = \int_0^1 x^{km} \mathscr{K}_n^{(\alpha,\beta)}(x;k|q)\,d\Omega(\alpha,\beta;x)\begin{cases} = 0, & 0 \leqq m < n \\ \neq 0, & m = n \end{cases}.$$

Formula (2.6a) can be verified easily. Indeed substituting in (2.6a) for $\mathscr{J}_n^{(\alpha,\beta)}(x;k|q)$ from (2.1) and using (1.12), we get, for $m < n$,

$$I_{n,m} = \frac{1}{[\alpha\beta q^2]_{n-1}}\sum_{j=0}^{n}\frac{(q^{-nk};q^k)_j}{(q^k;q^k)_j}\,q^{kj}[\alpha q^{1+kj}]_m[\alpha\beta q^{2+m+kj}]_{n-m-1}$$
$$= \frac{1}{[\alpha\beta q^2]_{n-1}}\,[D_{q^k}^n\{[\alpha q x]_m[\alpha\beta q^{2+m}x]_{n-m-1}\}]_{x=1}.$$

Since the n-th q-derivative of a polynomial of degree $k < n$ vanishes, it follows that $I_{n,m} = 0$ for $m = 0, 1, 2, \ldots, n-1$. In case $m = n+s-1$ $(s \geqq 1)$ we write

(2.7) $$I_{n,n+s-1} = \frac{1}{[\alpha\beta q^2]_{n-1}}\sum_{j=0}^{n}\frac{(q^{-nk};q^k)_j}{(q^k;q^k)_j}\,q^{kj}\,\frac{[\alpha q^{1+kj}]_{n+s-1}}{[\alpha\beta q^{1+n+kj}]_s}$$

But

(2.8) $$[x]_{n+s-1}/[\beta x q^n]_s = P_{n-1}(x) + \sum_{r=0}^{s-1} A_r/1 - \beta x q^{n+r}$$

where $P_{n-1}(x)$ is a polynomial of degree $n - 1$ and

$$A_r = ((-1)^n q^{-(1/2)n(n+1)-nr}/\beta^{n+r})([\beta q]_{n+r}[1/\beta]_{s-r-1}/[q]_r[q]_{s-r-1}) .$$

Substituting from (2.8) in (2.7) and observing that

$$\sum_{j=0}^{n} \frac{(q^{-nk}; q^k)_j}{(q^k; q^k)_j} q^{kj} P_{n-1}(\alpha q^{1+kj}) = [D_{q^k}^n P_{n-1}(x\alpha q)]_{x=1} = 0 ,$$

on interchanging the order of summation and then using (1.11) (with base q^k), we get

$$(2.9)\quad \begin{aligned} I_{n,n+s-1} &= (-\alpha)^n((q^k; q^k)_n/[\alpha\beta q^2]_{n-1})q^{n(n+1)/2} \\ &\cdot \sum_{r=0}^{s-1} \frac{[\beta q]_{n+r}[1/\beta]_{s-r-1}}{[q]_r[q]_{s-r-1}} \cdot \frac{\beta^{-r}}{(\alpha\beta q^{1+r+n}; q^k)_{n+1}} . \end{aligned}$$

In particular

$$(2.10)\quad I_{n,n} = (-\alpha)^n q^{n(n+1)/2}(q^k; q^k)_n[\beta]_n/[\alpha\beta q^2]_{n-1}(\alpha\beta q^{1+n}; q^k)_{n+1} .$$

This completes the verification of (2.6a).

Now we proceed to prove (2.6b). We first require the following formula which is a q-analog of a result of Carlitz [5]. Let $p = q^{-1}$, then

$$(2.11)\quad \begin{aligned} \alpha^n q^{n(1+ki)}(q^{-ki}; q^k)_n &= \sum_{r=0}^{n} ((1/\alpha)p^{1+ki}; p)_r/(p; p)_r/\alpha^r \cdot \\ &\quad p^{-r(a+ki)} \sum_{j=0}^{r} ((p^{-r}; p)_j/(p; p)_j) p^{j(1+n)}(\alpha q^{1+j}; q^k)_n , \end{aligned}$$

which can be obtained from (1.13) and (1.7) (with base $p = q^{-1}$) with $f(x) = x^n[\alpha q/x; q^k]_n$ evaluated at $x = \alpha q^{1+ki}$.

Furthermore using the q-binomial theorem (1.9), we can show that

$$(2.12)\quad \int_0^1 x^r[\beta q x]_m d\Omega(\alpha, \beta; x) = [\alpha q]_r[\beta q]_m/[\alpha\beta q^2]_{m+r} .$$

Hence the left hand side of (2.6b) with the aid of (2.2) and (2.12) is

$$I'_{n,m} = \frac{[\alpha q]_{km}}{[\alpha\beta q^2]_{n+km}} \sum_{r=0}^{n} \frac{[\alpha q^{1+km}]_r}{[q]_r} q^r \sum_{j=0}^{r} \frac{[q^{-r}]_j(\alpha q^{1+j}; q^k)_n}{[q]_j} q^{j(r-n)}$$

which in view of (2.11), with p replaced by $1/q$, becomes

$$(2.13)\quad I'_{n,m} = q^{n(km+1)}\alpha^n([\alpha q]_{km}/[\alpha\beta q^2]_{km+n}(q^{-km}; q^k)_n .$$

Formula (2.13) is valid for all integers m and in particular for $0 \leqq m < n$, $I'_{mn} = 0$. On the other hand if $n = m$, we get

$$(1.14)\quad I'_{n,m} = (-1)^n q^{1/2kn(n-1)+n} \alpha^n([\alpha q]_{kn}/[\alpha\beta q^2]_{kn+n})(q^k; q^k)_n .$$

This completes the verification of (2.6b). Now to verify (2.4) we only need to multiply I'_{nm} by the coefficient of x^{nk} in (2.1).

3. Some Properties. We shall obtain in this section some properties of our system of biorthogonal polynomials. All of these properties imply for $k = 1$ corresponding properties for the little q-Jacobi polynomials some of which, to our best knowledge, are new.

$$(3.1) \qquad \mathscr{J}_n^{(\alpha,\beta)}(x; k|q) = \sum_{m=0}^{k} c(n, m)\mathscr{J}_m^{(\gamma,\delta)}(x; k|q)$$

where

$$c(n, m) = \frac{(-1)^m q^{1/2km(m+1)}}{(q^k; q^k)_m[\gamma\delta q^{1+m}]_{km}} \sum_{j=0}^{n-m} \frac{(q^{-nk}; q^k)_{m+j}}{(q^k; q^k)_j} \cdot \frac{[\alpha\beta q^{n+1}]_{km+kj}[\gamma q]_{km+kj}}{[\alpha q]_{km+kj}[\gamma\delta q^{m+2+km}]_{kj}} q^{kj}$$

For $k = 1$ (3.1) reduces to the connection coefficient formula for the little q-Jacobi polynomials due to Andrews and Askey [3].

In the following we take $a = q^\alpha$, $b = q^\beta$ and $c = q^\gamma$. One can then show

$$(3.2) \qquad D_q^{n+\beta}[x^{\alpha+\beta+n}(x^k q^{k-kn}; q^k)_n] = ([aq]_\infty/[abq^{1+n}]_\infty x^\alpha \mathscr{J}_n^{(a,b)}(x; k|q).$$

If β is a positive integer this reduces to a Rodrigues formula for $\mathscr{J}_n^{(a,b)}(x; k|q)$

$$(3.3) \qquad D_q^{-\mu}[x^\lambda \mathscr{J}_n^{(a,b)}(x; k|q)] = \frac{x^{\lambda+\mu}}{[q^{1+\lambda}]_\mu} \sum_{j=0}^{n} \frac{(q^{-nk}; q^k)_j[q^{1+\alpha+\beta+n}]_{kj}[q^{1+\lambda}]_{kj}}{(q^k; q^k)_j[q^{1+\alpha}]_{kj}[q^{1+\lambda+\mu}]_{kj}} (qx)^{kj},$$

which for $\lambda = \alpha$ reduces to the following interesting formula

$$(3.4) \qquad D_q^{-\mu}[x^\alpha \mathscr{J}_n^{(a,b)}(x; k|q)] = (x^{\alpha+\mu}/[aq]_\mu)\mathscr{J}_n^{(\alpha+\mu,\beta-\mu)}(x; k|q)$$

whereas for $\mu = \beta - \gamma$, $\lambda = \alpha + \gamma + n$, (3.3) reduces to

$$(3.5) \qquad D_q^{\gamma-\beta}[x^{\alpha+\gamma+n}\mathscr{J}_n^{(a,b)}(x; k|q)] = x^{\alpha+\beta+n}([abq^{1+n}]_\infty/[acq^{1+n}]_\infty)\mathscr{J}_n^{(a,c)}(x; k|q).$$

(3.4) and (3.5) for $k = 1$ reduce to q-analog of formulas given by Askey and Fitch [4].

If $x^{kn} = \sum_{m=0}^{n} D(n, m)\mathscr{J}_m^{(\alpha,\beta)}(x; k|q)$, then

$$(3.6) \qquad D(n, m) = \frac{(-1)^m(q^k; q^k)_n[\alpha q]_{kn}q^{1/2km(m-1)}}{(q^k; q^k)_m(q^k; q^k)_{n-m}[\alpha\beta q^{m+1}]_{km}[\alpha\beta q^{2+m+km}]_{kn-km}}.$$

If $x^n = \sum_{m=0}^{n} E(n, m)\mathscr{K}_m^{(\alpha,\beta)}(x; k|q)$, then

$$E(n, m) = (-1)^m(1 - \alpha\beta q^{1+m+km}) \sum_{r=0}^{n-m} \frac{[\beta q]_{m+r}[1/\beta]_{n-m-r}}{[q]_r[q]_{n-m-1}} \cdot \frac{\beta^{-r} q^{-1/2m(m-1)}}{(\alpha\beta q^{1+r+m}; q^k)_{m+1}} . \tag{3.7}$$

Both formulas (3.6) and (3.7) reduce, for $k = 1$, to

$$x^n = [\alpha q]_n \sum_{j=0}^{n} \frac{[q_n]}{[q]_j[q]_{n-j}} \frac{(-1)^j q^{j(j-1)/2} P_j(x: \alpha, \beta|q)}{[\alpha\beta q^{j+1}]_j[\alpha\beta q^{2j+2}]_{n-j}} . \tag{3.8}$$

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